# Existence and uniqueness of solutions for the first order non-linear differential equations with multi-point boundary conditions

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**Abstract.** This article discusses the existence and uniqueness of solutions for the system of non-linear first order ordinary differential equations with multipoint boundary conditions. The Green function is constructed, and the problem is reduced to the equivalent integral equation. Existence and uniqueness of the solution to this problem is studied using the Banach contraction mapping principle and Schaefer's fixed point theorem.

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## 1. Introduction and Problem Statement

Multipoint boundary value problems for the ordinary differential equations (ODEs) arise in modeling the broad class of natural processes. For example, if to consider the dynamical system with $n$ degrees of freedom, exactly $n$ states observed at $n$ different instants of time, then the mathematical description of this system leads to the multipoint boundary value problem. As another example we can note the vibrations of a uniform cross-section string composed of $N$ parts of different densities and also some problems in the theory of elastic stability [29]. As another example we can note the vibrations of a uniform cross-section string composed of $N$ parts of different densities and also some problems in the theory of elastic stability [29]. In some cases multipoint boundary value problems also arise when discrediting the boundary value problems for the partial differential equations. Due to these and many other strong relation with a

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broad range of applications in different fields of physics and mathematics such problems are under the intensive focus of many researchers [9, 10].

It should be noted that the multipoint boundary value problems have been well studied for the second order differential equations (see [4, 11–13, 22, 24] and references therein). These works were mainly initiated by Ilin and Moiseev [12]. Since then, nonlinear multi-point boundary-value problems have been studied by several authors using the Leray-Schauder Continuation Theorem, Leray-Schaudern nonlinear alternatives, coincidence degree theory, and fixed point theorem in cones. However, for the first order differential equations, such problems have been less studied. Examples of such works can be shown [1, 3, 15–17, 19, 23, 25, 30, 31]

Similar problems for two-point and integral boundary value problems are considered in [2, 5–8, 14, 18, 20, 21, 24, 26–28].

Note that the problem under consideration in this work was also studied by M. Urabe. In [30] he gives the similar result. But those results were obtained under more strong conditions. Thus he requires the existence of the approximate solution of the considered problem with high enough accuracy that cannot be achieved in many cases. Moreover, the fundamental matrix of some quasilinear system also should be known in [30] that is difficult problem itself. The results in this work are obtained by only the initial data of the problem and we do not need solving any auxiliary problem.

In this work for the first time Green function is constructed for the multi-point boundary value problem. The considered problem is reduced to the equivalent integral equations. Then the existence and uniqueness result are studied using the Banach contraction mapping principle. The existence of the solution is also proved by applying Schaefer's fixed point theorem.

Consider the existence and uniqueness of solutions of the nonlinear differential equations of the type

$$\dot{x}(t) = f(t, x), \ t \in [0, T], \tag{1}$$

with multi-point boundary conditions

$$\sum_{i=0}^{m} l_i x(t_i) = \alpha, \tag{2}$$

where $l_i, i = 1, 2, ..., m$ are constant square matrices of order $n$ such that $\det N \neq 0$, $N = \sum_{i=0}^{m} l_i; f : [0, T] \times R^n \to R^n$ is a given function; points $t_i, i = 1, 2, ..., m$ satisfy the condition $0 = t_0 < t_1 < \cdots < t_m = T$. We denote by $C([0, T]; R^n)$ the Banach space of all continuous functions from $[0, T]$ into $R^n$ with the norm $\|x\| = \max \{|x(t)| : t \in [0, T]\}$ where $|\cdot|$ is the norm in the space $R^n$.

This paper is organized as follows. In Section 2 we introduce definitions and lemmas which are the key tools for our main result. Section 3 focuses the theorems on the existence and uniqueness of the solution of problem (1)-(2) established under some sufficient conditions on the nonlinear terms.

## 2. Preliminaries

We define the solution of problem (1)-(2) as follows:

**Definition 1.** *The function $x \in C([0,T], R^n)$ is called a solution of problem* (1)-(2) *if $\dot{x}(t) = f(t, x(t))$ for each $t \in [0,T]$, and boundary conditions* (2) *are satisfied.*

For the sake of simplicity, we can consider the following problem:

$$\dot{x} = y(t), \quad t \in [0,T], \tag{3}$$

$$\sum_{i=0}^{m} l_i x(t_i) = \alpha. \tag{4}$$

**Lemma 1.** *Let $y \in C([0,T], R^n)$. Then the unique solution $x(t) \in C([0,T], R^n)$ of the boundary value problem for differential equation* (3) *with boundary conditions* (4) *is given by*

