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Hyper Homomorphism and Hyper Product of Hyper UP-algebras

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Abstract. In this paper, we investigate the concept of regular congruence relation on hyper UP-algebras and establish some homomorphism theorems on such algebras. We also examine the notion of hyper product of hyper UP-algebras.

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1. Introduction

In 1934, F. Marty [7] first introduced the concept of hyperstructure theory at the 8th Congress of Scandinavian Mathematics. This led to the formulation of hyper BCK-algebra by Y. Jun et al. [11], hyper BCI-algebra by X. Long [6], and many other classes of algebras. R. Borzooei and H. Harizavi [1] defined the regular congruence relation on a hyper BCK-algebra, constructed a quotient hyper BCK-algebra, established some homomorphism theorems, and got some related results involving the hyper product of hyper BCK-algebras. G. Flores and G. Petalcorin [2] introduced regular congruence relation on a hyper BCI-algebra and presented some isomorphism theorems on hyper BCI-algebras.

In 2017, A. Iampan [4] defined a new algebraic structure called a UP-algebra and showed that the notion of UP-algebras is a generalization of KU-algebras that was introduced by C. Prabpayak and U. Leerawat [8]. Recently, D. Gomisong [3] applied hyperstuctures to UP-algebras in her graduate thesis following the structure of hyper KU-algebras by S. Mostafa et al. [5]. D. Romano gave an equivalent definition of hyper UP-algebra in [10] and proved that every hyper KU-algebra is a hyper UP-algebra. He also introduced the quotient of a hyper UP-algebra in [9]. In this paper, we investigate the concept of regular

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congruence relation on a hyper UP-algebra and present some homomorphism theorems on hyper UP-algebras. We also examine the concept of hyper product of hyper UP-algebras and extend it to the hyper product of an arbitrary family of hyper UP-algebras.

2. Preliminaries

Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a mapping from $H \times H$ into $\mathcal{P}^*(H)$.

Definition 1. [3] A *hyper* UP-*algebra* is a set H with constant 0 and hyperoperation $\circledast$ satisfying the following axioms: for all $x, y, z \in H$,

(HUP1) $[(x \circledast y) \circledast (x \circledast z)] \ll y \circledast z$,

(HUP2) $0 \circledast x = \{x\}$,

(HUP3) $x \circledast 0 = \{0\}$,

(HUP4) $x \ll y$ and $y \ll x$ imply $x = y$,

where $x \ll y$ is defined by $0 \in y \circledast x$ and for every $A, B \subseteq H$, $A \ll B$ is defined by: for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call "$\ll$" the *hyperorder* in H.

A hyper UP-algebra H with constant 0 and hyperoperation $\circledast$ is denoted by $(H; \circledast, 0)$. By (HUP2) or (HUP3), $x \circledast y \neq \varnothing$ for all $x, y \in H$.

Note that in [10], $x \ll y$ is defined by Romano as $0 \in x \circledast y$. Thus, (HUP1) in [3] and [10] are equivalent; that is, $0 \in (y \circledast z) \circledast [(x \circledast y) \circledast (x \circledast z)]$. Moreover, (HUP2) to (HUP4) are identical, with "$\circ$" denoted by "$\circledast$".

Example 1. [3] Let $H = \{0, a, b, c\}$ be a set. Define the hyperoperation $\circledast$ by the following Cayley table :

$\circledast$	0	a	b	c
0	{0}	{a}	{b}	{c}
a	{0}	{0,a}	{0,b}	{c}
b	{0}	{a}	{0,b}	{c}
c	{0}	{0,a}	{0,b}	{0,a,c}

Then, $(H; \circledast, 0)$ is a hyper UP-algebra.

Proposition 1. [3, 10] *Let H be a hyper* UP-*algebra. Then the following hold for all $x, y, z \in H$ and for every nonempty subsets $A, B, C \subseteq H$:*

(i) $0 \circledast 0 = \{0\}$

(ii) $0 \circledast A = A$

(iii) $z \ll z$

(iv) $A \subseteq B$ *implies* $A \ll B$

(v) $x \circledast z \ll z$

(vi) $A \circledast 0 = \{0\}$

(vii) $A \ll \{0\}$ *implies* $A = \{0\}$

$(viii)$ $A \circledast (B \circledast C) = B \circledast (A \circledast C)$

(ix) $0 \ll x$

(x) $x \in (0 \circledast y)$ *implies* $x \ll y$

Definition 2. [3] Let $(H; \circledast, 0)$ and $(H'; \circledast', 0')$ be hyper UP-algebras. A mapping $f : H \to K$ is called a *hyper homomorphism* if

(HH1) $f(0) = 0'$,

(HH2) $f(x \circledast y) = f(x) \circledast' f(y)$ for all $x, y \in H$.

The following definitions are analogous to the ones given by Borzooei and Harizavi [1] for regular congruence realtions on hyper BCK-algebras.

Definition 3. Let θ be an equivalence relation on a hyper UP-algebra H and $A, B \subseteq H$. Then

(i) $A\theta B$ if there exists $a \in A$ and $b \in B$ such that $a\theta b$;

(ii) $A\bar{\theta}B$ if for all $a \in A$, there exists $b \in B$ such that $a\theta b$ and for all $b \in B$, there exists $a \in A$ such that $a\theta b$;

(iii) θ is called a *congruence relation* on H if whenever $x\theta y$ and $x'\theta y'$, then $(x \circledast x')\bar{\theta}(y \circledast y')$, for all $x, y, x', y' \in H$;

(iv) θ is called a *regular congruence relation* on H if θ is a congruence relation on H and whenever $(x \circledast y)\theta\{0\}$ and $(y \circledast x)\theta\{0\}$, then $x\theta y$ for all $x, y \in H$.

The set $[x]_\theta = \{y \in H : y\theta x\}$ is called the *congruence class* determined by x.

3. Regular Congruence Relations and Hyper Homomorphisms on Hyper UP-algebras

All throughout, H, H', H'' are hyper UP-algebras.

