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Modular stabilities of a reciprocal second power functional equation

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Abstract. In the present work, we propose a different reciprocal second power Functional Equation (FE) which involves the arguments of functions in rational form and determine its stabilities in the setting of modular spaces with and without using Fatou property. We also prove the stabilities in β-homogenous spaces. As an application, we associate this equation with the electrostatic forces of attraction between unit charges in various cases using Coloumb's law.

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1. Introduction & Preliminaries

The hypothesis connected with linear spaces and the concepts of modular spaces were dealt in [20]. Later, this theory has been employed by many authors [1, 9, 16, 29, 32]. The significant application of modular theory is that it is useful in interpolation ([10, 17]) and in numerous Orlicz spaces [21]. The common notions and properties related to modular theory are available in [18, 19, 21].

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The detailed information about the evolution of theory of stability of FEs are available in [3, 5, 6, 24, 25, 30]. There are many techniques of solving stability problems of FEs, such as the technique through the attribute of shadowing [28], the technique via fixed averages [27], the technique by virtue of sandwich hypothesis [22]. The dominant tools to determine classical stability problems are the direct method and the fixed point method [6, 23].

Also, without the application of Δ_2-condition, proposed in [7], there are many stability problems via fixed point theorem of quasicontracion functions in the setting of modular spaces. By employing Khamis's invariant point theorem, the modular stabilities of additive FE alongwith with the Fatou property and Δ_2-condition are dealt in [26]. Moreover, the modular stability problems of quadratic FEs were discussed satisfying Fatou property without utilizing Δ_2-condition in [31]. One can refer [2, 4, 8, 11–15] for more details about stabilities of real and complex valued multiplicative inverse FEs.

In this present work, we propose a different reciprocal second power FE of the form

$$m_q\left(\frac{uv}{2u+v}\right)+m_q\left(\frac{uv}{2u-v}\right)=2m_q(u)+8m_q(v). \tag{1}$$

We solve equation (1) for its solution and investigate its various stability results in modular spaces with and without using Fatou property and in β-homogenous spaces.

2. Solution of equation (1) in the domain of non-zero real numbers

In this section, we impose the definition of reciprocal second power function and then we solve equation (1) for its solution in the setting of non-zero real numbers.

Definition 1. *A mapping $m_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ is called a reciprocal second power function if it satisfies* (1)*. Hence,* (1) *is said to be a reciprocal second power FE.*

Theorem 1. *Let $m_q : \mathbb{R}^\star \longrightarrow \mathbb{R}$ be a function. Then, m_q satisfies* (1) *if and only if there exists an identity function $I : \mathbb{R}^\star \longrightarrow \mathbb{R}$ such that $m_q(u) = [I(1/u)]^2$, for all $u \in \mathbb{R}^\star$.*

Proof. Let m_q satisfies (1). Then m_q is a reciprocal second power function and hence we can assume $m_q(u) = \dfrac{1}{u^2}$ for all $u \in \mathbb{R}^\star$. If I is an identity mapping, then $[I(1/u)]^2 = \dfrac{1}{u^2} = m_q(u)$ for all $u \in \mathbb{R}^\star$.

On the other hand, let there exists an identity function $I : \mathbb{R}^\star \longrightarrow \mathbb{R}$ such that $m_q(u) = [I(1/u)]^2$ for all $u \in \mathbb{R}^\star$. Thus, we have

$$\begin{aligned} m_q\left(\frac{uv}{2u+v}\right)+m_q\left(\frac{uv}{2u-v}\right) &= \left[I\left(\frac{2u+v}{uv}\right)\right]^2+\left[I\left(\frac{2u-v}{uv}\right)\right]^2 \\ &= \frac{(2u+v)^2}{u^2v^2}+\frac{(2u-v)^2}{u^2v^2} \\ &= \frac{8}{v^2}+\frac{2}{u^2} \end{aligned}$$

$$= 2m_q(u) + 8m_q(v)$$

for all $u, v \in \mathbb{R}^\star$, which indicates m_q satisfies (1).

In the following results, for the purpose of easy computation, let us consider the difference operator Γ_{m_q} defined as follows:

$$\Gamma_{m_q}(u,v) = m_q\left(\frac{uv}{2u+v}\right) + m_q\left(\frac{uv}{2u-v}\right) - 8m_q(u) - 2m_q(v).$$