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)y(\tau)d\tau, \tag{5}$$

*where*

$$G(t,\tau) = \begin{cases} G_1(t,\tau), & t \in [0, t_1], \\ G_2(t,\tau), & t \in (t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ G_m(t,\tau), & t \in (t_{m-1}, T], \end{cases}$$

*with*

$$G_i(t,\tau) = \begin{cases} N^{-1}l_0, & t_0 \leq \tau \leq t_1, \\ N^{-1}\left(\sum_{k=0}^{1} l_k\right), & t_1 < \tau \leq t_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ N^{-1}\left(\sum_{k=0}^{i-1} l_k\right), & t_{i-1} < \tau \leq t_i, \\ N^{-1}\left(\sum_{k=0}^{i} l_k\right), & t_i < \tau \leq t, \\ -N^{-1}\left(\sum_{k=i+1}^{m} l_i\right), & t < \tau \leq t_{i+1}, \\ -N^{-1}\left(\sum_{k=i+2}^{m} l_i\right), & t_{i+1} < \tau \leq t_{i+2}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -N^{-1}l_m, & t_{m-1} < \tau \leq T, \end{cases}$$

$i = 1, 2, ..., m.$

*Proof.* If the function $x = x(\cdot)$ is a solution of equation (3), then for $t \in (0, T)$

$$x(t) = x(0) + \int_0^t y(\tau)d\tau, \tag{6}$$

where $x_0$ is an arbitrary constant vector. Now we define $x_0$ so that, the function in equality (6) satisfies condition (4). Then we have

$$\sum_{i=0}^m l_i[x_0 + \int_0^{t_i} y(s)\,ds] = \alpha.$$

This obviously gives

$$x_0 = N^{-1}\alpha - N^{-1}\left[\sum_{i=1}^m l_i \int_0^{t_i} y(s)\,ds\right]. \tag{7}$$

Considering the value $x_0$ determined from the equality (7) in (6) we get

$$x(t) = N^{-1}\alpha - N^{-1}\left[\sum_{i=1}^m l_i \int_0^{t_i} y(s)\,ds\right] + \int_0^t y(s)\,ds. \tag{8}$$

Suppose that $t \in [0, t_1]$ Then equality (8) may be written as follows:

$$x(t) = N^{-1}\alpha - N^{-1}\left(l_1 \int_0^t y(\tau)d\tau + l_1 \int_t^{t_1} y(\tau)d\tau\right) - N^{-1}\left(l_2 \int_0^t y(\tau)d\tau + l_2 \int_t^{t_1} y(\tau)d\tau\right)$$

$$-N^{-1}l_2 \int_{t_1}^{t_2} y(\tau)d\tau - N^{-1}\left(l_3 \int_0^t y(\tau)\,d\tau + l_3 \int_t^{t_1} y(\tau)\,d\tau\right) - N^{-1}l_3\left(\sum_{i=1}^2 \int_{t_i}^{t_{i+1}} y(\tau)d\tau\right)$$

$$-\ldots - N^{-1}\left(l_m \int_0^t y(\tau)d\tau + l_m \int_t^{t_1} y(\tau)\,d\tau\right) - N^{-1}l_m\left(\sum_{i=1}^m \int_{t_i}^{t_{i+1}} y(\tau)\,d\tau\right) + \int_0^t y(\tau)\,d\tau.$$

One can easily rewrite this equality in the equivalent form:

$$x(t) = N^{-1}\alpha + \int_0^t \left(E - N^{-1}\sum_{i=1}^m l_i\right) y(\tau)d\tau - N^{-1}\int_t^{t_1}\left(\sum_{i=1}^m l_i\right) y(\tau)d\tau$$

$$-N^{-1}\left(\sum_{i=2}^m l_i\right)\int_{t_1}^{t_2} y(\tau)\,d\tau - N^{-1}\left(\sum_{i=3}^m l_i\right)\int_{t_2}^{t_3} y(\tau)d\tau - \ldots - N^{-1}l_m \int_{t_{m-1}}^T y(\tau)d\tau, \tag{9}$$

where $E$ is an identity matrix. Since equality

$$\left(E - N^{-1} \sum_{i=1}^{m} l_i\right) = N^{-1} l_0$$

is valid following function may be introduced

$$G_1(t, \tau) = \begin{cases} N^{-1} l_0, & t_0 \leq \tau \leq t, \\ -N^{-1}\left(\sum_{i=1}^{m} l_i\right), & t < \tau \leq t_1, \\ -N^{-1}\left(\sum_{i=2}^{m} l_i\right), & t_1 < \tau \leq t_2, \\ -N^{-1}\left(\sum_{i=3}^{m} l_i\right), & t_2 < \tau \leq t_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -N^{-1} l_m, & t_{m-1} < \tau \leq T. \end{cases}$$