Proposition 2. *If $f : H \longrightarrow H'$ is a hyper homomorphism, then for all nonempty subsets $A, B \subseteq H$ we have $f(A \circledast B) = f(A) \circledast' f(B)$.*

Proof. Let $f : H \longrightarrow H'$ be a hyper homomorphism and $\varnothing \neq A, B \subseteq H$. Let $x \in f(A \circledast B) = f\left(\bigcup_{a \in A, b \in B} a \circledast b\right)$. Then there exist $a \in A$ and $b \in B$ such that $x \in f(a \circledast b)$. Since f is a hyper homomorphism,

$$x \in f(a) \circledast' f(b) \subseteq \bigcup_{f(a) \in f(A), f(b) \in f(B)} f(a) \circledast' f(b) = f(A) \circledast' f(B).$$

Thus, $f(A \circledast B) \subseteq f(A) \circledast' f(B)$. Now, let

$$y \in f(A) \circledast' f(B) = \bigcup_{f(a) \in f(A), f(b) \in f(B)} f(a) \circledast' f(b).$$

Then there exist $f(a) \in f(A)$ and $f(b) \in f(B)$ such that $y \in f(a) \circledast' f(b)$. Since f is a hyper homomorphism,

$$y \in f(a \circledast b) \in f\left(\bigcup_{a \in A, b \in B} a \circledast b \right) = f(A \circledast B).$$

Thus, $f(A) \circledast' f(B) \subseteq f(A \circledast B)$. Therefore, $f(A \circledast B) = f(A) \circledast' f(B)$. □

Definition 4. Let $f : H \longrightarrow H'$ be a hyper homomorphism. We say that f is a *hyper monomorphism* if f is one-to-one, and f is a *hyper epimorphism* if f is onto; f is a *hyper isomorphism*, denoted by $\cong_{\mathcal{H}}$, if f is both one-to-one and onto.

Lemma 1. *Suppose $f : H \longrightarrow H'$ and $g : H' \longrightarrow H''$ are both hyper homomorphisms (epimorphisms) of hyper UP-algebras. Then $g \circ f$ is a hyper homomorphism (epimorphism) of hyper* UP*-algebras.*

The following result establishes the transitivity of the relation $\bar{\theta}$ on H.

Lemma 2. *Let θ be an equivalence relation on H and $A, B \subseteq H$. If $A\bar{\theta}B$ and $B\bar{\theta}C$, then $A\bar{\theta}C$.*

Proof. Suppose that $A\bar{\theta}B$ and $B\bar{\theta}C$. Since $A\bar{\theta}B$, by Definition 3(*ii*), for each $a \in A$ (respectively $b \in B$), there exists $b \in B$ (respectively $a \in A$) such that $a\theta b$. Similarly, since $B\bar{\theta}C$, for all $b \in B$ (respectively $c \in C$), there exists $c \in C$(respectively $b \in B$) such that $b\theta c$. Since by assumption θ is an equivalence relation for each $a \in A$ (respectively $c \in C$), there exists $c \in C$(respectively $a \in A$) such that $a\theta c$. Therefore, $A\bar{\theta}C$. □

Lemma 3. *Let θ be an equivalence relation on H. Then the following are equivalent:*

(*i*) *θ is a congruence relation on H;*

(*ii*) *if $x\theta y$, then $(x \circledast a)\bar{\theta}(y \circledast a)$ and $(a \circledast x)\bar{\theta}(a \circledast y)$ for all $a, x, y \in H$.*

Proof. (*i*) $\Longrightarrow$ (*ii*) Let θ be a congruence relation on H and $a, x, y \in H$. Suppose $x\theta y$. Since θ is a congruence relation on H and $a\theta a, (x \circledast a)\bar{\theta}(y \circledast a)$ and $(a \circledast x)\bar{\theta}(a \circledast y)$, by Definition 3(*iii*).

(*ii*) $\Longrightarrow$ (*i*) Assume $x\theta y$. Let $x, y, x', y' \in H$. Suppose that $x\theta y$ and $x'\theta y'$. By (*ii*), $(x \circledast x')\bar{\theta}(y \circledast x')$ and $(y \circledast x')\bar{\theta}(y \circledast y')$, so that by Lemma 2, $(x \circledast x')\bar{\theta}(y \circledast y')$. By Definition 3(*iii*), θ is a congruence relation on H. □

Theorem 1. *Suppose that θ and θ' are regular congruence relations on H with $[0]_\theta = [0]_{\theta'}$. Then $\theta = \theta'$.*

Proof. Let θ and θ' be regular conguence relations on H with $[0]_\theta = [0]_{\theta'}$. Since θ and θ' are both equivalence relations on H, it suffices to show that $x\theta y$ if and only if $x\theta' y$ for all $x, y \in H$. Let $x\theta y$. Since θ is a congruence relation on H, by Lemma 3, $(x \circledast x)\bar{\theta}(x \circledast y)$. Note that $0 \in x \circledast x$ by Proposition 1(iii). Thus by Definition 3(ii), there exists an element $s \in x \circledast y$ such that $0\theta s$. It follows that $s \in [0]_\theta = [0]_{\theta'}$. Hence, $(x \circledast y)\theta'\{0\}$.

In a similar manner, since $x\theta y, (y \circledast x)\bar{\theta}(y \circledast y)$. Also, $0 \in y \circledast y$ implies that there exists $t \in y \circledast x$ such that $0\theta t$. Hence, $t \in [0]_\theta = [0]_{\theta'}$. Thus, $(y \circledast x)\theta'\{0\}$. Now, since $(x \circledast y)\theta'\{0\}, (y \circledast x)\theta'\{0\}$, and θ' is a regular congruence relation, we have $x\theta' y$ by Definition 3(iv).

Similarly, let $x\theta' y$. Then $(x \circledast x)\bar{\theta}'(x \circledast y)$. Also, $0 \in x \circledast x$ implies that there exists an element $s \in x \circledast y$ such that $0\theta' s$. Furthermore, $s \in [0]_{\theta'} = [0]_\theta$. So, $(x \circledast y)\theta\{0\}$.