3. Modular stability of equation (1) with $\Delta_{\frac{1}{3}}$-condition

In this present section, we explore the investigate stability results of equation (1) connected with modular theory with modular space U_μ without applying the Fatou property. In this section, let P denote a linear space. In the following results, suppose there exists $\ell > 0$ so that $\mu(3u) \leq \dfrac{1}{\ell}\mu(u)$, for all $u \in U_\mu$, then the modular μ is said to satisfy the $\Delta_{\frac{1}{3}}$-condition. Also, we say this constant ℓ is a $\Delta_{\frac{1}{3}}$-constant related to $\Delta_{\frac{1}{3}}$-condition. One can notice that if μ is convex and satisfies $\Delta_{\frac{1}{3}}$-condition with $\Delta_{\frac{1}{3}}$-constant $\ell > 0$. If $\ell < \frac{1}{3}$, then $\mu(u) \leq \frac{1}{\ell}\mu\left(\frac{u}{3}\right) \leq \frac{1}{3\ell}\mu(u)$, which implies $\mu = 0$. When μ is convex modular, then we have $\Delta_{\frac{1}{3}}$-constant $\ell \geq \frac{1}{3}$. In the following main results, let us consider U to be a normed linear space over the set of real numbers.

Theorem 2. *Suppose U_μ satisfies the $\Delta_{\frac{1}{3}}$-condition. Let there exists a mapping $\phi : P \times P \longrightarrow [0,\infty)$ such that the mapping $m_q : P \longrightarrow U_\mu$ satisfies*

$$\mu\left(\Gamma_{m_q}(u,v)\right) \leq \phi(u,v), \tag{2}$$

$$\lim_{n\to\infty} \ell^{2n}\phi\left(\frac{u}{3^n}, \frac{v}{3^n}\right) = 0 \quad \text{and} \quad \sum_{i=0}^{\infty}\left(3\ell^3\right)^i \phi\left(\frac{u}{3^i}, \frac{u}{3^i}\right) < \infty$$

for all $u, v \in P$, then a unique reciprocal second power function $D : P \longrightarrow U_\mu$ exists and satisfies

$$\mu\left(m_q(u) - D(u)\right) \leq \frac{3}{\ell}\sum_{i=0}^{\infty}\left(3\ell^3\right)^i \phi\left(\frac{u}{3^i}, \frac{u}{3^i}\right) \tag{3}$$

for all $u \in P$.

Proof. By taking $v = u$ in (2), we obtain $\mu\left(m_q\left(\dfrac{u}{3}\right) - 9m_q(u)\right) \leq \phi(u,u)$ for all $u \in P$. Employing $\Delta_{\frac{1}{3}}$-condition of μ, one can find

$$\begin{aligned}\mu\left(m_q(u) - \frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)\right) &= \mu\left(\sum_{i=0}^{n} 3^i\left(\frac{1}{3^{3i-2}}m_q\left(\frac{u}{3^{i-1}}\right) - \frac{1}{3^{3i}}m_q\left(\frac{u}{3^i}\right)\right)\right)\\ &\leq \frac{1}{\ell^2}\sum_{i=0}^{n}(3\ell^3)^i\phi\left(\frac{u}{3^i}, \frac{u}{3^i}\right) \end{aligned} \tag{4}$$

for all $u \in P$. Now, shifting u to $3^{-m}u$ in (4), we obtain

$$\begin{aligned}\mu\left(\frac{1}{9^m}m_q\left(\frac{u}{3^m}\right)-\frac{1}{9^{n+m}}m_q\left(\frac{u}{3^{n+m}}\right)\right) &\leq \ell^{-2m}\mu\left(m_q\left(\frac{u}{3^m}\right)-\frac{1}{9^n}m_q\left(\frac{u}{3^{n+m}}\right)\right)\\ &\leq \ell^{-(2m+2)}\sum_{i=0}^{n}(3\ell^3)^i\phi\left(\frac{u}{3^{i+m}},\frac{u}{3^{i+m}}\right)\\ &\leq \frac{3^{-m}}{\ell^{m+2}}\sum_{i=m+1}^{n+m}(3\ell^3)^i\phi\left(\frac{u}{3^i},\frac{u}{3^i}\right)\end{aligned}$$

for all $u \in P$. The right-hand side of the above inequality tends to 0 when $m \to \infty$ since $\ell \geq \frac{1}{3}$, which indicates that the series is convergent. In lieu of completeness of U_μ, this sequence $\left\{\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)\right\}$ turns out to be Cauchy for all $u \in P$ and hence it is $\mu-$convergent in U_μ. Hence, we have a mapping $D : P \longrightarrow U_\mu$ given by

$$D(u) = \mu - \lim_{n\to\infty}\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right),$$

that is, $\lim_{n\to\infty}\mu\left(\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)-D(u)\right) = 0$ for all $u \in P$. So, without using Fatou property, we observe from $\Delta_{\frac{1}{3}}$-condition that the inequality

