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Identification of the Memory Kernels and Controllability for Parabolic Equations

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Abstract. This paper deals with the controllability and observability properties of the mathematical models (describing systems with thermal memory) consisting of boundary value problems of parabolic type, where the differential equation contains additional integral expressions including "memory functions" which describe the memory property of the material. The proof of controllability relies on a Carleman type estimate and duality arguments.

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1. Introduction

In many of the applications [4] we begin with a partial differential equation and, through simplifying assumptions, arrive at an integral or integrodifferential equation which takes the whole history into account. Lunardi [11] and Unger et al [14], for example, studied the problem concerned with materials with memory having the property that the mathematical-physical description of their state at a given point of time includes such states in which the materials have been at earlier points of time. In the linear theory of heat flow in a rigid homogeneous isotropic body consisting of material with thermal memory, the following system of constitutive relationships hold (see [9,14])

$$e(t,x) = \beta y(t,x) + \int_{-\infty}^{t} n(t,\tau) y(\tau,x) d\tau, \tag{1}$$

$$s(t,x) = -\zeta \nabla y(t,x) - \int_{-\infty}^{t} m(t,\tau) \nabla y(\tau,x) d\tau, \tag{2}$$

together with the heat-balance equation:

$$e_t(t,x) + divs(t,x) = f(t,x), \tag{3}$$

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where $e(t,x)$ is the internal energy, $s(t,x)$ is the heat flux, $y(t,x)$ is the body temperature with time $t \in [0,T]$ for fixed T, $x \in \Omega$ where $\Omega \subset \mathbb{R}^3$ is an open bounded domain with a smooth boundary $\partial\Omega$ of class C^1, $f(t,x)$ is the given heat source, $\beta = c\rho$ (c is the specific caloric constant; ρ is the density) and ζ is the heat-conduction coefficient.

If we assume that $y(t,x) \equiv 0$ for $-\infty < t < 0$, it can be immediately seen that the relations (1)-(3) lead to the system of the form,

$$\begin{aligned}
\beta y_t(t,x) &- \zeta\Delta y(t,x) - \int_0^t m(t,\tau)\Delta y(\tau,x)d\tau \\
&+\frac{\partial}{\partial t}\Big(\int_0^t n(t,\tau)y(\tau,x)d\tau\Big) = f(t,x) \text{ in } (0,T)\times\Omega, \\
y(0,x) &= y_0(x) \text{ in } \Omega, \\
y(t,x) &= 0 \text{ on } (0,T)\times\partial\Omega,
\end{aligned}$$

where $y_0(x)$ is the given initial temperature distribution. The memory kernels n and m are sufficiently smooth and have support in (t_0, t_1), where $0 < t_0 < t_1 < T$ satisfying $m(t,t) = n(t,t) = 0$ and represent the derivatives of the relaxation function of internal energy and heat flux respectively. Hereafter, for our convenience, assume that $\beta = 1$ and $\zeta = 1$ and set $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$.

We now consider the corresponding controlled parabolic system with memory kernels

$$\left.\begin{aligned}
&y_t - \Delta y(t,x) - M_0^t * \Delta y(t) + (N_0^t * y(t))_t = f(t,x) + \chi_\omega u(t,x) \text{ in } Q \\
&y(0,x) = y_0(x) \text{ in } \Omega \\
&y(t,x) = 0 \text{ on } \Sigma,
\end{aligned}\right\} \tag{4}$$

where χ_ω is the characteristic function of the open set $\omega \subset \Omega$, $u = u(t,x)$ is the control function to be determined which acts on the system through ω while $f \in L^2(Q)$ is the given source term. The notations $M_0^t * \Delta y$ and $N_0^t * y$ respectively stand for memory integrals from 0 to t, that is,

$$\begin{aligned}
M_0^t * \Delta y(t) &= \int_0^t m(t,\tau)\Delta y(\tau)d\tau, \\
N_0^t * y(t) &= \int_0^t n(t,\tau)y(\tau)d\tau.
\end{aligned}$$

The system is null controllable at time T if, for each $y_0 \in H_0^1(\Omega)$, there exists a control $u \in L^2(\omega\times(0,T))$ such that the associated solution satisfies

$$y(T,x) = 0 \;\; a.e. \; x \in \Omega.$$

The null controllability of linear parabolic equations without the memory kernels has been intensively studied by several authors; for instance see Barbu [2], Fernandez-Cara et al [6], Fursikov and Imanuvilov [7], Imanuvilov [8] and the references cited therein. Fernandez-Cara and Zuazua [5] studied the approximate controllability for heat equations and Barbu and Iannelli [3] discussed the approximate controllability for the system of the form (4) with the kernel $n(\cdot) = 0$. Sakthivel *et al* [12] obtained the exact null controllability result by establishing a Carleman type inequality for the linear parabolic equation (taking the history into account),

$$
\begin{aligned}
y_t - \Delta y \quad &+ \quad \int_0^t a(t-\tau)y(\tau,x)d\tau = u(t,x) + l(t,x) \text{ in } Q, \\
y(0,x) \quad &= \quad y_0(x) \text{ in } \Omega, \\
\alpha_1 \frac{\partial y}{\partial \nu} \quad &+ \quad \alpha_2 y = 0 \text{ on } \Sigma,
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^1$, the kernel $a \in C^1[0,T], a(0) = 0$ and $\alpha_1 \geq 0$ is a constant and $\alpha_2 \in C^1(\Sigma),\ \alpha_2 \geq 0$, while Fernandez-Cara et al [6] studied the exact controllability of the parabolic equation of the form,

$$
y_t - \Delta y + B(t,x)\nabla y + a(t,x)y = v(t,x)\chi_\omega \text{ in } Q
$$

with Fourier boundary conditions when the coefficients a, B and α_2 satisfy $a \in L^\infty(Q), B \in L^\infty(Q)$, and $\alpha_2 \in L^\infty(\Sigma)$. The problem under consideration is interesting and different from the previous works (see [9,10]) because the derivation of Carleman estimate containing a special type of integral term for the backward adjoint problem of (4) stated in (5) require a careful treatment of the surface integrals to guarantee the existence (ie., to settle the integral term properly so as to get the same upper bound) of this estimate for the parabolic integrodifferential equations.

Throughout this paper we shall use the following notations for general function spaces. For each positive integer m, we denote, by $H^m(\Omega)$, the Sobolev spaces of functions in $L^2(\Omega)$ whose weak derivatives of order less than or equal to m are also in $L^2(\Omega)$. We define $L^2(0,T;H^1(\Omega))$, the space of all equivalence classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$. The space $L^2(0,T;L^2(\Omega))$ is analogously defined. Moreover, we set

$$
\begin{aligned}
H^1(0,T;L^2(\Omega)) \quad &= \quad \{y \in L^2(0,T;L^2(\Omega)) : \ \frac{dy}{dt} \in L^2(0,T;L^2(\Omega))\}, \\
H^{2,1}(Q) \quad &= \quad \{y \in L^2(0,T;H_0^1(\Omega) \cap H^2(\Omega)),\ \frac{dy}{dt} \in L^2(0,T;L^2(\Omega))\},
\end{aligned}
$$

where $\frac{dy}{dt}$ is taken in the sense of distributions. For the definition and detailed discussion on these spaces one can refer [1,13].