By similar argument, we will obtain $(y \circledast x)\bar{\theta}'(y \circledast y)$. Since $0 \in y \circledast y$, there exists $v \in y \circledast x$ such that $0\theta' v$. So, $v \in [0]_{\theta'} = [0]_\theta$. Hence, $(y \circledast x)\theta\{0\}$. Since θ is a regular congruence relation, we have $x\theta y$. □

We now reformulate the quotient structure of a hyper UP-algebra presented in [9] via regular congruence relation on a hyper UP-algebra H.

Theorem 2. [9] *Let θ be a regular congruence relation on H, $I = I_0 = [0]_\theta$ and $H/I = \{I_x : x \in H\}$, where $I_x = [x]_\theta$ for all $x \in H$. Then H/I with the hyperoperation $\circledast$ and hyperorder $\ll$ which are defined as follows*

$$I_x \circledast I_y = \{I_z : z \in x \circledast y\} \text{ and } I_x \ll I_y \text{ if and only if } I \in I_y \circledast I_x$$

is a hyper UP-algebra which is called the quotient hyper UP-algebra.

Example 2. Let $H = \{0, 1, 2, 3\}$ be a set. Define the hyperoperation $\circledast$ by the following Cayley table:

$\circledast$	0	1	2	3
0	{0}	{1}	{2}	{3}
1	{0}	{0,1}	{0,2}	{1,3}
2	{0}	{1}	{0,2}	{3}
3	{0}	{0,1}	{0,2}	{0,1,3}

By routine calculations, $(H; \circledast, 0)$ is a hyper UP-algebra. Define a relation θ on H by $\theta = \{(0,0), (1,1), (0,2), (2,0), (2,2), (3,3)\}$. By Lemma 3, it can be verified that θ is a congruence relation on H. Moreover, by routine calculations, θ is a regular congruence relation. Consider $I_0 = I = [0]_\theta = \{0, 2\}, I_1 = \{1\}$, and $I_3 = \{3\}$. Then $H/I = \{I, I_1, I_3\}$. Thus, our Cayley table is as follows:

$\circledast$	I	I_1	I_3
I	$\{I\}$	$\{I_1\}$	$\{I_3\}$
I_1	$\{I\}$	$\{I, I_1\}$	$\{I_1, I_3\}$
I_3	$\{I\}$	$\{I, I_1\}$	$\{I, I_1, I_3\}$

By routine calculations, H/I is a hyper UP-algebra.

To establish the First Hyper Isomorphism Theorem on hyper UP-algebras, we first reformulate some results on hyper homomorphisms of hyper UP- algebras.

Lemma 4. [9] *Let θ be a regular congruence relation on H and $I = [0]_\theta$. Then the mapping $\pi : H \longrightarrow H/I$ which is defined by $\pi(x) = I_x$, for all $x \in H$, is a hyper epimorphism which is called the canonical epimorphism.*

Theorem 3. [9] **(Hyper Homomorphism Theorem)** *Let θ be a regular congruence on H and $I = [0]_\theta$. If $f : H \longrightarrow H'$ is a hyper homomorphism of hyper* UP*-algebras such that I is contained in the kernel of f, then $\bar{f} : H/I \longrightarrow H'$, which is defined by $\bar{f}(I_x) = f(x)$, for all $x \in H$, is a unique hyper homomorphism such that $\bar{f} \circ \pi = f$, where π denotes the canonical epimorphism and $\circ$ is the composition map.*

Theorem 4. (First Hyper Isomorphism Theorem) *Let θ be a regular congruence relation on H and $I = [0]_\theta$. If $f : H \longrightarrow H'$ is a hyper homomorphism of hyper* UP-*algebras such that $ker\ f = I$, then $H/ker\ f \cong_{\mathcal{H}} Imf$.*

Proof. Define $\bar{f} : H/I \longrightarrow H'$ by $\bar{f}(I_x) = f(x)$ for all $x \in H$. Let $x, y \in H$. Then $I_x, I_y \in H/I$. From Theorem 3, $\bar{f}$ is a hyper homomorphism. Thus, $\bar{f}(I_x \circledast I_y) = \bar{f}(I_x) \circledast' \bar{f}(I_y)$ and $\bar{f}(I) = 0'$.

Suppose that $\bar{f}(I_x) = \bar{f}(I_y)$ with $x, y \in H$. Then $f(x) = f(y)$. Since f is a hyper homomorphism, $0' = f(0) \in f(x \circledast x) = f(x) \circledast' f(x) = f(x) \circledast' f(y) = f(x \circledast y)$. So, there exists an element $u \in x \circledast y$ such that $f(u) = 0'$, that is, $u \in kerf = I = [0]_\theta$. Thus, $u\theta 0$ and $(x \circledast y)\theta\{0\}$. Also, $0' = f(0) \in f(x \circledast x) = f(x) \circledast' f(x) = f(y) \circledast' f(x) = f(y \circledast x)$. Thus, there exists an element $v \in y \circledast x$ such that $f(v) = 0'$. Moreover, $v \in kerf = I = [0]_\theta$ and $v\theta 0$. Thus, $(y \circledast x)\theta\{0\}$. Since θ is a regular congruence relation, it follows that $x\theta y$. Thus, $I_x = I_y$. Hence, $\bar{f}$ is one-to-one, thus $ker\ \bar{f} = (ker\ f)/I \subseteq H/I$ is trivial, which occurs if and only if $ker\ f = I$. Clearly, $Im\ \bar{f} = Im\ f$ and $\bar{f} : H/I \longrightarrow Im\ f$ is onto. Therefore, $H/ker\ f \cong_{\mathcal{H}} Imf$. □

Lemma 5. *Let $f : H \longrightarrow H'$ be a hyper homomorphism on hyper* UP-*algebras with $I = [0]_\theta$ and $J = [0']_{\theta'}$ where θ and θ' are regular congruence relations on H and H', respectively. Suppose that $I \subseteq ker\ f$. Then for all $x, y \in H, x\theta y$ implies that $f(x)\theta' f(y)$.*

Proof. Let $f : H \longrightarrow H'$ be a hyper homomorphism with $I = [0]_\theta \subseteq ker\ f$ and $J = [0']_{\theta'}$ where θ and θ' are regular congruence relations on H and H', respectively. Let $x, y \in H$ such that $x\theta y$. Since θ is a regular congruence relation, we have $x\theta x$ and $(x \circledast x)\bar{\theta}(x \circledast y)$ by Definition 3(iii). Since $0 \in x \circledast x$ by Proposition 1(iii), there exists an element $u \in x \circledast y$ such that $0\theta u$. Thus, $u \in I \subseteq ker\ f$, that is, $f(u) = 0'$. It follows that $f(u) \in H'$ and $f(u)\theta' 0'$. Since f is a hyper homomorphism, $f(u) \in f(x \circledast y) = f(x) \circledast' f(y)$, thus $(f(x) \circledast' f(y))\bar{\theta}'\{0'\}$.