$$\begin{aligned}&\mu\left(m_q(u)-D(u)\right)\\ &\leq 3\mu\left(\frac{1}{3}m_q(u)-\frac{1}{3}\cdot\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)\right)+3\mu\left(\frac{1}{3}\cdot\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)-\frac{1}{3}D(u)\right)\\ &\leq \frac{3}{k}\mu\left(m_q(u)-\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)\right)+3k\mu\left(\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)-D(u)\right)\\ &\leq \frac{3}{\ell}\sum_{i=0}^{n}\left(3\ell^3\right)^i\phi\left(\frac{u}{3^i},\frac{u}{3^i}\right)+3\ell\mu\left(\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)-D(u)\right)\end{aligned}$$

is true for $u \in P$ and all integers $n > 1$. Allowing $n \to \infty$ in the above inequality indicates that (4) holds. Plugging (u, v) by $(3^{-n}u, 3^{-n}v)$ in (2), we find that

$$\begin{aligned}&\mu\Bigg(3^{-n}m_q\left(\frac{3^{-2n}uv}{3^{-n}(2u+v)}\right)+3^{-n}m_q\left(\frac{3^{-2n}uv}{3^{-n}(2u-v)}\right)\\ &\qquad -8\cdot 3^{-n}m_q(3^{-n}u)-2\cdot 3^{-n}m_q(3^{-n}v)\Bigg)\\ &\leq \ell^{2n}\phi\left(\frac{u}{3^n},\frac{v}{3^n}\right)\end{aligned}$$

which approaches zero as $n \to \infty$ for all $u, v \in P$. Thus, in liue of the convexity of μ, we have

$$\mu\left(\frac{1}{13}D\left(\frac{uv}{2u+v}\right)+\frac{1}{13}D\left(\frac{uv}{2u-v}\right)-\frac{8}{13}D(u)-\frac{2}{13}D(v)\right)$$

$$
\begin{aligned}
&\leq \frac{1}{13}\mu\Bigg(\frac{1}{13}D\left(\frac{uv}{2u+v}\right) - 3^{-n}m_q\left(\frac{3^{-n}uv}{2u+v}\right) \\
&\quad + \frac{1}{13}D\left(\frac{uv}{2u-v}\right) - 3^{-n}m_q\left(\frac{3^{-n}uv}{2u-v}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{8}{13}\mu\left(D(u) - 3^{-n}m_q\left(3^{-n}u\right)\right) + \frac{2}{13}\mu\left(D(v) - 3^{-n}m_q\left(3^{-n}v\right)\right) \\
&\quad + \frac{1}{13}\mu\Bigg(3^{-n}m_q\left(\frac{3^{-n}uv}{2u+v}\right) + 3^{-n}m_q\left(\frac{3^{-n}uv}{2u-v}\right) \\
&\qquad\qquad - 8\cdot 3^{-n}m_q\left(3^{-n}u\right) - 2\cdot 3^{-n}m_q\left(3^{-n}v\right)\Bigg)
\end{aligned}
$$

for all $u, v \in P$ and all integer $n > 1$. Letting the limit $n \to \infty$, one obtains that D is reciprocal inverse second power function. To show the uniqueness of D, let us assume that there is another reciprocal second power function $D' : P \longrightarrow U_\mu$ satisfying

$$
\mu\left(m_q(u) - D^{'}(u)\right) \leq \frac{3}{k}\sum_{i=0}^{\infty}(3\ell^3)^i\phi\left(\frac{u}{3^i}, \frac{u}{3^i}\right).
$$

Then we see from the equalities: $D(3^{-n}u) = 9^n D(u)$ and $D^{'}(3^{-n}u) = 9^n D^{'}(u)$ that

$$
\begin{aligned}
&\mu\left(D(u) - D^{'}(u)\right) \\
&\leq 3\mu\left(\frac{1}{3}\cdot\frac{1}{9^n}D\left(\frac{u}{3^n}\right) - \frac{1}{3}\cdot\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right)\right) + 3\mu\left(\frac{1}{3}\cdot\frac{1}{9^n}m_q\left(\frac{u}{3^n}\right) - \frac{1}{3}\cdot\frac{1}{9^n}D'\left(\frac{u}{3^n}\right)\right) \\
&\leq 3\ell^{-(2n+1)}\mu\left(D\left(\frac{u}{3^n}\right) - m_q\left(\frac{u}{3^n}\right)\right) + 3\ell^{-(2n+1)}\mu\left(m_q\left(\frac{u}{3^n}\right) - D'\left(\frac{u}{3^n}\right)\right) \\
&\leq 3\ell^{-3n}\sum_{i=1}^{\infty}(3\ell^3)^i\phi\left(\frac{u}{3^{(n+i)}}, \frac{u}{3^{(n+i)}}\right) \\
&\leq \frac{3^{1-n}}{\ell^n}\sum_{i=0}^{\infty}(3\ell^3)^i\phi\left(\frac{u}{3^i}, \frac{u}{3^i}\right)
\end{aligned}
$$

for all $u \in P$. It indicates from the above inequality that D is distinctive by allowing $n \to \infty$. Hence the proof is complete.