Considering the last one we can transfer equality (9) to the following an integral equation

$$x(t) = N^{-1}\alpha + \int_0^T G_1(t, \tau) y(\tau) d\tau, \quad t \in [0, t_1].$$

Assuming $t \in (t_1, t_2]$ we can write equality (8) in the following form

$$x(t) = N^{-1}\alpha - N^{-1}\left(\sum_{i=1}^{m} l_i\right) \int_0^{t_1} y(t) dt - N^{-1}\left(\sum_{i=2}^{m} l_i\right)\left(\int_{t_1}^{t} y(\tau) d\tau + \int_t^{t_2} y(\tau) d\tau\right)$$

$$- N^{-1}\left(\sum_{i=3}^{m} l_i\right) \int_{t_2}^{t_3} y(\tau) d - N^{-1}\left(\sum_{i=4}^{m} l_i\right) \int_{t_3}^{t_4} y(\tau) d\tau - \ldots - N^{-1} l_m \int_{t_{m-1}}^{T} y(\tau) d\tau$$

$$+ \int_0^{t_1} y(t) dt + \int_{t_1}^{t} y(\tau) d\tau.$$

From this it is easy to derive

$$x(t) = N^{-1}\alpha + N^{-1} l_0 \int_0^{t_1} y(t) dt + N^{-1}\left(\sum_{i=0}^{1} l_i\right)\left(\int_{t_1}^{t} y(\tau) d\tau\right) - N^{-1}\left(\sum_{i=2}^{m} l\right) \int_t^{t_2} y(\tau) d\tau$$

$$- N^{-1}\left(\sum_{i=3}^{m} l_i\right) \int_{t_2}^{t_3} y(\tau) d - N^{-1}\left(\sum_{i=4}^{m} l_i\right) \int_{t_3}^{t_4} y(\tau) d\tau - \ldots - N^{-1} l_m \int_{t_{m-1}}^{T} y(\tau) d\tau.$$

In this step we again introduce a new function

$$G_2(t, \tau) = \begin{cases} N^{-1}l_0, & t_0 \le \tau \le t_1, \\ N^{-1}\left(\sum_{i=0}^{1} l_i\right), & t_1 < \tau \le t, \\ -N^{-1}\left(\sum_{i=2}^{m} l_i\right), & t < \tau \le t_2, \\ -N^{-1}\left(\sum_{i=3}^{m} l_i\right), & t_2 < \tau \le t_3, \\ \dots\dots\dots\dots\dots\dots\dots \\ -N^{-1}l_m, & t_{m-1} < \tau \le T. \end{cases}$$

Therefore we conclude that if $t \in (t_1, t_2]$ then the solution of the considered boundary value problem can be presented in the form

$$x(t) = N^{-1}\alpha + \int_0^T G_2(t, \tau)y(\tau)d\tau.$$

Continuing this process in a similar way, for the segment $t \in (t_i, t_{i+1}]$ we get

$$G_i(t, \tau) = \begin{cases} N^{-1}l_0, & t_0 \le \tau \le t_1, \\ N^{-1}\left(\sum_{i=0}^{1} l_i\right), & t_1 < \tau \le t_2, \\ \dots\dots\dots\dots\dots\dots\dots \\ N^{-1}\left(\sum_{k=0}^{i-1} l_k\right), & t_{i-1} < \tau \le t_i, \\ N^{-1}\left(\sum_{k=0}^{i} l_k\right), & t_i < \tau \le t, \\ -N^{-1}\left(\sum_{k=i+1}^{m} l_i\right), & t < \tau \le t_{i+1}, \\ -N^{-1}\left(\sum_{k=i+2}^{m} l_i\right), & t_{i+1} < \tau \le t_{i+2}, \\ \dots\dots\dots\dots\dots\dots\dots \\ -N^{-1}l_m, & t_{m-1} < \tau \le T. \end{cases}$$

Finally we see that the solution of boundary value problem (1)-(2) may be presented in the form

$$x(t) = N^{-1}\alpha + \int_0^T G(t, \tau)y(\tau)d\tau.$$

Proof is completed.

**Lemma 2.** *Let $f \in C([0, T] \times R^n; R^n)$. Then the function $x(t)$ is a solution of boundary value problem* (1)-(2) *if and only if $x(t)$ is a solution of the integral equation*

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau)f(\tau, x(\tau))d\tau. \tag{10}$$

*Proof.* Let $x(t)$ be a solution of boundary value problem (1)-(2). This lemma can be proved analogously to Lemma 1. By direct checking it is easy to justify that the solution of integral equation (10) satisfies also boundary value problem (1)-(2). Lemma 2 is proved.

