EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 5, 2020, 1097-1109
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Special Issue Dedicated to Professor Hari M. Srivastava On the Occasion of his 80th Birthday

On Tosha-degree of an edge in a graph

R. Rajendra [1], P. Siva Kota Reddy [2,*]

[1] *Department of Mathematics, Mangalore University, Mangalagangothri, Karnataka, India*
[2] *Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru, Karnataka, India*

Abstract. In an earlier paper, we have introduced the Tosha-degree of an edge in a graph without multiple edges and studied some properties. In this paper, we extend the definition of Tosha-degree of an edge in a graph in which multiple edges are allowed. Also, we introduce the concepts - zero edges in a graph, T-line graph of a multigraph, Tosha-adjacency matrix, Tosha-energy, edge-adjacency matrix and edge energy of a graph G and obtain some results.

2020 Mathematics Subject Classifications: 05Cxx, 05C07, 05C50.

Key Words and Phrases: Adjacency matrix, degree of a vertex, energy, line graph, Tosha-degree of an edge

1. Introduction

For standard terminology and notion in graphs and matrices, we refer the reader to the text-books of Harary [2] and Bapat [1]. The non-standard will be given in this paper as and when required.

Throughout this paper, $G = (V, E)$ denotes a graph (finite and undirected) and $V = V(G)$ and $E = E(G)$ denote vertex set and edge set of G, respectively. The degree of a vertex $v \in V(G)$, denoted by $d(v)$ or $d_G(v)$, is the number of edges incident on v, with self-loops counted twice. A vertex of degree one is a pendant vertex and an edge incident onto a pendant vertex is a pendant edge. A graph G is r-regular if every vertex

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i5.3710

Email addresses: `rrajendrar@gmail.com` (R. Rajendra), `pskreddy@jssstuniv.in` (P. S. K. Reddy)

of G has degree r. The minimum degree $\delta(G)$ of a graph G is the minimum degree among all the vertices of G and the maximum degree $\Delta(G)$ of G is the maximum degree among all the vertices of G.

Two non-distinct edges in a graph are adjacent if they are incident on a common vertex. We consider that an edge in a graph is not adjacent to itself. The letters k, l, m, n, and r denote positive integers or zero.

The line graph $L(G)$ of a simple graph with at least one edge is the graph (W, F), where there is a one-to-one correspondence ϕ from E to W such that there is an edge between $\phi(\alpha)$ and $\phi(\beta)$ if and only if the edges α and β are adjacent. We identify the set W by E.

The adjacency matrix of a graph G with n vertices is denoted by $A(G)$. If $A(G)$ is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$, the energy of G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In our earlier paper [4], we have introduced the Tosha-degree of an edge in a graph without multiple edges, Rajendra-Reddy index of a graph and Tosha-degree equivalence graph of a graph, and studied some properties. In this paper, we define Tosha-degree of an edge in a graph in which multiple edges are allowed. The aim of this paper is to introduce the concepts: zero edges in a graph, T-line graph of a multigraph, Tosha-adjacency matrix, Tosha-energy, edge-adjacency matrix and edge energy of a graph G and obtain some results.

A *signed graph* is an ordered pair $\Sigma = (G, \sigma)$, where $G = (V, E)$ is a graph called the *underlying graph* of Σ and $\sigma : E \to \{+, -\}$ is a function. A *marking* of Σ is a function $\mu : V(G) \to \{+, -\}$.

In [4], we have also defined the Tosha-degree equivalence graph of a graph which is motivated us to extend this notion to signed graphs as follows: The *Tosha-degree equivalence signed graph* (See [3]) of a signed graph $\Sigma = (G, \sigma)$ as a signed graph $T(\Sigma) = (T(G), \sigma')$, where $T(G)$ is the underlying graph of $T(\Sigma)$ is the Tosha-degree equivalence graph of G, where for any edge e_1e_2 in $T(\Sigma)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. Hence, we shall call a given signed graph Σ as *Tosha-degree equivalence signed graph* if it is isomorphic to the Tosha-degree equivalence signed graph $T(\Sigma')$ of some sigraph Σ' (See [3]). In [3], we offered a switching equivalence characterization of signed graphs that are switching equivalent to Tosha-degree equivalence signed graphs and k^{th} iterated Tosha-degree equivalence signed graphs. Further, we have presented the structural characterization of Tosha-degree equivalence signed graphs.