Using similar argument, with $y\theta y$, we have $(f(y) \circledast' f(x))\bar{\theta}'\{0'\}$. Since θ' is a regular congruence relation, by Definition 3(iv) we have $f(x)\theta' f(y)$. □

Theorem 5. *Let θ and θ' be regular congruence relations on hyper* UP*-algebras H and H', respectively, such that $I = [0]_\theta$ and $J = [0']_{\theta'}$. If $f : H \longrightarrow H'$ is a hyper homomorphism of hyper* UP*-algebras such that $x\theta y$ if and only if $f(x)\theta' f(y)$, for all $x, y \in H$, then there exists a unique hyper homomorphism $f^* : H/I \longrightarrow H'/J$ such that $\pi' \circ f = f^* \circ \pi$ where π and π' are the canonical epimorphisms and $\circ$ is the composition map.*

$$\begin{array}{ccc} H & \xrightarrow{f} & H' \\ \Big\downarrow \pi & & \Big\downarrow \pi' \\ H/I & \xrightarrow{f^*} & H'/J \end{array}$$

Proof. Consider the mapping $f^* : H/I \longrightarrow H'/J$ defined by $f^*(I_x) = J_{f(x)}$, for all $x \in H$. Let $x, y \in H$ such that $I_x = I_y$. Then $x\theta y$ and so $f(x)\theta' f(y)$ by assumption. Hence, $f^*(I_x) = J_{f(x)} = J_{f(y)} = f^*(I_y)$ and f^* is well-defined.

Let $I_x, I_y \in H/I$ and $J_t \in f^*(I_x \circledast I_y)$. Then there exists an element $t' \in x \circledast y$ such that $J_{f(t')} = f^*(I_{t'}) = J_t$. Now, $t' \in x \circledast y$ implies $f(t') \in f(x \circledast y) = f(x) \circledast' f(y)$. So, $J_t = J_{f(t')} \in J_{f(x)} \circledast' J_{f(y)} = f^*(I_x) \circledast' f^*(I_y)$. Hence, $f^*(I_x \circledast I_y) \subseteq f^*(I_x) \circledast' f^*(I_y)$.

Next, let $J_s \in f^*(I_x) \circledast' f^*(I_y) = J_{f(x)} \circledast' J_{f(y)}$. Then $s \in f(x) \circledast' f(y) = f(x \circledast y)$. Now, $s \in f(x \circledast y)$ implies there exists $w \in x \circledast y$ such that $f(w) = s$, that is, $I_w \in I_x \circledast I_y$ and $J_s = J_{f(w)} = f^*(I_w) \in f^*(I_x \circledast I_y)$. Therefore, $f^*(I_x) \circledast' f^*(I_y) \subseteq f^*(I_x \circledast I_y)$ and so $f^*(I_x \circledast I_y) = f^*(I_x) \circledast' f^*(I_y)$. Moreover, $f^*(I) = J_{f(0)} = J_{0'} = J$. Also, $dom(\pi' \circ f) = H = dom(f^* \circ \pi)$. Let $x \in H$. Then

$$(\pi' \circ f)(x) = \pi'(f(x)) = J_{f(x)} = f^*(I_x) = f^*(\pi(x)) = (f^* \circ \pi)(x).$$

Thus, $\pi' \circ f = f^* \circ \pi$. Next, we let $\phi : H/I \longrightarrow H'/J$ be a homomorphism such that $\pi' \circ f = \phi \circ \pi$. Note that $dom(\pi' \circ f) = H = dom(\phi \circ \pi)$. Then $\phi = f^*$ since for all $x \in H$, we have $\phi(I_x) = \phi(\pi(x)) = J_{\pi(x)} = \pi'(f(x)) = (\pi' \circ f)(x) = (f^* \circ \pi)(x) = f^*(I_x)$. □

Theorem 6. *Suppose $f : H \longrightarrow H'$ is a hyper epimorphism of hyper* UP*-algebras, θ' is a regular conruence relation on H' and $J = [0']_{\theta'}$. Then there exists a regular congruence relation θ on H such that $H/I \cong_{\mathcal{H}} H'/J$, where $I = [0]_\theta$.*

Proof. Define θ on H by $x\theta y$ if and only if $f(x)\theta' f(y)$, for all $x, y \in H$. Let $x \in H$. Then $f(x) \in H'$ and so, by reflexivity of θ' on H', we have $f(x)\theta' f(x)$. It follows that $x\theta x$ and θ is a reflexive relation on H. Assume that $x\theta y$, where $x, y \in H$. So, $f(x), f(y) \in H'$ and $f(x)\theta' f(y)$. Hence, $f(y)\theta' f(x)$ which will imply that $y\theta x$. Thus, θ is a symmetric relation on H. Suppose $x\theta y$ and $y\theta z$, where $x, y, z \in H$. Then $f(x)\theta' f(y)$ and $f(y)\theta' f(z)$, for all $x, y, z \in H$. Note that $f(x), f(y), f(z) \in H'$ and by transitivity of θ' on H', we have $f(x)\theta' f(z)$. Thus, $x\theta z$ on H and θ is a transitive relation on H. Therefore, θ is an equivalence relation on H.