The paper is organized as follows: In section 2 we establish a Carleman estimate for the dual problem stated in (5) and we deduce an observability inequality . In section 3, we prove the null controllability of the system (4) making use of observability inequality and an *a priori* estimate for the solution of the system (4).

2. Carleman and Observability Inequalities

In this section we shall obtain a Carleman inequality and an observability estimate for the following adjoint system associated with (4),

$$\left.\begin{array}{l} q_t + \Delta q + M_t^T * \Delta q(t) + N_t^T * q_t(t) = g \text{ in } Q \\ q(T,x) = q_T(x) \text{ in } \Omega \\ q(t,x) = 0 \text{ on } \Sigma, \end{array}\right\} \tag{5}$$

where $q_T \in L^2(\Omega)$, $g \in L^2(Q)$ and $M_t^T * \Delta q$, $N_t^T * q_t$ are the corresponding adjoint integrals, that is,

$$\begin{aligned} M_t^T * \Delta q(t) &= \int_t^T m(\tau,t)\Delta q(\tau)d\tau, \\ N_t^T * q_t(t) &= \int_t^T n(\tau,t) q_\tau(\tau)d\tau. \end{aligned}$$

To formulate our results, we give some of the frequently used notations, following the idea used in [7], which provide a fundamental tool in proving the Carleman type estimates. Let $\omega_0 \Subset \omega$ be a suitably fixed sub domain. Then there exists a function $\psi \in C^2(\overline{\Omega})$ such that

$$\psi(x) > 0 \ \ \forall \ \ x \in \Omega, \ \ \psi|_{\partial\Omega} = 0, \ \ |\nabla\psi(x)| > 0 \ \ \forall \ \ x \in \Omega \backslash \omega_0.$$

We define two weight functions that will be used throughout this paper as follows: For fixed $\lambda > 0$ and the function ψ defined above, we introduce functions $\phi, \alpha : Q \to \mathbb{R}$ defined by the formulas

$$\phi(t,x) = \frac{e^{\lambda\psi(x)}}{\xi(t)}, \quad \alpha(t,x) = \frac{e^{2\lambda\Psi(x)} - e^{\lambda\psi}}{\xi(t)},$$

where

$$\xi(t) = t(T-t) \quad \text{and} \quad \Psi = \|\psi(x)\|_{C(\overline{\Omega}).}$$

Moreover, in proving the main inequality, we need the following estimates for the functions ϕ and α:

$$\left.\begin{array}{lcl} \left|\frac{\partial\phi}{\partial t}\right| &=& \frac{|T-2t|}{t^2(T-t)^2} e^{\lambda\psi} \le C(\Omega,\omega) T\phi^2 \\ \left|\frac{\partial\alpha}{\partial t}\right| &=& \frac{|T-2t|}{t^2(T-t)^2} |e^{2\lambda\Psi} - e^{\lambda\psi}| \le C(\Omega,\omega) \frac{Te^{\lambda\psi}}{t^2(T-t)^2)} \le C(\Omega,\omega) T\phi^2 \\ \left|\frac{\partial^2\alpha}{\partial t^2}\right| &=& \frac{|2T^2-6Tt+6t^2|}{t^3(T-t)^3} |e^{2\lambda\Psi} - e^{\lambda\psi}| \le C(\Omega,\omega) T^2\phi^3 \end{array}\right\} \tag{6}$$

where $C(\Omega,\omega)$ is a generic constant. Throughout the proof of the estimate, we use $C(\Omega,\omega)$, the generic constant for all the space derivatives of ψ. One can also easily verify the identities which will be used in the sequel are

$$\nabla\phi = \lambda\phi\nabla\psi, \quad \nabla\alpha = -\lambda\phi\nabla\psi.$$

Now we are ready to state and prove the main estimate of this section. Though the proof of this estimate follows standard technique for general parabolic equations without memory, we have to do careful calculations on the memory integrals which involves second derivative in spatial variable as well as first derivative in time variable.

Theorem 1. *For any solution q of the dual problem (5) with the kernels $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$ have support in (t_0,t_1), where $0<t_0<t_1<T$, there exist λ_0, s_0 and C, the constant depending on Ω,ω,λ and T such that for every $\lambda\geq\lambda_0, s\geq s_0$ the following inequality holds:*

$$L_{Q,s,\phi}(q)\leq C\Big(\iint\limits_{(0,T)\times\omega} e^{-2s\alpha}s^3\phi^3|q|^2dxdt+\iint\limits_{Q} e^{-2s\alpha}|g|^2dxdt\Big),\tag{7}$$

where we used the notation

$$\begin{aligned}L_{Q,s,\phi}(q) &= \iint\limits_{Q}(s\phi)^{-1}\big(|q_t|^2+|\Delta q|^2+\Big|M_t^T*\Delta q(t)\Big|^2+\Big|N_t^T*q_t(t)\Big|^2\big)e^{-2s\alpha}dxdt\\ &\quad+\iint\limits_{Q}(s^3\phi^3|q|^2+s\phi|\nabla q|^2)e^{-2s\alpha}dxdt.\end{aligned}$$

Moreover, the constants λ_0, s_0 take the form $\lambda_0=C(\Omega,\omega)[1+\sqrt{T}+T^2+T^4]$ and $s_0=C(\Omega,\omega)[T+T\sqrt{T}+T^2+T^4]$.

To prove this theorem we need the following lemma in terms of the new transformed variable $p=e^{-s\alpha}q$ which essentially completes the first part of Theorem 1.

Lemma 1. *Let the kernels $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$ have support in (t_0,t_1), where $0<t_0<t_1<T$ and $g\in L^2(Q)$ be given. There exist $\widetilde{\lambda}_0,\widetilde{s}_0$ and C only depending on Ω,ω and T such that, for any $\lambda\geq\widetilde{\lambda}_0=C(\Omega,\omega)(1+T^4)$, any $s\geq\widetilde{s}_0=C(\Omega,\omega)(T+T^2+T^4)$, the weak solution of (5) satisfies*

$$\begin{aligned}\widetilde{L}_{Q,s,\lambda}(p) &\leq C\big(\|e^{-s\alpha}g\|^2_{L^2(Q)}+\widetilde{L}_{Q^{\omega_0},s,\lambda}(p)+M_{Q,s,\lambda}(m;p)+N_{Q,s,\lambda}(n;p)\\ &+ M_{Q,s,\lambda}(m_t;p)+N_{Q,s,\lambda}(n_t;p)\big),\end{aligned}\tag{8}$$