2. Tosha-degree of an edge in a graph

In [4], R. Rajendra and P.S.K. Reddy have defined the Tosha-degree of an edge in a graph without multiple edges as follows: *The Tosha-degree of an edge α in a graph G without multiple edges, denoted by $T(\alpha)$, is the number of edges adjacent to α in G, with self-loops counted twice.* Here we allow graphs with multiple edges (multi-graphs) and the new definition of the Tosha-degree of an edge in a graph (with or without multiple edges) is given below:

Definition 1. *Let α be an edge in a graph G. The Tosha-degree of α, denoted by $T(\alpha)$ or $T_G(\alpha)$, is the number of edges adjacent to α in G, where self-loops and edges parallel to α are counted twice.*

By the Definition 1, for any edge α in a graph G, $T(\alpha) \geq 0$.

Definition 2. *A graph G is said to be a Tosha-regular graph if all edges are of equal Tosha-degree. We say that G is l-Tosha-regular, if $T(\alpha) = l$, for all $\alpha \in E(G)$.*

The following proposition is proved for graphs without parallel edges in [4]. This result is true for graphs having parallel edges also with respect to the Definition 1.

Proposition 1. *[4] Let α be an edge in a graph G with end vertices u and v.*

(i) If α is not a self-loop, then

$$T(\alpha) = d(u) + d(v) - 2 \tag{1}$$

(ii) If α is a self-loop, then $u = v$ and

$$T(\alpha) = d(u) - 2 \tag{2}$$

Proof. The proof follows by the definition 1, and the definition of degree of a vertex.

Observation: By the Proposition 1, for an edge α in a graph G, it follows that,

(a) if α is not a self-loop, then

$$2(\delta(G) - 1) \leq T(\alpha) \leq 2(\Delta(G) - 1);$$

(b) if α is a self-loop, then

$$\delta(G) - 2 \leq T(\alpha) \leq \Delta(G) - 2.$$

Corollary 1. *[4] If G is a simple graph and α is an edge in G, then*

$$T(\alpha) = d_{L(G)}(\alpha) \tag{3}$$

where $d_{L(G)}(\alpha)$ is the degree of α as a vertex in the line graph $L(G)$ of G.

Proof. Follows from the definition of $L(G)$ and Eq.(1).

Corollary 2. *In a simple graph G, the number of odd Tosha-degree edges is even.*

Proof. In any graph the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph $L(G)$ of G is even. Since the vertices in $L(G)$ are corresponding to the edges in G, by Eq.(3) it follows that, the number of odd Tosha-degree edges in G is even.

Remark 1. *The Corollary 2 may not be true for the graphs having self-loops. There are graphs with odd number of edges and all edges are of odd Tosha-degree. For eg., consider the graph G given in Figure 1. The graph G has three edges namely, α, β and γ. We observe that $T(\alpha) = 1$, $T(\beta) = 3$, $T(\gamma) = 1$ and hence all the edges in G are of odd Tosha-degree.*

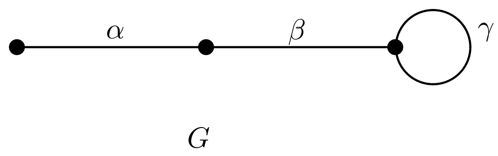

Figure 1: Graph containing odd number of odd Tosha-degree edges.