4. Modular stability of equation (1) without $\Delta_{\frac{1}{3}}$-condition

In this present section, we provide a different result related to modular stability of equation (1) without $\Delta_{\frac{1}{3}}$-condition.

Theorem 3. *Assume that U_p is a p-complex modular space where p is convex. Also, let $\phi : U \times U \longrightarrow [0, \infty)$ be a function with the condition*

$$\widehat{\phi}(u,v) = \sum_{i=0}^{\infty} \frac{1}{9^{i+1}} \phi(3^{-i}u, 3^{-i}u) < \infty \tag{5}$$

for all $u, v \in U$. Assume that $m_q : U \longrightarrow U_p$ is a mapping such that

$$p\left(\Gamma_{m_q}(u,v)\right) \leq \phi(u,v) \tag{6}$$

for all $u, v \in U$. Then a unique reciprocal second power function $T : U \longrightarrow U_p$ exists and satisfies

$$p\left(m_q(u) - T(u)\right) \leq \widehat{\phi}(u,v) \tag{7}$$

for all $u, v \in U$.

Proof. Putting (u, v) as (u, u) in (6) and then dividing by 9 on both sides, we obtain

$$p\left(\frac{1}{9} m_q(3^{-1}u) - m_q(u)\right) \leq \frac{1}{9} m_q(u,u) \tag{8}$$

for all $u \in U$. Then by induction arguments, we arrive at

$$p\left(\frac{m_q(3^{-n}u)}{9^n} - m_q(u)\right) \leq \frac{1}{9} \sum_{i=0}^{n-1} \frac{1}{9^i} \phi(3^{-i}u, 3^{-i}u) \tag{9}$$

for all $u \in U$. It is clear that the case $n = 1$ follows directly from (8). Assume that (9) is true for $n \in \mathbb{N}$. Then, we obtain the ensuing inequality:

$$\begin{aligned}
&p\left(\frac{m_q(3^{-(n+1)}u)}{9^{n+1}} - m_q(u)\right) \\
&\quad = p\left(\frac{1}{9}\left(\frac{m_q(3^{-n}u)}{9^n} - m_q(3^{-1}u)\right) + \frac{1}{9}\left(m_q(3^{-1}u) - 9m_q(u)\right)\right) \\
&\quad \leq \frac{1}{9} p\left(m_q(3^{-n}u) - m_q(3^{-1}u)\right) + \frac{1}{9} p\left(m_q(3^{-1}u) - 9m_q(u)\right) \\
&\quad \leq \frac{1}{9} . \frac{1}{9} \sum_{i=0}^{n-1} \frac{\phi(3^{-(i+1)}u, 3^{-(i+1)}u)}{9^i} + \frac{1}{9} \phi(u,u)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{9}\left(\sum_{i=0}^{n-1} \frac{\phi(3^{-(i+1)}u, 3^{-(i+1)}u)}{9^{i+1}}\right) + \frac{1}{9}\phi(u,u) \\
&= \frac{1}{9} \sum_{i=0}^{n} \frac{\phi\left(3^{-i}u, 3^{-i}u\right)}{9^n}
\end{aligned}$$

for all $u \in U$. Hence (9) holds for every $k \in \mathbb{N}$. Let m and n be non-negative integers with $n > m$. Then (9), we have

$$\begin{aligned} p\left(\frac{m_q(3^{-n}u)}{9^n} - \frac{m_q(3^{-m}u)}{9^m}\right) &= p\left(\frac{1}{9^m}\left(\frac{m_q(3^{-n}u)}{9^{n-m}} - m_q(3^{-m}u)\right)\right) \\ &\leq \frac{1}{9^m}\cdot\frac{1}{9}\sum_{i=0}^{n-m-1}\frac{1}{9^i}\phi\left(3^{-(m+i)}u, 3^{-(m+i)}u\right) \\ &\leq \frac{1}{9}\sum_{i=0}^{n-m-1}\frac{1}{9^{m+i}}\phi\left(3^{-(m+i)}u, 3^{-(m+i)}u\right) \\ &\leq \frac{1}{9}\sum_{k=m}^{n-1}\frac{1}{9^k}\phi\left(3^{-k}u, 3^{-k}u\right) \end{aligned} \tag{10}$$

for all $u \in U$. By the application of (5) and (10), we observe that the the sequence $\left\{\dfrac{m_q(3^{-n}u)}{9^n}\right\}$ turns out to be Cauchy in U_p. By virtue of completeness of U_p, the sequence is convergent. This formulates that there exists a function $T : U \longrightarrow U_p$ defined by