where

$$\begin{aligned}\widetilde{L}_{Q,s,\lambda}(p) &= \iint\limits_{Q}s^3\lambda^4\phi^3|p|^2dxdt+\iint\limits_{Q}s\lambda^2\phi|\nabla p|^2dxdt,\\ M_{Q,s,\lambda}(m;p) &= \iint\limits_{Q}e^{-2s\alpha}s\lambda\phi\Big|M_t^T*\Delta(e^{s\alpha}p)(t)\Big|^2dxdt,\\ N_{Q,s,\lambda}(n;p) &= \iint\limits_{Q}e^{-2s\alpha}s\lambda\phi\Big|N_t^T*(e^{s\alpha}p)_t(t)\Big|^2dxdt,\end{aligned}$$

and the notations $M_{Q,s,\lambda}(m_t;p), N_{Q,s,\lambda}(n_t;p)$ denote the time derivative of the kernels respectively in $M_{Q,s,\lambda}(m;p)$, $N_{Q,s,\lambda}(n;p)$ and $Q_{\omega_0}=(0,T)\times\omega_0$.

The proof of Lemma 1 is quite similar to the detailed proof given in [9],[10]. The explicit dependence of the constant on time and space is not obtained in [12] and we refer to [10],[6] where the explicit dependence has been computed. Now we need to estimate the memory integrals appearing on the right hand side of the estimate (8) and in fact this will complete the proof of Theorem 1.

Proof. First we write the inequality (8) in terms of the original variable by substituting $p=e^{-s\alpha}q$ to have

$$\begin{aligned}\iint\limits_Q e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2dxdt+\iint\limits_Q s\lambda^2\phi|\nabla(e^{-s\alpha}q)|^2dxdt \le C\Big(\iint\limits_Q e^{-2s\alpha}|g|^2dxdt\\ +\iint\limits_{Q_{\omega_0}} e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2dxdt+\iint\limits_{Q_{\omega_0}} s\lambda^2\phi|\nabla(e^{-s\alpha}q)|^2dxdt\\ +M_{Q,s,\lambda}(m;q)+M_{Q,s,\lambda}(m_t;q)+N_{Q,s,\lambda}(n;q)+N_{Q,s,\lambda}(n_t;q)\Big).\end{aligned}$$

Note that $\nabla(e^{-s\alpha}q)=e^{-s\alpha}s\lambda\phi\nabla\psi q+e^{-s\alpha}\nabla q$ and

$$\begin{aligned}2\iint\limits_Q e^{-2s\alpha}s^2\lambda^3\phi^2\nabla\psi q\nabla q dxdt \quad &\ge\quad -\rho\iint\limits_Q e^{-2s\alpha}s\lambda^2\phi|\nabla q|^2dxdt\\ &\quad -\frac{1}{\rho}\iint\limits_Q e^{-2s\alpha}s^3\lambda^4\phi^3|\nabla\psi|^2|q|^2dxdt,\end{aligned}$$

where the parameter $\rho\in(0,1)$. Choose $\|\nabla\psi\|_{C(\bar\Omega)}\le\rho$ to obtain

$$\begin{aligned}\widetilde{L}_{Q,s,\lambda}(q)\le C\Big(\iint\limits_Q e^{-2s\alpha}|g|^2dxdt+\widetilde{L}_{Q_{\omega_0},s,\lambda}(q)+M_{Q,s,\lambda}(m;q)\\ +M_{Q,s,\lambda}(m_t;q)+N_{Q,s,\lambda}(n;q)+N_{Q,s,\lambda}(n_t;q)\Big),\end{aligned}\tag{9}$$

since we have redefined the notations $\widetilde{L},M,N$ as follows:

$$\begin{aligned}\widetilde{L}_{Q,s,\lambda}(q) &= \iint\limits_Q e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2dxdt+\iint\limits_Q e^{-2s\alpha}s\lambda^2\phi|\nabla q|^2dxdt,\\ M_{Q,s,\lambda}(m;q) &= \iint\limits_Q e^{-2s\alpha}s\lambda\phi\left|M_t^T*\Delta q(t)\right|^2dxdt,\end{aligned}$$

$$N_{Q,s,\lambda}(n;q) \;=\; \iint\limits_{Q} e^{-2s\alpha}s\lambda\phi\left|N_t^T * q_t(t)\right|^2 dxdt.$$

Next we shall express the term $|\nabla q|^2$ over Q_{ω_0} on the right hand side of (9), in terms of $|q|^2$ in the larger domain ω(since $\omega_0 \Subset \omega \subset \Omega$). To attain this, let us introduce a truncating function $\theta = \theta(x), 0 \leq \theta \leq 1$ satisfying

$$\theta \in C_0^2(\omega), \quad \theta = 1 \text{ in } \bar{\omega}_0 \text{ and } \theta = 0 \text{ in } \Omega\backslash\omega.$$

Multiplying (5) by $e^{-2s\alpha}\theta s\lambda^2\phi q$ and integrating over Q, we obtain that

$$\begin{aligned}\iint\limits_{Q} e^{-2s\alpha}\theta s\lambda^2\phi|\nabla q|^2 dxdt &\leq \frac{1}{4}\iint\limits_{Q} e^{-2s\alpha}|g|^2 dxdt + \frac{1}{4}M_{Q^\omega,s,\lambda}(m;q)\\ &+\frac{1}{4}N_{Q^\omega,s,\lambda}(n;q) - \iint\limits_{Q} s\lambda^2\nabla(e^{-2s\alpha}\theta\phi)q\nabla q dxdt\\ &+\iint\limits_{Q_\omega} e^{-2s\alpha}(s^2\lambda^4\phi^2 + 2s\lambda^3\phi)|q|^2 dxdt - \frac{1}{2}\iint\limits_{Q_\omega} s\lambda^2(e^{-2s\alpha}\phi)_t|q|^2 dxdt = \sum_{i=1}^{6} I_i. \quad (10)\end{aligned}$$

Now a simple computation yields the following estimates: The integral I_4 can be estimated by

$$C(\Omega,\omega)\iint\limits_{Q_{\omega_0}} e^{-2s\alpha}(s^3\lambda^4\phi^3 + s\phi(\lambda^4+\lambda^2))|q|^2 dxdt + \frac{1}{4}\iint\limits_{Q_{\omega_0}} e^{-2s\alpha}s\lambda^2\phi|\nabla q|^2 dxdt,$$

where the first integral can be bounded by $\iint\limits_{Q_\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2 dxdt$, if $\lambda \geq C(\Omega,\omega)T^2,\ s \geq 1$. The integral I_6 has also the same bound,

$$I_6 \leq C(\Omega,\omega)T\iint\limits_{Q_\omega} e^{-2s\alpha}(s^2\lambda^2\phi^3 + s\lambda^2\phi^2)|q|^2 dxdt \leq \iint\limits_{Q_\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2 dxdt$$

for the choice of $\lambda \geq 1,\ \ s \geq C(\Omega,\omega)(T + T^{3/2})$. Thus, combining all the preceding inequality, we obtain

$$\begin{aligned}\iint\limits_{Q_{\omega_0}} e^{-2s\alpha}s\lambda^2\phi|\nabla q|^2 dxdt \;&\leq\; C\Big(\iint\limits_{Q} e^{-2s\alpha}|g|^2 dxdt + \iint\limits_{Q_\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2 dxdt \quad (11)\\ &+\; M_{Q^\omega,s,\lambda}(m;q) + N_{Q^\omega,s,\lambda}(n;q)\Big).\end{aligned}$$