Next, we will show that θ is a congruence relation. Let $a, x, y \in H$ such that $x\theta y$. Then $f(x)\theta' f(y)$. Since $f(a), f(x), f(y) \in H'$ and θ' is a congruence relation on H', from

Lemma 3 it follows that $(f(x) \circledast f(a))\bar{\theta'}(f(y) \circledast f(a))$ and $(f(a) \circledast f(x))\bar{\theta'}(f(a) \circledast f(y))$. Thus, $(x \circledast a)\bar{\theta}(y \circledast a)$ and $(a \circledast x)\bar{\theta}(a \circledast y)$. Therefore, by Lemma 3, θ is a congruence relation on H.

Let $x, y \in H$ such that $(x \circledast y)\theta\{0\}$ and $(y \circledast x)\theta\{0\}$. Then $f(x), f(y) \in H'$ and there exist $a \in (x \circledast y)$ and $b \in (y \circledast x)$ such that $a\theta 0$ and $b\theta 0$. Since f is a hyper homomorphism and $f(0) = 0', f(a) \in f(x \circledast y) = f(x) \circledast' f(y)$ and $f(b) \in f(y \circledast x) = f(y) \circledast' f(x)$ such that $f(a)\theta' 0'$ and $f(b)\theta' 0'$. Thus, $(f(x) \circledast' f(y))\theta'\{0'\}$ and $(f(y) \circledast' f(x))\theta'\{0'\}$. Since θ' is a regular congruence relation on H', $f(x)\theta' f(y)$, implying that $x\theta y$. Therefore, θ is a regular congruence relation on H.

Next, let $x \in I = [0]_\theta$. Since $x\theta 0$ and $f(0) = 0'$, $f(x)\theta' 0'$. It follows that $f(x) \in [0']_{\theta'} = J$, so $x \in f^{-1}(J)$. Thus, $I \subseteq f^{-1}(J)$. On the other hand, let $y \in f^{-1}(J)$. Then $f(y) \in J = [0']_{\theta'}$ and $f(y)\theta' 0'$. Hence, $y\theta 0$ and $y \in [0]_\theta = I$, implying that $f^{-1}(J) \subseteq I$. Thus, $I = f^{-1}(J)$.

Now, let $\pi : H' \longrightarrow H'/J$ be the canonical hyper epimorphism and define $\bar{f} : H \longrightarrow H'/J$ by $\bar{f} = \pi \circ f$. Since π and f are both hyper epimorphisms of hyper UP-algebras, by Lemma 1, $\bar{f}$ is a hyper epimorphism. Observe that

$$\begin{aligned}
ker\ \bar{f} &= \{x \in H : \bar{f}(x) = J\} \\
&= \{x \in H : \pi(f(x)) = J\} \\
&= \{x \in H : J_{f(x)} = J\} \\
&= \{x \in H : f(x) \in J\} \\
&= \{x \in H : x \in f^{-1}(J)\} \\
&= \{x \in H : x \in I\} \\
&= I.
\end{aligned}$$

Therefore, by the First Hyper Isomorphism Theorem, $H/I \cong_{\mathcal{H}} H'/J$. □

Theorem 7. *Let $f : H \longrightarrow H'$ be a hyper epimorphism on hyper UP-algebras and let Θ and Ω be relations on H and H', respectively, defined by $x\Theta y \iff f(x)\Omega f(y)$ for all $x, y \in H$. Then Θ is a regular congruence relation on H if and only if Ω is a regular congruence relation on H'.*

Proof. Utilizing the proof of Theorem 6, we only need to show that Θ is a regular congruence relation on H implies that Ω is a regular congruence relation on H'. Suppose Θ is a regular congruence relation on H. Let $u, v, w \in H'$. Then there exist $x, y, z \in H$ such that $f(x) = u, f(y) = v$, and $f(z) = w$. Since Θ is an equivalence relation on H, $x\Theta x$, thus $u = f(x)\Omega f(x) = u$ and Ω is a reflexive relation on H'. Suppose $u\Omega v$. Then $x\Theta y$ and since Θ is a symmetric relation on $H, y\Theta x$, so $v\Omega u$ and Ω is a symmetric relation on H'. Suppose $u\Omega v$ and $v\Omega w$. Then $x\Theta y$ and $y\Theta z$. Since Θ is a transitive relation on H, $x\Theta z$, that is, $u\Omega w$. Thus, Ω is an equivalence relation on H'.

Let $b, u, v \in H'$ and $u\Omega v$. Then there exist $a, x, y \in H$ such that $b = f(a), u = f(x), v = f(y)$, and $x\Theta y$. Since Θ is a congruence relation on H and $a \in H, (a \circledast x)\bar{\Theta}(a \circledast y)$ by

Lemma 3. Hence, $f(a) \circledast' f(x) = f(a \circledast x)\bar{\Omega} f(a \circledast y) = f(a) \circledast' f(y)$, that is, $(b \circledast' u)\bar{\Omega}(b \circledast' v)$. Similarly, since Θ is a congruence relation on H and $a \in H$, $(x \circledast a)\bar{\Theta}(y \circledast a)$. So, $f(x) \circledast' f(a) = f(x \circledast a)\bar{\Omega} f(y \circledast a) = f(y) \circledast' f(a)$, that is, $(u \circledast' b)\bar{\Omega}(v \circledast' b)$. Hence, Ω is a congruence relation on H'.

Now, let $u, v \in H'$ such that $(u \circledast' v)\Omega\{0'\}$ and $(v \circledast' u)\Omega\{0'\}$. Since $(u \circledast' v)\Omega\{0'\}$ and f is a hyper epimorphism, it follows that there exist $s, t \in H$ such that $f(s) = u, f(t) = v, f(s \circledast t) = f(s) \circledast' f(t) = (u \circledast' v)\Omega\{0'\}$. Similarly, $(v \circledast' u)\Omega\{0'\}$ implies $f(t \circledast s) = f(t) \circledast' f(s) = (v \circledast' u)\Omega\{0'\}$. Hence, $(s \circledast t)\Theta\{0\}$ and $(t \circledast s)\Theta\{0\}$. Since Θ is a regular congruence relation on H, it follows that $s\Theta t$ and $u\Omega v$. Therefore, Ω is a regular congruence relation on H'. □