$$T(u) = p - \lim \frac{m_q(3^{-n}u)}{9^n}. \tag{11}$$

To confirm that T satisfies (1), plugging (u, v) into $(3^{-n}u, 3^{-n}v)$ in (6) and then multiplying by 9^{-n} on both sides, we obtain

$$\begin{aligned} 9^{-n}p\Bigg(m_q\left(3^{-n}\left(\frac{uv}{2u+v}\right)\right) + m_q\left(3^{-n}\left(\frac{uv}{2u-v}\right)\right) - 8m_q(3^{-n}u) - 2m_q(3^{-n}v)\Bigg) \\ \leq 9^{-n}\phi(3^{-n}u, 3^{-n}u) \end{aligned} \tag{12}$$

for all $u, v \in U$. We can find that T satisfies (1) by letting $n \to \infty$ in the above inequality. To prove that T is unique reciprocal second power function which satisfies (1) and also (7). It is clear that both T' and T satisfy (7). Hence, we obtain

$$\begin{aligned} p\left(T'(u) - T(u)\right) &= 9^{-n}p\left(T'(3^{-n}u) - T(3^{-n}u)\right) \\ &\leq 9^{-n}\left(p\left(T'(3^{-n}u) - f(3^{-n}u)\right) + p\left(f(3^{-n}u) - T(3^{-n}u)\right)\right) \\ &\leq \sum_{i=n+1}^{\infty}\frac{1}{9^i}\phi(3^{-i}u, 3^{-i}u) \end{aligned} \tag{13}$$

for all $u, v \in U$. It is easy to find that T is distinctive by allowing $n \to \infty$ in (13) and employing (5), which completes the proof.

Corollary 1. *Let $m_q : U \longrightarrow U_p$ be a mapping with a constant $c \geq 0$, not depending on the values of u, v such that the inequality*

$$p\left(\Gamma_{m_q}(u, v)\right) \leq c$$

holds for all $u, v \in U$. *Then,* $T : U \longrightarrow U_p$ *is a unique reciprocal second power function satisfying* (1) *and* $p(m_q(u) - T(u)) \leq \frac{c}{8}$, *for all* $u \in U$.

Proof. It is easy to prove this corollary by taking $\phi(u, v) = c$, for all $u, v \in U$ in Theorem 3.

Corollary 2. *Let* $\lambda_1 \geq 0$ *be fixed and* $s \neq -2$ *if a function* $m_q : U \longrightarrow U_p$ *fulfills the inequality*

$$p\left(\Gamma_{m_q}(u, v)\right) \leq \lambda_1(|u|^s + |v|^s)$$

holds for all $u, v \in U$. *Then, there exists a unique reciprocal second power function* $T : U \longrightarrow U_p$ *satisfying* (1) *and*

$$p(m_q(u) - T(u)) \leq \frac{2\lambda_1}{(9 - 3^{-s})}|u|^s$$

for all $u \in U$.

Proof. The proof is obtained by taking $\phi(u, v) = \lambda_1(|u|^s + |v|^s)$ in Theorem 3.

Corollary 3. *Let* $m_q : U \longrightarrow U_p$ *be a mapping. If there exist* $x, y : s = x + y \neq -2$ *and* $\lambda_2 \geq 0$ *such that*

$$p\left(\Gamma_{m_q}(u, v)\right) \leq \lambda_2(|u|^x|v|^y)$$

holds for all $u, v \in U$. *Then, there exists a unique reciprocal second power function* $T : U \longrightarrow U_p$ *satisfying* (1) *and*

$$p(m_q(u) - T(u)) \leq \frac{\lambda_2}{(9 - 3^{-s})}|u|^s$$

for all $u \in U$.

Proof. The proof directly follows by taking $\phi(u, v) = c_2(|u|^a|v|^b)$ in Theorem 3.

Corollary 4. *Let* $\lambda_3 \geq 0$ *be fixed and* $s \neq -1$. *If a function* $m_q : U \longrightarrow U_p$ *satisfies the inequality*

$$p\left(\Gamma_{m_q}(u, v)\right) \leq \lambda_3(|u|^s|v|^s + (|u|^{2s} + |v|^{2s}))$$

for all $u, v \in U$. *Then, a unique reciprocal second power function* $T : U \longrightarrow U_p$ *exists and satisfies* (1) *and*

$$p(m_q(u) - m_q(v)) \leq \frac{3\lambda_3}{(9 - 3^{-2s})}|u|^{2s}$$

for all $u \in U$.

Proof. The proof is achieved by considering $\phi(u, v) = \lambda_3(|u|^s|v|^s + (|u|^{2s} + |v|^{2s}))$ in Theorem 3.

5. Stability of equation (1) in β-homogeneous spaces

In this section, we obtain the stability results of equation (1) in β-homogenous spaces.