Using (11), the inequality (9) can be re-estimated as

$$\widetilde{L}_{Q,s,\lambda}(q) \leq C\Big(\iint\limits_{Q} e^{-2s\alpha}|g|^2 dxdt + \iint\limits_{(0,T)\times\omega} e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2 dxdt$$

$$+M_{Q,s,\lambda}(m;q)+M_{Q,s,\lambda}(m_t;q)+N_{Q,s,\lambda}(n;q)+N_{Q,s,\lambda}(n_t;q)\Big), \tag{12}$$

for any $\lambda \geq \lambda_0 = C(\Omega,\omega)[1+\sqrt{T}+T^2+T^4]$ and $s \geq \tilde{s}_0 = C(\Omega,\omega)[T+T^2+T\sqrt{T}+T^4]$. Making use of the assumptions on the kernel, Hölder's inequality and changing the order of integration, we have

$$\begin{aligned} M_{Q,s,\lambda}(m;q) &= \iint\limits_Q e^{-2s\alpha}s\lambda\phi\left|M_t^T * \Delta q(t)\right|^2 dxdt \leq \iint\limits_Q e^{-2s\alpha}s\lambda\phi\left|M_0^T * \Delta q(t)\right|^2 dxdt \\ &\leq \iint\limits_Q e^{-2s\alpha}\lambda\phi\Big(\int_{t_0}^{t_1}|m(\tau,t)|^2 e^{(s^2+2s\alpha)}\phi(\tau)d\tau\Big)\Big(\int_{t_0}^{t_1}e^{-2s\alpha}s^{-1}\phi^{-1}(\tau)|\Delta q(\tau)|^2 d\tau\Big)dxdt \\ &\leq C\|m\|_{L^\infty}^2 \iint\limits_{(t_0,t_1)\times\Omega} e^{-2s\alpha}(s\phi)^{-1}\lambda|\Delta q|^2\Big(\int_0^T e^{-2s\alpha}\phi(\tau)d\tau\Big)dxdt \\ &\leq C\iint\limits_{(t_0,t_1)\times\Omega} e^{-2s\alpha}\lambda(s\phi)^{-1}|\Delta q|^2 dxdt \leq C\iint\limits_Q e^{-2s\alpha}\lambda(s\phi)^{-1}|\Delta q|^2 dxdt, \end{aligned} \tag{13}$$

where C depends on Ω,ω,t_0,t_1,T, and m. Similarly, estimating the integral $N_{Q,s,\lambda}(n;q)$, one can have

$$N_{Q,s,\lambda}(n;q) = \iint\limits_Q e^{-2s\alpha}s\lambda\phi\left|N_t^T * q_t(t)\right|^2 dxdt \leq C\iint\limits_Q e^{-2s\alpha}\lambda(s\phi)^{-1}|q_t|^2 dxdt, \tag{14}$$

where C depends on Ω,ω,t_0,t_1,T, and n. The similar estimates holds true for $M_{Q,s,\lambda}(m_t;q)$ and $N_{Q,s,\lambda}(n_t;q)$.

Indeed one can obtain a sharp estimate for the weight functions (used above) as follows: Following certain standard analysis used in [5], we obtain

$$e^{s\alpha}\phi \leq C(\Omega,\omega)(t(t-T))^{-1}e^{-s\tilde{\alpha}/t(t-T)} \leq 4T^{-2}e^{-\sigma(\Omega,\omega)sT^{-2}},$$

where $\widetilde{\alpha} = e^{2\lambda\Psi}-e^{\lambda\psi}$ and $\sigma = 4\min\limits_{x\in\Omega}\widetilde{\alpha}$ for $s \geq s_0 = \max(\tilde{s}_0,(\sigma(\Omega,\omega))^{-1}T^2)$.

In order to complete the theorem, it remains to obtain an estimate for the terms involving first order derivative in time and second in space variable. To obtain this, first of all multiplying (5) by $e^{\sqrt{-2s\alpha}}\lambda\sqrt{(s\phi)^{-1}}$, squaring and then integrating on Q, we get

$$\begin{aligned} \widehat{L}_{Q,s,\lambda}(q) &= \iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2|g|^2 dxdt + 2(D+E)+2(F+G)+2H \\ &\quad -2\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2 q_t\Delta q dxdt, \end{aligned} \tag{15}$$

where

$$\begin{aligned}
\widehat{L}_{Q,s,\lambda}(q) &= \iint_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2\big(|q_t|^2+|\Delta q|^2+\big|M_t^T*\Delta q(t)\big|^2+\big|N_t^T*q_t(t)\big|^2\big)dxdt,\\
(D+E) &= -\iint_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2 q_t\big(M_t^T*\Delta q(t)+N_t^T*q_t(t)\big)dxdt,\\
(F+G) &= -\iint_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2\Delta q\big(M_t^T*\Delta q(t)+N_t^T*q_t(t)\big)dxdt,\\
H &= -\iint_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2\big(M_t^T*\Delta q(t)\big)\big(N_t^T*q_t(t)\big)dxdt.
\end{aligned}$$

Now we have the following estimates by choosing the constants carefully and applying Young's inequality followed by Green's theorem and integration by parts. Integrating by parts with respect to time in $D+E$, we obtain

$$\begin{aligned}
2(D+E) &= -\iint_Q e^{-2s\alpha}\lambda^2(4\alpha_t\phi^{-1}+2s^{-1}\phi^{-2}\phi_t)q\big(M_t^T*\Delta q(t)+N_t^T*q_t(t)\big)dxdt\\
&\quad +2\iint_Q e^{-2s\alpha}\lambda^2(s\phi)^{-1}q\Big(\int_t^T m_t(\tau,t)\Delta q(\tau)d\tau+\int_t^T n_t(\tau,t)q_\tau(\tau)d\tau\Big)dxdt\\
&= D_1+D_2, \qquad (16)
\end{aligned}$$

where we used the assumption $m(t,t)=n(t,t)=0$. Since we observe that

$$\begin{aligned}
D_1 \leq &\iint_Q e^{-2s\alpha}s^3\lambda^4\phi^3|q|^2dxdt\\
&+\frac{1}{2}\iint_Q e^{-2s\alpha}\lambda^2(s\phi)^{-1}\big(\big|M_t^T*\Delta q(t)\big|^2+\big|N_t^T*q_t(t)\big|^2\big)dxdt
\end{aligned}$$