## 3. Main results

Let us set the following conditions:
(H1) The function $f \in C([0,T] \times R^n; R^n)$ is continuous;
(H2) There exist a constant $M \geq 0$ such that

$$|f(t,x) - f(t,y)| \leq M |x - y|$$

for $t \in [0,T]$ each and all $x, y \in R^n$;
(H3) There exists a constant $K \geq 0$ such that $|f(t,x)| \leq K$ for each $t \in [0,T]$ and all $x \in R^n$.

We give here the following uniqueness result.

**Theorem 1.** *Assume that, assumptions(H1) and (H2) hold and*

$$L = TSM < 1, \tag{11}$$

*where*

$$S = \max_{[0,T]\times[0,T]} \|G(t,\tau)\|.$$

*Then boundary value problem* (1)-(2) *has a unique solution on* $[0,T]$.

*Proof.* To prove the statement of the above theorem we transform the boundary value problem (1)- (2) into a fixed point problem. Consider the operator

$$(Fx)(t) = N^{-1}\alpha + \int_0^T G(t,\tau)f(\tau, x(\tau))d\tau. \tag{12}$$

It is not difficult to see that

$$F : C([0,T];R^n) \to C([0,T];R^n)$$

Obviously, the fixed points of the operator $F$ are solutions of boundary problem (1)-(2).
Setting

$$\max_{[0,T]} |f(t,0)| = M_f$$

we take

$$r \geq \frac{\left\|N^{-1}d\right\| + M_f TS}{1 - L}$$

We show that $FB_r \subset B_r$, where

$$B_r = \{x \in C([0,T]R^n) : \|x\| \leq r\}$$

For $x \in B_r$, using (H1), we get

$$\|(Fx)(t)\| \leq \left\|N^{-1}\alpha\right\| + \int_0^T |G(t,\tau)| \left(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|\right)d\tau$$

$$\leq \left\|N^{-1}d\right\| + S\int_0^T (M|x| + M_f)dt \leq \left\|N^{-1}d\right\| + SMrT + M_f TS \leq \frac{\left\|N^{-1}\alpha\right\| + M_f TS}{1 - L} \leq r.$$

In order to show that the operator $F$ is a contraction, for any $x, y \in B_r$ we have

$$|Fx - Fy| \leq \int_0^T |G(t,\tau)\left(f(\tau,x(\tau)) - f(\tau,y(\tau))\right)|d\tau \leq \int_0^T |G(t,\tau)| |f(\tau,x(\tau)) - f(\tau,y(\tau))| d\tau$$

$$\leq SM\int_0^T |x(t) - y(t)| dt \leq SMT \max_{[0,T]} |x(t) - y(t)| \leq SMT \|x - y\|$$

or

$$\|Fx - Fy\| \leq L \|x - y\|.$$

As one can see $F$ is contraction by condition (11). So, boundary value problem (1)-(2) has a unique solution.

Now we give a theorem on the existence of solutions for the considered problem.

**Theorem 2.** *Assume conditions(H1) and (H3) hold. Then boundary value problem* (1)-(2) *has at least one solution on* $[0,T]$.

*Proof.* Let $F$ be the operator defined by (12). We use Schaefer's fixed point theorem to prove that $F$ has a fixed point.

First we show that $F$ is continuous. To do this suppose that be $\{x_n\}$ a sequence such that $x_n \to x$ in $C\left([0,T]; R^n\right)$. Then for each $t \in [0,T]$

$$|(Fx)(t) - (Fx_n)(t)| = \left|\int_0^T G(t,\tau)\left(f(\tau, x(\tau)) - f(\tau, x_n(\tau))\right)d\tau\right|$$

$$\leq TSM |x(t) - x_n(t)| \leq L \|x - x_n\|.$$

It gives $\|(Fx)(t) - (Fx_n)(t)\| \to 0$ as $n \to \infty$, which implies that the operator $F$ is continuous.