Theorem 4. *Let V be a β-homogeneous complex Banach space $(0 < \beta \leq 1)$, and $\phi : U \times U \longrightarrow (0, \infty]$ be a function with*

$$\widehat{\phi}(u,v) = \frac{1}{9^{\beta}} \sum_{i=0}^{\infty} \frac{1}{9^{\beta i}} \phi\left(3^{-i}u, 3^{-i}u\right) < \infty \tag{14}$$

for all $u, v \in U$. Assume that $m_q : U \longrightarrow V$ is a mapping such that

$$\left\|\Gamma_{m_q}(u,v)\right\| \leq \phi(u,v) \tag{15}$$

holds for all $u, v \in U$. Then there exists a unique reciprocal second power function $T : U \longrightarrow V$ such that

$$\|m_q(u) - T(u)\| \leq \widehat{\phi}(u,u) \tag{16}$$

for all $u \in U$.

Proof. Firstly, let us substitute $v = u$ in (15). Then, we get

$$\left\|m_q(3^{-1}u) - 9m_q(u)\right\| \leq \phi(u,u) \tag{17}$$

for all $u \in U$. By employing induction technique on $k \in \mathbb{N}$ and using (17), we acquire

$$\left\|\frac{m_q(3^{-n}u)}{9^n} - m_q(u)\right\| \leq \frac{1}{9^{\beta}} \sum_{i=0}^{n-1} \frac{\phi(3^{-i}u, 3^{-i}u)}{9^{i\beta}} \tag{18}$$

for all $u \in U$. Let m and n be non-negative integers with $n > m$, Then using (18), we have

$$\begin{aligned} \left\|\frac{m_q(3^{-n}u)}{9^n} - \frac{m_q(3^{-n}u)}{9^m}\right\| &= \left\|\frac{1}{9^m}\left(\frac{m_q(3^{-n})}{9^{n-m}} - m_q(3^{-m})\right)\right\| \\ &\leq \frac{1}{9^{m\beta}} \frac{1}{9^{\beta}} \sum_{i=0}^{n-m-1} \frac{\phi(3^{-(i+m)}u, 3^{-(i+m)}u)}{9^{i\beta}} \\ &= \frac{1}{9^{\beta}} \sum_{i=m}^{n-1} \frac{1}{9^{i\beta}} \phi\left(3^{-i}u, 3^{-i}u\right) \end{aligned} \tag{19}$$

for all $u \in U$. Letting $n \to \infty$ in the above inequality, we find that the sequence $\left\{\dfrac{m_q(3^{-n}u)}{9^n}\right\}$ becomes Cauchy in U. Due to the completeness of U, the sequence is convergent. Hence there exists a mapping $T : U \longrightarrow V$ defined by

$$T(u) = \lim_{n\to\infty} \frac{m_q(3^{-n}u)}{9^n} \tag{20}$$

for all $u \in U$. Putting $m = 0$ and taking the limit $n \to \infty$ in the above inequality, we obtain (16) using (20). Next, consider an additional function $S : U \longrightarrow V$ satisfying (16) and (20). Then, we get

$$\begin{aligned}\|T(u) - S(u)\| &\leq \left\|\frac{T(3^{-n}) - m_q(3^{-n})}{9^n}\right\| + \left\|\frac{m_q(3^{-n}) - S(3^{-n})}{9^n}\right\| \\ &\leq \frac{1}{9^\beta}\sum_{i=0}^{\infty}\frac{1}{9^{(i+n)\beta}}\phi\left(3^{-(n+i)}u, 3^{-(n+i)}u\right) \\ &= \frac{1}{9^\beta}\sum_{i=n}^{\infty}\frac{1}{9^{i\beta}}\phi\left(3^{-i}u, 3^{-i}u\right).\end{aligned}$$

From the above inequality, weobserve T is unique by letting $n \to \infty$, which completes the proof.

Corollary 5. *Let $m_q : U \longrightarrow V$ be a function with a constant $\lambda_4 \geq 0$, not depending on the values of u, v such that the inequality*

$$\left\|\Gamma_{m_q}(u, v)\right\| \leq \lambda_4$$

holds for all $u, v \in U$. Then, $T : U \longrightarrow V$ is a unique reciprocal second power function satisfying (1) *and*

$$\|m_q(u) - T(u)\| \leq \frac{\lambda_4}{9^\beta - 1}$$

for all $u \in U$.

Proof. Taking $\phi(u, v) = \lambda_4$ in Theorem 4, we arrive at the required result.

Corollary 6. *Let $\lambda_5 \geq 0$ be fixed and $s \neq -2\beta$. Suppose a function $m_q : U \longrightarrow V$ satisfies the inequality*

$$\left\|\Gamma_{m_q}(u, v)\right\| \leq \lambda_5(||u||^s + ||v||^s)$$

for all $u, v \in U$. Then, a unique reciprocal second power function $T : U \longrightarrow V$ exists and satisfies (1) *and*

$$\|m_q(u) - T(u)\| \leq \frac{2\lambda_5}{(9^\beta - 3^{-s})}||u||^s$$

for all $u \in U$.