for any $\lambda\geq 1$ and $s\geq C(\Omega,\omega)(T+T\sqrt{T})$. The integral D_2 can be bounded by

$$\begin{aligned}
D_2 \leq &\iint_Q e^{-2s\alpha}s^3\lambda^3\phi^3|q|^2dxdt+\iint_Q e^{-2s\alpha}\lambda(s\phi)^{-1}\Big(\Big|\int_t^T m_t(\tau,t)\Delta q(\tau)d\tau\Big|^2\\
&+\Big|\int_t^T n_t(\tau,t)q_\tau(\tau)d\tau\Big|^2\Big)dxdt = D_{21}+D_{22} \qquad (17)
\end{aligned}$$

for $s \geq C(\Omega,\omega)T^2$. Computation similar to (13) gives further that,

$$\begin{aligned}
D_{22} \quad &\leq \quad \iint\limits_Q e^{-2s\alpha}\lambda(s\phi)^{-1}\Big[\Big(\int_{t_0}^{t_1}|m_t(\tau,t)|^2e^{2s\alpha}\phi d\tau\Big)\Big(\int_{t_0}^{t_1}e^{-2s\alpha}\phi^{-1}|\Delta q(\tau)|^2d\tau\Big)\\
&\quad +\Big(\int_{t_0}^{t_1}|n_t(\tau,t)|^2e^{2s\alpha}\phi d\tau\Big)\Big(\int_{t_0}^{t_1}e^{-2s\alpha}\phi^{-1}|q_\tau(\tau)|^2d\tau\Big)\Big]dxdt\\
&\leq \quad C\|m_t\|^2_{L^\infty}\iint\limits_Q e^{-2s\alpha}\lambda(s\phi)^{-1}|\Delta q|^2dxdt+C\|n_t\|^2_{L^\infty}\iint\limits_Q e^{-2s\alpha}\lambda(s\phi)^{-1}|q_t|^2dxdt.
\end{aligned}$$

Here we abserve that for any $\lambda \geq \lambda_0$ sufficiently large, the last two integrals can be absorbed in $\widehat{L}_{Q,s,\lambda}(q)$. Now the simple calculation using Green's formula yields

$$\begin{aligned}
-2\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2q_t\Delta q dxdt &= \iint\limits_Q e^{-2s\alpha}\lambda^3(4-2(s\phi)^{-1})q_t(\nabla\psi\cdot\nabla q)dxdt\\
&\quad +\iint\limits_Q e^{-2s\alpha}\lambda^2(s^{-1}\phi^{-2}\phi_t+2\phi^{-1}\alpha_t)|\nabla q|^2dxdt\\
&\leq \quad \frac{1}{4}\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2|q_t|^2dxdt+\iint\limits_Q e^{-2s\alpha}s\lambda^2\phi|\nabla q|^2dxdt, \qquad (18)
\end{aligned}$$

for any $s \geq C(\Omega,\omega)(T+T^2+T\sqrt{T})$. Since we have chosen (if necessarily by normalizing) that $\|\nabla\psi\|_{C(\bar{\Omega})} \leq 1/\lambda$ and used the fact that $\alpha(0)=\alpha(T)=+\infty$. Moreover, we have

$$2H \leq \iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2\Big(\Big|M_0^T * \Delta q(t)\Big|^2+\Big|N_0^T * q_t(t)\Big|^2\Big)dxdt, \qquad (19)$$

and

$$\begin{aligned}
2(F+G) \quad &\leq \quad \frac{1}{4}\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2|\Delta q|^2dxdt\\
&\quad +8\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2\Big(\Big|M_0^T * \Delta q(t)\Big|^2+\Big|N_0^T * q_t(t)\Big|^2\Big)dxdt. \qquad (20)
\end{aligned}$$

Proceeding calculations similar to (13) and (14), we note that the integrals in (19) and the last integral in (20) can further be estimated as

$$C\|m\|^2_{L^\infty}\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2|\Delta q|^2dxdt+C\|n\|^2_{L^\infty}\iint\limits_Q e^{-2s\alpha}(s\phi)^{-1}\lambda^2|q_t|^2dxdt. \qquad (21)$$

Consequently, if $C\|m\|^2_{L^\infty} \leq \frac{1}{4}$ and $C\|n\|^2_{L^\infty} \leq \frac{1}{4}$, the estimations (16)-(21) yield

$$\widehat{L}_{Q,s,\lambda}(q) \leq C\Big(\iint\limits_{Q} e^{-2s\alpha}|g|^2 dxdt + \widetilde{L}_{Q,s,\lambda}(q)\Big). \tag{22}$$

Eventually, making use of the estimations (13), (14) and choosing $\lambda \geq \lambda_0, s \geq s_0$ large enough, recall that the powers of λ in $\widehat{L}_{Q,s,\lambda}(q)$, dominates the powers in (13),(14)), we get

$$\begin{aligned} &\widehat{L}_{Q,s,\lambda}(q) + \widetilde{L}_{Q,s,\lambda}(q) \\ \leq \quad & C\Big(\iint\limits_{Q} e^{-2s\alpha}|g|^2 dxdt + \iint\limits_{(0,T)\times\omega} e^{-2s\alpha}(s\phi)^3\lambda^4|q|^2 dxdt\Big). \end{aligned} \tag{23}$$

This completes the proof of the theorem.

Remark 1. *Smallness condition on the memory kernels $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$ imposed in the estimate (22) is indeed necessary to arrive at such an estimate as the integral involves the second spatial derivative as the absorption is not possible otherwise. In practice the memory kernels are exponential functions (with negative exponents, in general) and so the assumption is valid for appropriate weights.*

An important consequence of Theorem 1 is the following observability estimate. The proof of this estimate is similar to that of the estimate derived for various problems in Fursikov et al [7] and Fernandez-Cara et al [5]. This estimate essentially gives the unique continuation property for the solutions of the system (5), precisely, $q = 0$ in $(0,T)\times\omega$ implies $q \equiv 0$ in $(0,T)\times\Omega$; in particular $q(0) = 0$ in Ω. Now we state the observability inequality for the adjoint system (5).

Corollary 1. *Under the assumptions of theorem 1, there exists a positive constant W depending on Ω, ω, m, n and T such that*

$$\|q(0)\|^2_{L^2(\Omega)} \leq W(\Omega,\omega,T)\Big(\iint\limits_{(0,T)\times\omega} |q|^2 dxdt + \iint\limits_{Q} |g|^2 dxdt\Big) \tag{24}$$

where $W(\cdot) = \exp\big[C(1+\frac{1}{T}+\frac{1}{\sqrt{T}}+T+T^2+T^2(\|m\|^2_{L^\infty}+\|n\|^2_{L^\infty})\big]$ and q is the weak solution of the problem (5).

