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On regular hypersemigroups

Niovi Kehayopulu

Abstract. It is shown that an hypersemigroup $(S, \circ)$ is regular if and only if the set of all quasi-ideals of S with the operation "$*$" is a von Neumann regular semigroup. It is both regular and intra-regular if and only if the set of all quasi-ideals of S with the operation "$*$" is a band.

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It has been shown in Semigroup Forum [2] that an le-semigroup $(S, \cdot, \leq)$ is regular if and only if the set Q of all quasi-ideal elements of S with the multiplication "$\cdot$" of S is a von Neumann regular semigroup. Moreover, it has been proved that if S is both regular and intra-regular, then $(Q, \cdot)$ is a band. "Conversely", if the quasi-ideal elements of S are idempotent, then S is both regular and intra-regular. As a consequence, an le-semigroup S is both regular and intra-regular if and only if $(Q, \cdot)$ is a band.

As an example to the paper in Turkish J. Math. [7], we examine the above results on lattice ordered semigroups in case of an hypersemigroup. An hypersemigroup $(S, \circ)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ x) * \{a\}$; that is, for every $a \in S$ there exists $y \in a \circ x$ such that $a \in y \circ a$. It is called *intra-regular* if for every $a \in S$ there exist $x, y \in S$ such that $a \in (x \circ a) * (a \circ y)$; that is, for every $a \in S$ there exist $x, y \in S$, $u \in x \circ a$ and $v \in a \circ y$ such that $a \in u \circ v$. A subset A of an hypersemigroup $(S, \circ)$ is called *idempotent* if $A * A = A$. For notations and definitions not given in the present paper we refer to [7].

Lemma 1 [3] *Let $(S, \circ)$ be an hypersemigroup. If S is regular, then the right ideals and the left ideals of S are idempotent and for every right ideal A and every left ideal B of S, the product $A * B$ is a quasi-ideal of S.*

Lemma 2 [4, 5] *An hypersemigroup $(S, \circ)$ is regular if and only if, for any nonempty subset A of S, we have $A \subseteq A * S * A$.*

Lemma 3 *Let $(S, \circ)$ be an hypersemigroup, A a right ideal and B a left ideal of S. Then the intersection $A \cap B$ is a quasi-ideal of S.*

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Email address: nkehayop@math.uoa.gr (N. Kehayopulu)

Proof First of all, since A is a right ideal and B is a left ideal of S, the intersection $A \cap B$ is nonempty. Indeed: Take an element $a \in A$ and an element $b \in B$ $(A, B \neq \emptyset)$; then $a \circ b \subseteq A * B \subseteq A * S \subseteq A$ and $a \circ b \subseteq A * B \subseteq S * B \subseteq B$, so $a \circ b \subseteq A \cap B$. Since $a \circ b$ is a nonempty set, the set $A \cap B$ is nonempty as well (see also [5]). We also have

$$\Big((A \cap B) * S\Big) \cap \Big(S * (A \cap B)\Big) \subseteq (A * S) \cap (S * B) \subseteq A \cap B,$$

thus $A \cap B$ is a quasi-ideal of S. □

Lemma 4 [4, 5] *An hypersemigroup $(S, \circ)$ is regular if and only if, for every right ideal A and every left ideal B of S, we have $A \cap B \subseteq A * B$ (equivalently, $A \cap B = A * B$).*

Lemma 5 *If $(S, \circ)$ is a regular hypersemigroup, then $S * S = S$.*

Proof Since S is regular, for every nonempty subset A of S, by Lemma 2, we have $A \subseteq A * S * A$. Thus we have $S \subseteq (S * S) * S \subseteq S * S \subseteq S$ and so $S * S = S$. □

A semigroup $(S, \cdot)$ is called *von Neumann regular* (or just *regular*) if for each $a \in S$ there exists $x \in S$ such that $a = axa$ [1, 8].

As always, $\mathcal{P}^*(S)$ denotes the set of all nonempty subsets of S.