Proof. Replacing $\phi(u, v) = \lambda_5(||u||^s + ||v||^s)$ in Theorem 4 and proceeding further, we obtain the desired result.

Corollary 7. *Let $m_q : U \longrightarrow V$ be a function. If there exist $x, y : s = x + y \neq -2\beta$ and $\lambda_6 \geq 0$ such that*

$$\left\|\Gamma_{m_q}(u, v)\right\| \leq \lambda_6(||u||^x||v||^y)$$

holds for all $u, v \in U$. Then, there exists a unique reciprocal second power function $T : U \longrightarrow V$ satisfying (1) *and*

$$\|m_q(u) - T(u)\| \leq \frac{\lambda_6}{(9^\beta - 3^{-s})}||u||^s$$

for all $u \in U$.

Proof. Choosing $\phi(u,v) = \lambda_6(|u|^x|v|^y)$ in Theorem 4, we achieve the result.

Corollary 8. *Let $\lambda_7 \geq 0$ be fixed and $s \neq -\beta$. Let a function $m_q : U \longrightarrow V$ satisfies the inequality*

$$\left\|\Gamma_{m_q}(u,v)\right\| \leq \lambda_7(||u||^s||v||^s + (||u||^{2s} + ||v||^{2s}))$$

for all $u, v \in U$. Then, a unique reciprocal second power function $T : U \longrightarrow V$ exists and satisfies (1) *and*

$$\|m_q(u) - T(u)\| \leq \frac{3\lambda_7}{(9^\beta - 3^{-2s})}||u||^{2s}$$

for all $u \in U$.

Proof. Selecting $\phi(u,v) = \lambda_7(||u||^s||v||^s + (||u||^{2s} + ||v||^{2s}))$ in Theorem 4, we get the required result.

6. Application of equation (1)

We close our investigation with an application of equation (1) using Coloumb's law. According to Coloumb, the electrostatic force of attraction between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

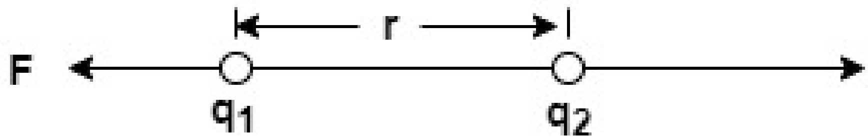

Figure 1: Electrostatic force of attraction F between two point charges q_1 and q_2

That is,

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{r^2}$$

where F and r, respectively, are the force of attraction and distance between the point charges q_1 and q_2. Suppose the constant $\frac{1}{4\pi\epsilon_0}$ is taken as a constant c and unit point charges are assumed, then the electrocstatic force of attraction is given by

$$F = \frac{c}{r^2}$$

which is a reciprocal second power function. Suppose the distance between two unit point charges is $\frac{uv}{2u+v}$, then the electrocstatic force of attraction is given by

$$m_q\left(\frac{uv}{2u+v}\right) = \frac{c(2u+v)^2}{u^2v^2}.$$

Also, if the distance is $\frac{uv}{2u-v}$, then the electrocstatic force of attraction is given by

$$m_q\left(\frac{uv}{2u-v}\right) = \frac{c(2u-v)^2}{u^2v^2}.$$

Then using equation (1), we can relate that the sum of the above electrocstatic forces of attraction $m_q\left(\frac{uv}{2u+v}\right)$ and $m_q\left(\frac{uv}{2u-v}\right)$ is given by the sum of electrocstatic forces of attraction $2m_q(u) = \frac{2c}{u^2}$ and $8m_q(v) = \frac{8c}{v^2}$. Hence equation (1) dealt in this study can be associated with the electrocstatic forces of attraction between the charges in different situations.

7. Conclusion

In this investigation, we introduced a new reciprocal second power FE (1) and investigated its various classical stability results in modular spaces and β-homogenous spaces. We solved equation (1) for its solution in the setting of non-zero real numbers. We associated equation (1) with Coloumb's law to employ it in various situations to connect the electrocstatic forces of attraction in different assumptions.

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References

[1] I Amemiya. On the representation of complemented modular lattices. *J. Math. Soc.*, 9:263–279, 1957.

[2] A Bodaghi and B V Senthil Kumar. Estimation of inexact reciprocal-quintic and reciprocal-sextic functional equations. *Mathematica*, 49(82)1-2:3–14, 2017.

[3] K Cieplinski. Applications of fixed point theorems to the hyers-ulam stability of functional equations-a survey. *Ann. Funct. Anal.*, 3:151–164, 2012.