Proof. Let q be the solution of (5) and $g \in L^2(Q)$. We shall first prove the variant of the inequality (24), namely,

$$\|q(0)\|^2_{L^2(\Omega)} \leq W^*(T)\Big(\iint\limits_{(T/4,3T/4)\times\Omega} |q|^2 dxdt + \iint\limits_{Q} |g|^2 dxdt$$

$$+\iint_{(t_0,t_1)\times\Omega} e^{-2s\alpha}(s\phi)^{-1}(|\Delta q|^2+|q_t|^2)dxdt\Big) \tag{25}$$

where $W^*(\cdot)=\exp[C(\frac{1}{T}+T+T^2(\|m\|^2_{L^\infty}+\|n\|^2_{L^\infty})]$. Multiplying (5) by q and integrating on Ω, we get

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |q|^2dx+\int_\Omega |\nabla q|^2dx \le \frac{3}{2}\int_\Omega |q|^2dx+\frac{1}{2}\int_\Omega \big(|g|^2+\left|M_t^T*\Delta q(t)\right|^2+\left|N_t^T*q_t(t)\right|^2\big)dx.$$

It follows that

$$-\frac{d}{dt}\Big(\exp[3t]\int_\Omega |q|^2dx\Big)\le \exp[3t]\int_\Omega \big(|g|^2+\left|M_t^T*\Delta q(t)\right|^2+\left|N_t^T*q_t(t)\right|^2\big)dx. \tag{26}$$

Integrating (26) with respect to time in $0\le t\le T/4$, we have

$$\int_\Omega |q(0)|^2dx \le \exp[3T/4]\int_\Omega |q(T/4,x)|^2dx$$
$$+\exp[3T]\int_0^{T/4}\int_\Omega \big(|g|^2+\left|M_t^T*\Delta q(t)\right|^2+\left|N_t^T*q_t(t)\right|^2\big)dxd\tau.$$

Again integrating (26) from $T/4$ to t, we get

$$\exp[3T/4]\int_\Omega |q(T/4,x)|^2dx\le \exp[3T]\int_\Omega |q|^2dx$$
$$+\exp[3T]\int_{T/4}^{t}\int_\Omega \big(|g|^2+\left|M_t^T*\Delta q(t)\right|^2+\left|N_t^T*q_t(t)\right|^2\big)dxd\tau$$

for all $t\in\big[T/4,3T/4\big]$. Thus we have

$$\int_\Omega |q(0)|^2dx\le C\Big(\int_\Omega |q|^2dx+\int_0^t\int_\Omega \big(|g|^2+\left|M_t^T*\Delta q(t)\right|^2+\left|N_t^T*q_t(t)\right|^2\big)dxd\tau\Big), \tag{27}$$

where $C=\exp[3T]$. By the assumption on the kernel, we have

$$\int_0^t\int_\Omega \left|M_t^T*\Delta q(t)\right|^2dxd\tau$$
$$\le \int_0^T\int_\Omega\Big(\int_{t_0}^{t_1}|m(\tau,t)|^2e^{s(1+2\alpha(\tau))}\phi(\tau)d\tau\Big)\Big(\int_{t_0}^{t_1}e^{-2s\alpha(\tau)}s^{-1}\phi^{-1}(\tau)|\Delta q(\tau)|^2d\tau\Big)dxdt$$
$$\le CT\|m\|^2_{L^\infty}\iint_{(t_0,t_1)\times\Omega}e^{-2s\alpha}(s\phi)^{-1}|\Delta q|^2dxdt. \tag{28}$$

Now estimating the integral $\int_0^t \int_\Omega \left| N_t^T * q_t(t) \right|^2 dx d\tau$ similar to the above and substituting the preceding estimates into (27) and integrating the resulting inequality with respect to time in $(T/4, 3T/4)$, one can obtain the inequality (25). To complete the proof it suffices to obtain an estimate for the right hand side integrals of (25) in terms of the L^2 integral of q over $(0,T) \times \omega$. From the Carleman estimate for the adjoint system (5), we obtain

$$L_{Q,s,1}(q) \quad \leq \quad C\Big(\iint\limits_Q e^{-2s\alpha} |g|^2 dx dt + \iint\limits_{(0,T)\times\omega} e^{-2s\alpha} s^3 \phi^3 |q|^2 dx dt \Big). \tag{29}$$

Here one can easily verify the following weight function estimates using certain standard analysis (see [5]):

$$e^{-2s\alpha}\phi^3 \quad \leq \quad C(\Omega,\omega) \frac{1}{(t(T-t))^3} e^{2s\tilde{\alpha}/t(T-t)} \leq C(\Omega,\omega) \Big(\frac{2}{T}\Big)^6 e^{-\sigma(\Omega,\omega) s T^{-2}} \quad \forall \quad (t,x) \in \bar{Q},$$

provided $s \geq s_1 = \max(s_0, 3(\sigma(\Omega,\omega))^{-1} T^2)$, where the constant $\sigma(\Omega,\omega) = 8 \min \widetilde{\alpha}$. If we look at the constants s_0 and s_1, then we get

$$s_1 \leq s_2 = C(\Omega,\omega) \big[T + T^2 + T\sqrt{T} + T^4 \big].$$

For $s \geq s_2$, we have

$$e^{-2s\alpha}\phi^3 \geq C(\Omega,\omega) \Big(\frac{16}{3T^2} \Big)^3 e^{-C(\Omega,\omega) s T^{-2}} \quad \forall \quad (t,x) \in [T/4, 3T/4] \times \bar{\Omega}.$$

Let us fix the constant $s = s_2$ and making use of the above weight function estimates, and from (29), we deduce the following estimate

$$\begin{aligned} \iint\limits_{(T/4,3T/4)\times\Omega} |q|^2 dx dt + \iint\limits_{(t_0,t_1)\times\Omega} e^{-2s\alpha} (s\phi)^{-1} (|\Delta q|^2 + |q_t|^2) dx dt \\ \leq \widetilde{W}(\cdot) \Big(\iint\limits_{(0,T)\times\omega} |q|^2 dx dt + \iint\limits_Q |g|^2 dx dt \Big), \end{aligned}$$

where $\widetilde{W}(\cdot) = \exp\big[C(1 + \frac{1}{T} + \frac{1}{\sqrt{T}} + T^2) \big]$. Coupling the above estimate with (25), one can obtain the observability estimate (24).

3. Controllability Results

In this section, we prove a null controllability result for the problem stated in (4). We shall obtain a solution to the global controllability problem for the equation (4) as a limit of an approximation process with the aid of certain suitably defined optimal control problem. To derive the estimate, we use the maximum principle and the observability inequality which is derived in the previous section for the dual problem (5). We first obtain an explicit bound for the weak solution of the system

$$\left.\begin{array}{l} y_t - \Delta y - M_0^t * \Delta y(t) + (N_0^t * y(t))_t = f \text{ in } Q \\ y(0,x) = y_0(x) \text{ in } \Omega \\ y(t,x) = 0 \text{ on } \Sigma, \end{array}\right\} \tag{30}$$

where $f \in L^2(Q), y_0 \in H_0^1(\Omega)$ are given. The above problem has a unique solution $y \in L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))\cap H^1([0,T];L^2(\Omega))$ whenever $y_0 \in H_0^1(\Omega)$. The existence and uniqueness of a solution to this problem is well known, see for example [9]. The following proposition gives an *a priori* estimate for the solution of the system (30).