Remark 1. *Let $f : H \longrightarrow H'$ be a hyper epimorphism on hyper UP-algebras and let Θ and Ω be the relations on H and H', respectively, as defined in Theorem 7. Then*

(*i*) *Ω is called the regular congruence relation induced by f and Θ, and*

(*ii*) *Θ is called the regular congruence relation induced by f and Ω.*

Theorem 8. *Let $f : H \longrightarrow H'$ be a hyper epimorphism on hyper UP-algebras. Then there is a one-to-one correspondence between the regular congruence relations on H' and the regular congruence relations on H such that ker f is contained in the regular congruence class containing 0.*

Proof. Let $f : H \longrightarrow H'$ be a hyper epimorphism of hyper UP-algebras and

$$\mathcal{A} = \{\Theta : \Theta \text{ is a regular congruence relation on } H \text{ with } ker\ f \subseteq [0]_\Theta\}$$
$$\mathcal{B} = \{\Omega : \Omega \text{ is a regular congruence relation on } H'\}.$$

Define $\gamma : \mathcal{A} \longrightarrow \mathcal{B}$ by $\gamma(\Theta) = \Omega$, where Ω is the regular congruence relation on H' induced by f and Θ. Then $\Omega \in \mathcal{B}$. Let $\Theta_1, \Theta_2 \in \mathcal{A}$ such that $\Omega_1 = \gamma(\Theta_1) = \gamma(\Theta_2) = \Omega_2$. Then for all $x, y \in H, x\Theta_1 y \Leftrightarrow f(x)\Omega_1 f(y) \Leftrightarrow f(x)\Omega_2 f(y) \Leftrightarrow x\Theta_2 y$. Hence, $\Theta_1 = \Theta_2$ and γ is well-defined and one-to-one.

Now, let $\Omega \in \mathcal{B}$ and consider the induced regular congruence relation Θ on H. If $x \in ker\ f$, then $f(x) = f(0.)$ So, $f(x)\Omega f(0)$ implies $x\Theta 0$. Thus, $ker\ f \subseteq [0]_\Theta$ and so, $\Theta \in \mathcal{A}$. Lastly, we show that γ is onto, that is, $\gamma(\Theta) = \Omega$. Suppose $\gamma(\Theta) = \Omega'$ for some $\Omega' \in \mathcal{B}$. Then by the definitions of Ω and Θ, for each $t \in H'$,

$$t\Omega' 0' \Leftrightarrow t = f(x) \text{ and } x\Theta 0 \text{ for some } x \in H \Leftrightarrow f(x)\Omega f(0) \Leftrightarrow t\Omega 0'.$$

Thus, $[0']_\Omega = [0']_{\Omega'}$ and by Theorem 1, $\Omega = \Omega'$. Hence, $\gamma(\Theta) = \Omega$. Therefore, γ is a bijection. □

4. Hyper Product of Hyper UP-algebras

Throughout this section, H and K shall mean the hyper UP-algebras $(H, \circledast_H, 0_H)$ and $(K, \circledast_K, 0_k)$ with $\ll_H$ and $\ll_K$ as their hyper orders, respectively.

The following introduction of the hyper product of two hyper UP-algebras is influenced by the construction of the hyper product of two hyper BCK-algebras by Borzooei et al. [12], as cited in [1].

Suppose H and K are hyper UP-algebras. Then

$$H \times K = \{(a,b) | a \in H \text{ and } b \in K\}.$$

Define the hyperoperation " $\circledast$ " on $H \times K$ by

$$(a,b) \circledast (c,d) = (a \circledast_H c, b \circledast_K d)$$

and hyperorder " $\ll$ " by $(a,b) \ll (c,d) \iff a \ll_H c$ and $b \ll_K d$ for all $(a,b),(c,d) \in H \times K$. For every $(A,B),(C,D) \subseteq H \times K$, $(A,B) \ll (C,D)$ if and only if for all $(a,b) \in (A,B)$, there exists $(c,d) \in (C,D)$ such that $(a,b) \ll (c,d)$. Then $(H \times K; \circledast, (0_H, 0_K))$ is called the *hyper product* of H and K.

Theorem 9. [9] *Let H and K be hyper UP-algebras. Then $H \times K$ is a hyper UP-algebra.*

Theorem 10. *Let $\alpha_1 : H_1 \longrightarrow K_1$ and $\alpha_2 : H_2 \longrightarrow K_2$ be hyper homomorphisms of hyper UP-algebras. Define $\alpha : H_1 \times H_2 \longrightarrow K_1 \times K_2$ by $\alpha((a,b)) = (\alpha_1(a), \alpha_2(b))$ for all $(a,b) \in H_1 \times H_2$. Then*

- (i) *α is a hyper homomorphism;*
- (ii) *$ker\ \alpha = ker\ \alpha_1 \times ker\ \alpha_2$;*
- (iii) *$Im\ \alpha = Im\ \alpha_1 \times Im\ \alpha_2$; and*
- (iv) *α is a hyper monomorphism (respectively, hyper epimorphism) if and only if α_i is a hyper monomorphism (respectively, hyper epimorphism) for each $i = 1, 2$.*

Proof. Define $\alpha : H_1 \times H_2 \longrightarrow K_1 \times K_2$ by $\alpha((a,b)) = (\alpha_1(a), \alpha_2(b))$ for all $(a,b) \in H_1 \times H_2$.

(i) Let $(a,b),(c,d) \in H_1 \times H_2$ such that $(a,b) = (c,d)$. Then $a = c$ and $b = d$. Now, since α_1 and α_2 are well-defined maps, it follows that

$$\begin{aligned}\alpha((a,b)) &= (\alpha_1(a), \alpha_2(b)) \\ &= (\alpha_1(c), \alpha_2(d)) \\ &= \alpha((c,d)).\end{aligned}$$

So, α is a well-defined map. Observe that $(0_{H_1}, 0_{H_2}) \in H_1 \times H_2$. Since α_1 and α_2 are hyper homomorphisms, by (HH1) we have

$$\alpha((0_{H_1}, 0_{H_2})) = (\alpha_1(0_{H_1}), \alpha_2(0_{H_2})) = (0_{K_1}, 0_{K_2})$$

and by (HH2),

$$\begin{aligned}\alpha((a,b)\circledast(c,d)) &= \alpha((a\circledast c, b\circledast d))\\ &= \{\alpha((u,v))|u\in a\circledast c, v\in b\circledast d\}\\ &= \{(\alpha_1(u),\alpha_2(v))|u\in a\circledast c, v\in b\circledast d\}\\ &= (\alpha_1(a\circledast c),\alpha_2(b\circledast d))\\ &= (\alpha_1(a)\circledast\alpha_1(c),\alpha_2(b)\circledast\alpha_2(d))\\ &= \alpha(a,b)\circledast\alpha(c,d).\end{aligned}$$

Hence, α is a hyper homomorphism.