[4] H Dutta and B V Senthil Kumar. Geometrical elucidations and approximation of some functional equations in numerous variables. *Proc. Indian Natn. Sc. Acad.*, 85(3):603–611, 2019.

[5] P Găvruta. A generalization of the hyers-ulam-rassias stability of approximately additive mapppings. *J. Math. Anal. Appl.*, 184:431–436, 1994.

[6] D H Hyers. On the stability of the linear functional equation. *Proc. Nat. Acad. Sci. U.S.A.*, 27:222–224, 1941.

[7] M A Khamsi. Quasicontraction mappings in modular spaces without δ_2-condition. *Fixed Point Theory Appl.*, Art. ID 916187:1–6, 2008.

[8] S O Kim, B V Senthil Kumar, and A Bodaghi. Stability and non-stability of the reciprocal-cubic and reciprocal-quartic functional equations in non-archimedean fields. *Adv. Difference Equ.*, 77:1–12, 2017.

[9] S Koshi and T Shimogaki. On f-norms of quasi-modular spaces. *J. Fac. Sci., Hokkaido Univ., Ser.*, 15:202–218, 1961.

[10] M Krbec. Modular interpolation spaces. *I. Z. Anal. Anwend.*, 1:25–40, 1982.

[11] B V Senthil Kumar and H Dutta. Non-archimedean stability of a generalized reciprocal-quadratic functional equation in several variables by direct and fixed point methods. *Filomat.*, 32(9):3199–3209, 2018.

[12] B V Senthil Kumar and H Dutta. Approximation of multiplicative inverse undecic and duodecic functional equations. *Math. Meth. Appl. Sci.*, 42:1073–1081, 2019.

[13] B V Senthil Kumar and H Dutta. Fuzzy stability of a rational functional equation and its relevance to system design. *Int. J. General Syst.*, 38(2):157–169, 2019.

[14] B V Senthil Kumar and H Dutta. Fundamental stabilities of various forms of complex valued functional equations. In H. Dutta and J. Peters, editors, *Applied Mathematical Analysis: Theory, Methods, and Applications. Studies in Systems, Decision and Contro.*, volume 177. Springer, Cham, 2020.

[15] B V Senthil Kumar, H Dutta, and S. Sabarinathan. Approximation of a system of rational functional equations of three variables. *Int. J. Appl. Comput. Math.*, 5(3):1–16, 2019.

[16] W A Luxemburg. *Banach function spaces.* PhD thesis, Delft University of Technology, Delft, The Netherlands, 1959.

[17] L Maligranda. *Orlicz Spaces and Interpolation.* Seminarios de Mathematica, 5, Universidade Estadual de Campinas, Departamento de Matematica, Campinas, 1989.

[18] B Mazur. Modular cuves and the eisenstein ideal. *Publ. Math. IHES.*, 47:33–186, 1978.

[19] J Musielak. *Orlicz Spaces and Modular Spaces.* Springer, Berlin, 1983.

[20] H Nakano. *Modulared Semi-Ordered Linear Spaces.* Maruzen, Tokyo, 1950.

[21] W Orlicz. *Collected Papers.* PWN, Warszawa, 1988.

[22] Z Pales. Generalized stability of the cauchy functional equation. *Aequ. Math.*, 56:222–232, 1998.

[23] V Radu. The fixed point alternative and the stability of functional equations. *Fixed Point Theory.*, 4:91–96, 2003.

[24] J M Rassias. On approximately of approximately linear mappings by linear mappings. *J. Funct. Anal.*, 46:126–130, 1982.

[25] T M Rassias. On the stability of the linear mapping in banach spaces. *Proc. Amer. Math. Soc.*, 72:297–300, 1978.

[26] G Sadeghi. A fixed point approach to stability of functional equations in modular spaces. *Bull. Malays. Math. Sci. Soc.*, 37(2):333–344, 2014.

[27] L Szekelyhidi. Note on a stability theorem. *Can. Math. Bull.*, 25:500–501, 1982.

[28] J Tabor. General stability of functional equations of linear type. *J. Math. Anal. Appl.*, 328:192–200, 2007.

[29] P Turpin. Fubini inequalities and bounded multiplier property in generalized modular spaces. *Comment. Math.*, 1:331–353, 1978.

[30] S M Ulam. *Problems in Modern Mathematics.* Wiley-Interscience, New York, 1964.

[31] K Wongkum, P Chaipunya, and P Kumam. On the generalized ulam-hyers-rassias stability of quadratic mappings in modular spaces without δ_2-conditions. *J. Funct. Spaces.*, Art. ID 461719:1–6, 2015.

[32] S Yamamuro. On conjugate spaces of nakano spaces. *Trans Am. Math. Soc.*, 90:291–311, 1959.