The next step is to show that $F$ maps bounded sets from $C([0,T];R^n)$ into bounded sets in $C([0,T];R^n)$. To do this it is enough to show that for any $\eta > 0$ there exists a positive constant $\omega$ such that for each $x \in B_\eta = \{x \in C([0,T];R^n) : \|x\| \le \eta\}$ we have $\|F(x)\| \le \omega$. For each $t \in [0,T]$ we have

$$|(Fx)(t)| \le |N^{-1}\alpha\| + TSK.$$

From this we obtain

$$\|(Fx)(t)\| \le \|N^{-1}\alpha\| + TSK = \omega.$$

Now we show that $F$ maps bounded sets from $C([0,T];R^n)$ into equicontinuous sets in $C([0,T];R^n)$. Take $\xi_1, \xi_2 \in [0,T], \xi_1 < \xi_2$, and assume that $B_\eta$ is a bounded set in $C([0,T];R^n)$ and let $x \in B_\eta$.

Here two cases should be considered

**Case 1.** Let $\xi_1, \xi_2 \in [t_i, t_i + 1]$. Then

$$F(x(\xi_2)) - F(x(\xi_1)) = \int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau, x(\tau))d\tau$$

$$-\int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau, x(\tau))d\tau - \int_{t_i}^{\xi_1} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau, x(\tau))d\tau$$

$$+\int_{\xi_1}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau, x(\tau))d\tau = \int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau, x(\tau))d\tau$$

$$+\int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau, x(\tau))d\tau = \int_{\xi_1}^{\xi_2} f(\tau, x(\tau))d\tau.$$

**Case 2.** In this case let $\xi_1 \in [t_{i-1}, t_i), \xi_2 \in [t_i, t_{i+1}]$. Then

$$F(x(\xi_2)) - F(x(\xi_1)) = \int_{t_{i-1}}^{t_i} N^{-1}\left(\sum_{k=0}^{i-1} l_i\right) f(\tau, x(\tau))d\tau$$

$$+\int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_i\right) f(\tau, x(\tau))d\tau - \int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_i\right) f(\tau, x(\tau))d\tau$$

$$-\int_{t_{i-1}}^{\xi_1} N^{-1}\left(\sum_{k=0}^{i-1} l_i\right) f(\tau, x(\tau))d\tau + \int_{\xi_1}^{t_i} N^{-1}\left(\sum_{k=i}^{m} l_i\right) f(\tau, x(\tau))d\tau$$

$$+ \int_{t_i}^{t_{i+1}} N^{-1} \left( \sum_{k=i+1}^{m} l_i \right) f\left(\tau, x\left(\tau\right)\right) d\tau = \int_{\xi_1}^{t_i} f\left(\tau, x\left(\tau\right)\right) d\tau$$

$$+ \int_{t_i}^{\xi_2} f\left(\tau, x\left(\tau\right)\right) d\tau = \int_{\xi_1}^{\xi_2} f\left(\tau, x\left(\tau\right)\right) d\tau.$$

As $t_2 \to t_1$, the right-hand side of both above equalities tends to zero. Considering the above results and the Arzela-Ascoli theorem, we can conclude that $F : C\left([0, T]; R^n\right) \to C\left([0, T]; R^n\right)$ is completely continuous.

Here we establish apriori bounds i.e. we show that the set $\Delta = \{x \in C\left([0, T]; R^n\right) : x = \lambda F\left(x\right)\}$ for some $0 < \lambda < 1$ is bounded. Let $x \in \Delta$. Then $x = \lambda F\left(x\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in [0, T]$ we have

$$x(t) = \lambda N^{-1} \alpha + \lambda \int_0^T G(t, \tau) f(\tau, x(\tau)) d\tau.$$

From here

$$\|x\| \leq \left\| N^{-1} \alpha \right\| + SKT.$$

Therefore, the set $\Delta$ is bounded. The statement of the Schaefer's fixed point theorem may be applied and derived that the operator $F$ has at least one fixed point. So, there exists at least one solution for problems (1)- (2) on $[0, T]$.

## 4. Conclusion

It should be noted that the method considered in this paper are general enough and can be transformed to the different forms to cover a wide class of problems. We established here the results on the existence and uniqueness of the solutions for the first order nonlinear differential equations with multi-point boundary conditions. Given in the paper method can be used in similar multi-point problems for the ordinary differential equations

$$\dot{x} = f(t, x), \ t \in [0, T],$$

with multi-point and integral boundary conditions of the form

$$\sum_{i=0}^{m} l_i x\left(t_i\right) + \int_0^T n\left(t\right) x\left(t\right) dt = \alpha.$$

Here $0 = t_0 < t_1 < ... < t_{m-1} < t_m = T$; $n\left(t\right) \in R^{n \times n}$ is a given function; $l_i \in R^{n \times n}, i = 1, 2, ..., m$ are given matrices; $\alpha \in R^n$ is a given vector; and

$$\det N \neq 0, N = \sum_{i=0}^{m} l_i + \int_0^T n\left(t\right) dt.$$

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