Theorem 6 *An hypersemigroup $(S, \circ)$ is regular if and only if the set $\mathcal{Q}$ of all quasi-ideals of S with the multiplication "$*$" of $\mathcal{P}^*(S)$ is a von Neumann regular semigroup.*

Proof $\Longrightarrow$. First of all, for every quasi-ideal Q of S, we have

$$Q = (Q * S) \cap (S * Q) \tag{1}$$

In fact: Since S is regular, $R(Q)$ is a right ideal and $L(Q)$ is a left ideal of $(S, \circ)$, by Lemma 1, they are idempotent and we have

$$\begin{aligned} Q &\subseteq Q \cup (Q * S) = R(Q) = R(Q) * R(Q) = \Big(Q \cup (Q * S)\Big) * \Big(Q \cup (Q * S)\Big) \\ &= Q * Q \cup Q * S * Q \cup Q * Q * S \cup Q * S * Q * S \subseteq Q * S \end{aligned}$$

and

$$\begin{aligned} Q &\subseteq Q \cup (S * Q) = L(Q) = L(Q) * L(Q) = \Big(Q \cup (S * Q)\Big) * \Big(Q \cup (S * Q)\Big) \\ &= Q * Q \cup S * Q * Q \cup Q * S * Q \cup S * Q * S * Q \subseteq S * Q. \end{aligned}$$

Thus we have $Q \subseteq (Q * S) \cap (S * Q) \subseteq Q$, then $Q = (Q * S) \cap (S * Q)$ and property (1) is satisfied.

In addition, since S is regular, A is a right ideal and B is a left ideal of S, by Lemmas 3 and 4, $A * B$ is a quasi-ideal of S. So, by (1), we have

$$A * B = (A * B * S) \cap (S * A * B) \tag{2}$$

We are ready now to prove that $(\mathcal{Q}, *)$ is a von Neumann regular semigroup. In this respect, we prove the following:

$(\mathcal{Q}, *)$ is semigroup. Indeed: First of all, in an hypersemigroup, the operation "$*$" is associative (see [5], also [6; p. 22]). Let now Q_1, Q_2 be quasi-ideals of S. Then $Q_1 * Q_2$ is a quasi-ideal of S. Indeed: Since S is regular, $Q_1 * Q_2 * S$ is a right ideal and $S * Q_1 * Q_2$ is a left ideal of S, by Lemma 1, they are idempotent and we have

$$\begin{aligned}
\Big((Q_1 * Q_2) * S\Big) \quad \cap \quad & \Big(S * (Q_1 * Q_2)\Big) \\
= \quad & (Q_1 * Q_2 * S) * (Q_1 * Q_2 * S) \cap (S * Q_1 * Q_2) * (S * Q_1 * Q_2) \\
= \quad & (Q_1 * Q_2 * S * S) * (Q_1 * Q_2 * S) \cap (S * Q_1 * Q_2) * (S * S * Q_1 * Q_2) \\
& \text{(since } S * S = S) \\
= \quad & (Q_1 * Q_2 * S) * (S * Q_1 * Q_2) * S \cap S * (Q_1 * Q_2 * S) * (S * Q_1 * Q_2) \\
= \quad & (Q_1 * Q_2 * S) * (S * Q_1 * Q_2) \text{ (by (2))} \\
\subseteq \quad & Q_1 * (Q_2 * S * Q_2) \\
\subseteq \quad & Q_1 * (Q_2 * S \cap S * Q_2) \\
\subseteq \quad & Q_1 * Q_2 \text{ (since } Q_2 \text{ is a quasi-ideal of } S).
\end{aligned}$$

Hence $Q_1 * Q_2$ is a quasi-ideal of S. Thus $(\mathcal{Q}, *)$ is semigroup.

The semigroup $(\mathcal{Q}, *)$ is a von Neumann regular semigroup. In fact: Let $Q \in \mathcal{Q}$. Since $(S, \circ)$ is regular, by Lemma 2, we have

$$Q \subseteq Q * S * Q \subseteq (Q * S) \cap (S * Q) \subseteq Q.$$

Then $Q = Q * S * Q$, where $S \in \mathcal{Q}$ and so $(\mathcal{Q}, *)$ is a von Neumann regular semigroup.

$\Longleftarrow$. We remark first that for each quasi-ideal Q of S, we have

$$Q = Q * S * Q \tag{3}$$

In fact: Let Q be a quasi-ideal of S. Since $(\mathcal{Q}, *)$ is von Neumann regular semigroup, there exists $X \in \mathcal{Q}$ such that $Q = Q * X * Q$. Then

$$Q = Q * X * Q \subseteq Q * S * Q \subseteq (Q * S) \cap (S * Q) \subseteq Q.$$

Thus we have $Q = Q * S * Q$ and property (3) holds.