(ii) By definition,

$$\begin{aligned}ker\ \alpha &= \{(a,b)\in H_1\times H_2|\alpha((a,b))=(0_{K_1},0_{K_2})\}\\ &= \{(a,b)\in H_1\times H_2|(\alpha_1(a),\alpha_2(b))=(0_{K_1},0_{K_2})\}\\ &= \{(a,b)\in H_1\times H_2|\alpha_1(a)=0_{K_1} \text{ and } \alpha_2(b)=0_{K_2}\}\\ &= \{(a,b)\in H_1\times H_2|a\in\ ker\ \alpha_1, b\in ker\ \alpha_2\}\\ &= \ ker\ \alpha_1\times\ ker\ \alpha_2.\end{aligned}$$

(iii) By definition,

$$\begin{aligned}Im\ \alpha &= \{\alpha((a,b))|(a,b)\in H_1\times H_2\}\\ &= \{(\alpha_1(a),\alpha_2(b))|(a,b)\in H_1\times H_2\}\\ &= \{(\alpha_1(a),\alpha_2(b))|\alpha_1(a)\in\ Im\ \alpha_1,\ \alpha_2(b)\in\ Im\ \alpha_2\}\\ &= Im\ \alpha_1\times\ Im\ \alpha_2.\end{aligned}$$

(iv) Suppose that α is one-to-one. Let $a,c\in H_1$ and $b,d\in H_2$ such that $\alpha_1(a)=\alpha_1(c)$ and $\alpha_2(b)=\alpha_2(d)$. Then

$$\alpha((a,b))=(\alpha_1(a),\alpha_2(b))=(\alpha_1(c),\alpha_2(d))=\alpha((c,d)).$$

Since α is one-to-one, $(a,b)=(c,d)$, that is, $a=c$ and $b=d$. Thus, α_1 and α_2 are one-to-one maps.

Conversely, assume that α_1 and α_2 are one-to-one maps. Suppose $(a,b),(c,d)\in H_1\times H_2$ such that $\alpha((a,b))=\alpha((c,d))$. Then $(\alpha_1(a),\alpha_2(b))=\alpha((a,b))=\alpha((c,d))=(\alpha_1(c),\alpha_2(d))$. This means that $\alpha_1(a)=\alpha_1(c)$ and $\alpha_2(b)=\alpha_2(d)$ and since α_1 and α_2 are both one-to-one, it follows that $a=c$ and $b=d$. Hence, $(a,b)=(c,d)$. Therefore, α is one-to-one.

Suppose α is onto. Let $x\in K_1$ and $y\in K_2$. It follows that $(x,y)\in K_1\times K_2$. Since α is onto, there exists $(a,b)\in H_1\times H_2$ such that $(\alpha_1(a),\alpha_2(b))=\alpha((a,b))=(x,y)$, that is, $\alpha_1(a)=x$ and $\alpha_2(b)=y$ for some $a\in H_1$ and $b\in H_2$. So, α_1 and α_2 are onto maps. Next, suppose α_1 and α_2 are onto maps. Let $(x, y) \in K_1 \times K_2$. Then $x \in K_1$ and $y \in K_2$. Since α_1 and α_2 are onto maps, we can pick some elements $a \in H_1$ and $b \in H_2$ such that $\alpha_1(a) = x$ and $\alpha_2(b) = y$, that is, $\alpha((a, b)) = (\alpha_1(a), \alpha_2(b)) = (x, y)$ for some $(a, b) \in H_1 \times H_2$. Therefore, α is onto and (iv) holds. □

Recall that if $\{A_k : k \in \mathcal{I}\}$ is a family of sets, the Cartesian product $\prod_{k\in\mathcal{I}} A_k$ is the set of all functions $p : \mathcal{I} \longrightarrow \bigcup_{k\in\mathcal{I}} A_k$ such that $p(k) \in A_k$, for all $k \in \mathcal{I}$. If $p \in \prod_{k\in\mathcal{I}} A_k$ such that $p(i) = a_i \in A_i$ for all $i \in \mathcal{I}$, then we will denote p as $\{a_i\}$.

We now extend the hyper product $H \times K$ of H and K to the hyper product of an arbitrary family of hyper UP-algebras.

Let $\{H_k : k \in \mathcal{I}\}$ be a family of hyper UP-algebras. For each $k \in \mathcal{I}$, let $\circledast_k, 0_k$, and $\ll_k$ be the hyperoperation, the zero element, and the hyperorder of H_k, respectively. Let $G = \prod_{k\in\mathcal{I}} H_k$ and define the hyperoperation $\circledast$ as follows: for $\{x_k\}, \{y_k\} \in G, \{x_k\} \circledast \{y_k\} = \prod_{k\in\mathcal{I}} (x_k \circledast y_k)$. Since $x_k \circledast y_k \neq \varnothing$ for each $k \in \mathcal{I}$, the Axiom of Choice ensures us that $\prod_{k\in\mathcal{I}} (x_k \circledast y_k) \neq \varnothing$, and so $\circledast$ is indeed a hyperoperation. The zero element of G is $\{0_k\}$, and under the hyperoperation $\circledast$, the hyperorder $\ll$ is established as follows: for $\{x_k\}, \{y_k\} \in G$,