Proposition 1. *Let $f \in L^2(Q)$ and $y_0 \in H_0^1(\Omega)$ be given. Then the weak solution $y \in H^{2,1}(Q)$ of the problem (30) satisfies the estimate*

$$\|y\|^2_{H^{2,1}(Q)} \le V(\cdot)\big(\|y_0\|^2_{H_0^1(\Omega)} + \|f\|^2_{L^2(Q)}\big), \tag{31}$$

where $V(\cdot) = \exp[C(1 + T + \|n_t\|_{L^\infty} + T^2(\|n\|^2_{L^\infty} + \|n_t\|^2_{L^\infty} + \|n_{tt}\|^2_{L^\infty}))]$.

Proof. The proof follows the standard technique. First multiplying (30) by y and integrating on Ω, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |y|^2 dx + \int_\Omega |\nabla y|^2 dx \le \frac{3}{2}\int_\Omega |y|^2 dx + \frac{1}{2}\int_\Omega \big(|f|^2 + \big|M_0^t * \Delta y(t)\big|^2 + \big|(N_0^t * y(t))_t\big|^2\big) dx.$$

Applying the differential version of Gronwall's inequality in the interval 0 to t with $0 \le t \le t_1$ for some fixed $t_1 \in (0,T)$, we have

$$\begin{aligned} \iint\limits_{(0,t_1)\times\Omega} |\nabla y|^2 dx dt + \int_\Omega |y(t_1)|^2 dx \le \exp[3t_1]\Big(\int_\Omega |y_0|^2 dx \\ + \iint\limits_{(0,t_1)\times\Omega} \big(|f|^2 + \big|M_0^t * \Delta y(t)\big|^2 + \big|(N_0^t * y(t))_t\big|^2\big) dx dt\Big). \end{aligned} \tag{32}$$

Squaring both sides of the equation (30), and integrating on Ω, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\nabla y|^2 + \int_\Omega \big(|y_t|^2 + |\Delta y|^2 + \big|M_0^t * \Delta y(t)\big|^2 + \big|(N_0^t * y(t))_t\big|^2\big) dx = \|f\|^2_{L^2(\Omega)} \\ +2\int_\Omega \Delta y\big((N_0^t * y(t))_t - M_0^t * \Delta y(t)\big) dx - 2\int_\Omega y_t\big((N_0^t * y(t))_t - M_0^t * \Delta y(t)\big) dx \\ +2\int_\Omega \big(M_0^t * \Delta y(t)\big)(N_0^t * y(t))_t dx = \sum_{i=1}^4 I_i \end{aligned} \tag{33}$$

for all $t \in (0, T)$. Using Cauchy inequality one can easily see that

$$\int_0^{t_1} I_2 dt \le \frac{1}{4} \iint\limits_{(0,t_1)\times\Omega} |\Delta y|^2 dxdt + 8 \iint\limits_{(0,t_1)\times\Omega} \Big(\big|M_0^t * \Delta y(t)\big|^2 + \Big|\int_0^t n_t(t,\tau)y(\tau)d\tau\Big|^2\Big)dxdt. \quad (34)$$

Applying Hölder's inequality, the last integral can further be estimated as

$$\|m\|_{L^\infty}^2 t_1^2 \iint\limits_{(0,t_1)\times\Omega} |\Delta y|^2 dxdt + \|n_t\|_{L^\infty}^2 t_1^2 \iint\limits_{(0,t_1)\times\Omega} |y|^2 dxdt.$$

Integration by parts in time together with the assumptions on the kernel yields,

$$\begin{aligned}\int_0^{t_1} I_3 dt &= 2 \iint\limits_{(0,t_1)\times\Omega} y\Big(\int_0^t n_{tt}(t,\tau)y(\tau)d\tau + n_t(t,t)y(t)\Big)dxdt \\ &\quad -2 \iint\limits_{(0,t_1)\times\Omega} y\Big(\int_0^t m_t(t,\tau)\Delta y(\tau)d\tau\Big)dxdt = I_{31} + I_{32}. \end{aligned} \quad (35)$$

Since using Young's and Hölder's inequality, we further obtain that

$$I_{31} \le (1 + 2\|n_t\|_{L^\infty}) \iint\limits_{(0,t_1)\times\Omega} |y|^2 dxdt + \|n_{tt}\|_{L^\infty}^2 t_1^2 \iint\limits_{(0,t_1)\times\Omega} |y|^2 dxdt$$

and

$$I_{32} \le \eta t_1^2 \|m_t\|_{L^\infty}^2 \iint\limits_{(0,t_1)\times\Omega} |\Delta y|^2 dxdt + \frac{1}{\eta} \iint\limits_{(0,t_1)\times\Omega} |y|^2 dxdt.$$

Finally, we note that the estimation similar to I_{32} yields

$$\int_0^{t_1} I_4 dt \le \eta t_1^2 \|m_t\|_{L^\infty}^2 \iint\limits_{(0,t_1)\times\Omega} |\Delta y|^2 dxdt + \frac{1}{\eta}\|n\|_{L^\infty}^2 t_1^2 \iint\limits_{(0,t_1)\times\Omega} |y|^2 dxdt. \quad (36)$$

If $\|m\|_{L^\infty(0,T)}^2 t_1^2 \le \frac{1}{16}$, integrating (33) in the interval (0,t) and substituting (34)-(36) and using the Poincaré inequality $\int_\Omega |y|^2 dx \le C(\Omega) \int_\Omega |\nabla y|^2 dx$, one can have the following

$$\begin{aligned}&\|y(t_1)\|_{H_0^1(\Omega)}^2 + \iint\limits_{(0,t_1)\times\Omega} \Big(|y_t|^2 + |\Delta y|^2 + \big|M_0^t * \Delta y(t)\big|^2 \\ &\quad + \big|(N_0^t * y(t))_t\big|^2\Big)dxdt \le M(\cdot)\Big(\|y_0\|_{H_0^1(\Omega)}^2 + \iint\limits_{(0,t_1)\times\Omega} |f|^2 dxdt\Big), \end{aligned} \quad (37)$$

where $M(\cdot) = \exp[C(1+\|n_t\|_{L^\infty} + t_1^2(\|n\|^2_{L^\infty} + \|n_t\|^2_{L^\infty} + \|n_{tt}\|^2_{L^\infty}))]$. Since we have also chosen that $\eta \leq \frac{1}{16t_1^2\|m_t\|^2_{L^\infty}}$. With the estimate (37) together with (32) and the Sobolev estimate, one can conclude the proof.