We are ready now to prove that $(S, \circ)$ is regular. For this, let A be a nonempty subset of S. By Lemma 2, it is enough to prove that $A \subseteq A * S * A$.

Since $R(A)$ is a right ideal and $L(A)$ is a left ideal of S, by Lemma 3, $R(A) \cap L(A)$ is a quasi-ideal of S. Then, by (3), we have

$$\begin{aligned}
A \quad \subseteq \quad & R(A) \cap L(A) = \Big(R(A) \cap L(A)\Big) * S * \Big(R(A) \cap L(A)\Big) \\
\subseteq \quad & \Big(R(A) * S\Big) * L(A) \subseteq R(A) * L(A) \\
= \quad & \Big(A \cup (A * S)\Big) * \Big(A \cup (S * A)\Big)
\end{aligned}$$

$$\begin{aligned} &= A * A \cup A * S * A \cup A * S * S * A \\ &= A * A \cup A * S * A, \end{aligned}$$

then $A * A \subseteq A * A * A \cup A * S * A * A \subseteq A * S * A$, thus we obtain $A \subseteq A * S * A$ and so the hypersemigroup $(S, \circ)$ is regular. □

Lemma 7 [4, 5] *An hypersemigroup $(S, \circ)$ is intra-regular if and only if, for every right ideal A and every left ideal B of S, we have $A \cap B \subseteq B * A$.*

An element a of a semigroup S is called *idempotent* if $a^2 = a$. An *idempotent semigroup* or shorter a *band* is a semigroup in which all elements are idempotent.

Theorem 8 *Let $(S, \circ)$ is an hypersemigroup. If $(S, \circ)$ is both regular and intra-regular, then the set $\mathcal{Q}$ of all quasi-ideals of S with the operation "$*$" is a band. "Conversely", if the quasi-ideals of $(S, \circ)$ are idempotent, then S is both regular and intra-regular.*

Proof $\Longrightarrow$. Let $(S, \circ)$ be both regular and intra-regular. Since $(S, \circ)$ is regular, by Theorem 6, $(\mathcal{Q}, *)$ is a semigroup. Moreover, the elements of the semigroup $\mathcal{Q}$ are idempotent. In fact: Let Q be a quasi-ideal of S. Since S is regular, we have $Q = Q * S * Q$ (cf. the proof of Theorem 6). Hence we have

$$\begin{aligned} Q &= Q * S * Q = (Q * S * Q) * S * (Q * S * Q) \\ &= (Q * S * Q) * S * S * (Q * S * Q) \text{ (by Lemma 5)} \\ &= (Q * S) * (Q * S) * (S * Q) * (S * Q). \end{aligned}$$

Since S is intra-regular and $Q * S$ is a right ideal and $S * Q$ is a left ideal of S, by Lemma 7, we have $(Q * S) \cap (S * Q) \subseteq (S * Q) * (Q * S)$. Thus we have

$$\begin{aligned} Q &= (Q * S) * (Q * S) * (S * Q) * (S * Q) \\ &\subseteq (Q * S) * (S * Q) * (Q * S) * (S * Q) \\ &= (Q * S * S * Q) * (Q * S * S * Q) \\ &= (Q * S * Q) * (Q * S * Q) \text{ (by Lemma 5)} \\ &= Q * Q \subseteq (Q * S) \cap (S * Q) \subseteq Q, \end{aligned}$$

and $Q * Q = Q$. Hence $(\mathcal{Q}, *)$ is an idempotent semigroup and so is a band.

$\Longleftarrow$. Let A be a right ideal and B a left ideal of S. By Lemma 3, $A \cap B$ is a quasi-ideal of S. By hypothesis, we have $A \cap B = (A \cap B) * (A \cap B) \subseteq A * B,\ B * A$. Since $A \cap B \subseteq A * B$, by Lemma 4, S is regular. Since $A \cap B \subseteq B * A$, by Lemma 7, S is intra-regular. □

Corollary 9 *An hypersemigroup $(S, \circ)$ is both regular and intra-regular if and only if the set $\mathcal{Q}$ of all quasi-ideals of S with the operation "$*$" is a band.*

Proof If $(\mathcal{Q}, *)$ is a band, that is an idempotent semigroup, then for every $Q \in \mathcal{Q}$, we have $Q * Q = Q$, that means that the quasi-ideals of $(S, \circ)$ are idempotent so, by Theorem 8, S is both regular and intra-regular. □

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