$$\begin{aligned}
\{x_k\} \ll \{y_k\} &\iff \{0_k\} \in \{y_k\} \circledast \{x_k\} \\
&\iff \{0_k\} \in \prod_{k\in\mathcal{I}} (y_k \circledast x_k) \\
&\iff \text{for all } k \in \mathcal{I}, 0_k \in y_k \circledast x_k \\
&\iff \text{for all } k \in \mathcal{I}, x_k \ll_k y_k,
\end{aligned}$$

and for all $\prod_{k\in\mathcal{I}} A_k, \prod_{k\in\mathcal{I}} B_k \subseteq \prod_{k\in\mathcal{I}} H_k$,

$$\begin{aligned}
\prod_{k\in\mathcal{I}} A_k \ll \prod_{k\in\mathcal{I}} B_k &\iff \forall \{a_k\} \in \prod_{k\in\mathcal{I}} A_k, \exists \{b_k\} \in \prod_{k\in\mathcal{I}} B_k \text{ such that } \{a_k\} \ll \{b_k\} \\
&\iff \forall k \in \mathcal{I}, \forall a_k \in A_k, \exists b_k \in B_k \text{ such that } a_k \ll_k b_k \\
&\iff \forall k \in \mathcal{I}, A_k \ll_k B_k.
\end{aligned}$$

Then $(G, \circledast, \{0_K\})$ is called the *hyper product* of $\{H_k : k \in \mathcal{I}\}$.

Lemma 6. *Let $\{H_k : k \in \mathcal{I}\}$ be a nonempty family of hyper UP-algebras. Suppose that $A_k, B_k \subseteq H_k$, for all $k \in \mathcal{I}$. Then for each $k \in \mathcal{I}$,*

$$\prod_{k\in\mathcal{I}} A_k \circledast \prod_{k\in\mathcal{I}} B_k = \prod_{k\in\mathcal{I}} (A_k \circledast B_k).$$

Theorem 11. *Suppose that* $\{H_k : k \in \mathcal{I}\}$ *is a nonempty family of hyper UP-algebras. Then* $\left(\prod_{k\in\mathcal{I}} H_k, \circledast, \{0_k\}\right)$ *is a hyper UP-algebra.*

Proof. Suppose $\{H_k : k \in \mathcal{I}\}$ is a nonempty family of hyper UP-algebras. Let $\{a_k\}, \{b_k\}, \{c_k\}, \{d_k\} \in \prod_{k\in\mathcal{I}} H_k$. Then $a_k, b_k, c_k, d_k \in H_k$ for all $k \in \mathcal{I}$. We will show first that $\circledast$ is a well-defined hyperoperation on $\prod_{k\in\mathcal{I}} H_k$.

Assume that $\{a_k\} = \{b_k\}$ and $\{c_k\} = \{d_k\}$, for all $k \in \mathcal{I}$. Then $a_k = b_k$ and $c_k = d_k$ for all $k \in \mathcal{I}$. So,

$$\{a_k\} \circledast \{c_k\} = \prod_{k\in\mathcal{I}} (a_k \circledast_k c_k) = \prod_{k\in\mathcal{I}} (b_k \circledast_k d_k) = \{b_k\} \circledast \{d_k\}$$

for all $k \in \mathcal{I}$. Thus, $\circledast$ is a well-defined hyperoperation on $\prod_{k\in\mathcal{I}} H_k$. Let $\{x_k\}, \{y_k\}, \{z_k\} \in \prod_{k\in\mathcal{I}} H_k$. Then $x_k, y_k, z_k \in H_k$ for all $k \in \mathcal{I}$. Now, for each $k \in \mathcal{I}$, we have

$$\begin{aligned}(\{x_k\} \circledast \{y_k\}) \circledast (\{x_k\} \circledast \{z_k\}) &= \left(\prod_{k\in\mathcal{I}} (x_k \circledast_k y_k)\right) \circledast \left(\prod_{k\in\mathcal{I}} (x_k \circledast_k z_k)\right) \\ &= \left(\prod_{k\in\mathcal{I}} (x_k \circledast_k y_k) \circledast (x_k \circledast_k z_k)\right).\end{aligned}$$

Since for each $k \in \mathcal{I}, (x_k \circledast_k y_k) \circledast (x_k \circledast_k z_k) \ll_k y_k \circledast_k z_k$, it follows that

$$\prod_{k\in\mathcal{I}} (x_k \circledast_k y_k) \circledast (x_k \circledast_k z_k) \ll \prod_{k\in\mathcal{I}} (y_k \circledast_k z_k),$$

that is,

$$(\{x_k \circledast y_k\}) \circledast (\{x_k \circledast z_k\}) \ll \{y_k\} \circledast \{z_k\}.$$

This means that (HUP1) holds on $\prod_{k\in\mathcal{I}} H_k$.

Since for each $k \in \mathcal{I}, 0_k \circledast_k x_k = \{x_k\}$, it follows that

$$\{0_k\} \circledast \{x_k\} = \prod_{k\in\mathcal{I}} (0_k \circledast_k x_k) = \prod_{k\in\mathcal{I}} \{x_k\}.$$

Thus, (HUP2) holds on $\prod_{k\in\mathcal{I}} H_k$.

Moreover, since for each $k \in \mathcal{I}, x_k \circledast_k 0_k = \{0_k\}$, it follows that

$$\{x_k\} \circledast \{0_k\} = \prod_{k\in\mathcal{I}} (x_k \circledast_k 0_k) = \prod_{k\in\mathcal{I}} \{0_k\}.$$

Hence, (HUP3) holds on $\prod_{k\in\mathcal{I}} H_k$.

Furthermore, suppose $\{x_k\} \ll \{y_k\}$ and $\{y_k\} \ll \{x_k\}$ for all $k \in \mathcal{I}$. Then $x_k \ll_k y_k$ and $y_k \ll_k x_k$ for all $k \in \mathcal{I}$. Hence, $x_k = y_k$ for all $k \in \mathcal{I}$ and so $\{x_k\} = \{y_k\}$. This means that (HUP4) holds on $\prod_{k\in\mathcal{I}} H_k$. Therefore, $\left(\prod_{k\in\mathcal{I}} H_k, \circledast, \{0_k\}\right)$ is a hyper UP-algebra. $\square$

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