Now we prove the main result of this work.

Theorem 2. *Assume that $T > 0$ is fixed and $y_0 \in H_0^1(\Omega)$ is given and the kernels $m(\cdot,\cdot)$ and $n(\cdot,\cdot)$ have support in (t_0, t_1) where $0 < t_0 < t_1 < T$. Then there exists a control $u \in L^2(0,T;L^2(\omega))$ such that the corresponding solution of (4) satisfies*

$$y(T,x) = 0 \ \ a.e. \ \ x \in \Omega.$$

Moreover, the control u can be chosen in such a way that

$$||u||^2_{L^2(0,T;L^2(\omega))} \leq W(\Omega,\omega,T)||y_0||^2_{L^2(\Omega)},$$

where the constant $W(\cdot)$ is explicitly given in (24).

Proof. Let us fix $T > 0$ and $y_0 \in H_0^1(\Omega)$. For every $\epsilon > 0$, let us consider the problem

$$\min\left\{J_\epsilon(u) : u \in L^2(0,T;L^2(\omega))\right\},$$

where the functional J_ϵ is defined by

$$J_\epsilon(u) = \frac{1}{2}\iint\limits_{(0,T)\times\omega} |u|^2 dxdt + \frac{1}{2\epsilon}\int_\Omega |y(T,x)|^2 dx, \tag{38}$$

where y is the solution of (4) associated with the control u. In order to solve this control problem, it is enough to prove that the functional J_ϵ has a unique solution (see Fernandez-Cara et al [5]). Since, J_ϵ is a continuous strictly convex functional in $L^2(Q)$ and coercive, that is,

$$\liminf_{||u||_{L^2((0,T)\times\omega}\to\infty} J_\epsilon(u) = \infty,$$

J_ϵ has a unique solution (u_ϵ, y_ϵ) for every $\epsilon > 0$. Next, we shall obtain the necessary condition for optimality via maximum principle. We can verify that it is characterized by

$$u_\epsilon = -\chi_\omega q_\epsilon \tag{39}$$

where q_ϵ is the solution to the adjoint problem

$$\left.\begin{array}{l} (q_\epsilon)_t + \Delta q_\epsilon + M_0^t * \Delta q_\epsilon(t) + (N_0^t * q_\epsilon(t))_t = 0 \text{ in } Q \\ q_\epsilon(T,x) = \frac{1}{\epsilon} y_\epsilon(T,x) \text{ in } \Omega \\ q_\epsilon(t,x) = 0 \text{ on } \Sigma. \end{array}\right\} \tag{40}$$

Let us put $y = w + \varphi$. If y is the solution of (4) associated with u, and w is the weak solution of the homogeneous problem corresponding to (30), then φ satisfies

$$\left.\begin{array}{l}\varphi_t - \Delta\varphi - M_0^t * \Delta\phi_\epsilon(t) + (N_0^t * \phi_\epsilon(t))_t = \chi_\omega u \text{ in } Q \\ \varphi(0,x) = 0 \text{ in } \Omega \\ \varphi(t,x) = 0 \text{ on } \Sigma.\end{array}\right\} \tag{41}$$

Now the functional J_ϵ is differentiable at the point u. For $u, v \in L^2(0,T;L^2(\omega))$, we obtain

$$\langle J_\epsilon^{'}(u_\epsilon), v\rangle_{L^2(Q)} = \iint\limits_{(0,T)\times\omega} u_\epsilon v\,dx\,dt + \frac{1}{\epsilon}\int_\Omega y(T)\varphi(T)dx, \tag{42}$$

where φ is the solution to (41) associated with the control v. For the pair (u_ϵ, y_ϵ) to be a unique solution of J_ϵ, we must have

$$\langle J_\epsilon^{'}(u_\epsilon), v\rangle_{L^2(Q)} = 0.$$

The duality between φ and q gives the following

$$\iint\limits_{(0,T)\times\omega} q_\epsilon v\,dx\,dt = \int_\Omega q_\epsilon(T)\varphi(T)dx = \frac{1}{\epsilon}\int_\Omega y_\epsilon(T)\varphi(T)dx. \tag{43}$$

In view of (42) and (43), we can identify $u_\epsilon = -\chi_\omega q_\epsilon$, the optimal control stated in (39). Next we shall show that (u_ϵ, y_ϵ) converges along a subsequence of ϵ in a certain topology. In order to prove this, we need a suitable estimate for (u_ϵ, y_ϵ). In particular, we get L^2 estimate for u_ϵ. Multiplying (4) by q_ϵ(replace y by y_ϵ) and (40) by y_ϵ and adding and then integrating on $(0,T)\times\Omega$, we have

$$\int_\Omega q_\epsilon(T)y_\epsilon(T)dx = \int_\Omega y_0(x)q_\epsilon(0)dx + \iint\limits_{(0,T)\times\omega} u_\epsilon q_\epsilon\,dx\,dt.$$

Making use of the optimality condition $q_\epsilon(T,x) = \frac{1}{\epsilon}y_\epsilon(T,x)$ and the Young's inequality, we obtain

$$\iint\limits_{(0,T)\times\omega} |u_\epsilon|^2 dx\,dt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx \leq \frac{\eta}{2}||y_0||^2_{L^2(\Omega)} + \frac{1}{2\eta}||q_\epsilon(0)||^2_{L^2(\Omega)} \forall \;\; \eta > 0.$$

Using Corollary 1, we can choose η appropriately, for instance $\eta = W(\Omega,\omega,T)$; then we have

$$\frac{1}{2}\iint\limits_{(0,T)\times\omega} |u_\epsilon|^2 dx\,dt + \frac{1}{\epsilon}\int_\Omega |y_\epsilon(T,x)|^2 dx \leq W(\Omega,\omega,T)||y_0||^2_{L^2(\Omega)}, \tag{44}$$

where the constant $W(\cdot)$ is given by (24). The Proposition 1 and the estimate (44) allow us to pass to the weak limit in (4) (after replacing (u,y) by (u_ϵ, y_ϵ)) as $\epsilon \to 0$, which gives the solution of the null controllability problem (4). Since u_ϵ is bounded in $L^2(0,T;L^2(\omega))$, there exists a subsequence of ϵ still indexed by ϵ such that $u_\epsilon \to u$ weakly in $L^2(0,T;L^2(\omega))$, $y_\epsilon \to y$ weakly in $H^{2,1}(Q)$ as $\epsilon \to 0$. From (44) and Fatou's lemma for any constant C independent of ϵ, we have

$$||y(T,x)||^2_{L^2(\Omega)} \leq \liminf_{\epsilon \to 0} \int_\Omega |y_\epsilon(T,x)|^2 dx \leq \liminf_{\epsilon \to 0} C\epsilon = 0.$$

It follows that

$$y(T,x) \equiv 0 \ \ a.e. \ \ x \in \ \Omega.$$

The estimate for the control u follows from (44) and the proof is thus completed.

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