Observation: Let α be an edge in a simple graph G. The addition of a parallel edge β to α gives a count plus two to the Tosha-degree of α and to the edges parallel to α, and a count plus one to non-parallel edges adjacent to α in the new graph $G+\beta$ and Tosha-degrees of all other edges are unaltered $G+\beta$. Hence an odd (even) Thosha-degree edge γ remains odd (even) Tosha-degree in $G+\beta$, if it is not adjacent to α or $\gamma = \alpha$ in G.

Corollary 3. *If α and β are parallel edges in a graph G, then $T(\alpha) = T(\beta)$ in G.*

Proof. The proof follows by Proposition 1.

2.1. T-line graph of a multigraph

Definition 3. *A multigraph is a graph in which multiple edges (parallel edges) are permitted between any pair of vertices. All multigraphs in this paper are loopless.*

We say that two distinct edges α and β in a multigraph G are k-adjacent if they are adjacent and share k end vertices.

We say that two distinct vertices u and v in a multigraph G are r-adjacent if they are adjacent and the number of edges between them is r (i.e., r edges have common end vertices u and v).

From the Definition 3, it follows that, when two distinct edges α and β are k-adjacent in a multigraph G, we have,

$$k = \begin{cases} 1, & \text{if } \alpha \text{ and } \beta \text{ are not parallel;} \\ 2, & \text{if } \alpha \text{ and } \beta \text{ are parallel.} \end{cases}.$$

Definition 4. *Given a multigraph $G = (V, E)$, the T-line graph of G denoted by $TL(G)$, is a graph with vertex set E; two distinct vertices α and β are k-adjacent in $TL(G)$ if and only if their corresponding edges in G are k-adjacent.*

From the Definition 4, it is clear that,

(a) $TL(G)$ is also a multigraph,

(b) if G is a simple graph, then $TL(G)$ is nothing but $L(G)$.

Proposition 2. *Let G be a multigraph and α be a vertex in $TL(G)$ (so α is an edge in G). Then*

$$d_{TL(G)}(\alpha) = d_G(u) + d_G(v) - 2 = T_G(\alpha) \tag{4}$$

where u and v are end vertices of α in G.

Proof. Proof follows by the definitions 1, 3 and 4, and propositions 1 and 2.

Corollary 4. *In a multigraph G, the number of odd Tosha-degree edges is even.*

Proof. In any graph(multigraph) the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph $TL(G)$ of G is even. Since the vertices in $TL(G)$ are corresponding to the edges in G, by Eq.(4), the number of odd Tosha-degree edges in G is even.

3. Zero edges in a graph

Definition 5. *In a graph G, an edge α is said to be a zero edge if its Tosha degree is zero i.e., $T(\alpha) = 0$.*

Observations: The edge in the complete graph K_2 is a zero edge. The self-loop in the graph containing only one vertex and a self-loop attached to that vertex, is a zero edge.

Proposition 3. *A simple connected graph G has a zero edge if and only if $G \cong K_2$.*

Proof. Suppose that G is a simple connected graph having a zero edge, say $\alpha = uv$, where u and v are end vertices of α. Then

$$d(u) + d(v) - 2 = 0 \tag{5}$$

Since G is connected, $d(u) \geq 1$ and $d(v) \geq 1$; from Eq.(5), $d(u) = 1$ and $d(v) = 1$. Therefore, there is no other edge in G incident to u and v. So G has only one edge α. Since G is connected, $G \cong K_2$.

Conversely, if $G \cong K_2$, then clearly G is a simple connected graph having only one edge whose Tosha-degree is zero.

Corollary 5. *A simple connected graph G with two or more edges, has no zero edge. Hence $T(\alpha) \geq 1,\ \forall \alpha \in E(G)$.*

Proof. Follows from Proposition 3.

Corollary 6. *A simple graph G has no zero edge if and only if either $G \ncong K_2$ or no component of G is isomorphic to K_2 or no component of G is of only one vertex with a self-loop.*

Proof. Follows from Proposition 3.

4. Degree colorable graphs

In this section we consider self-loop free graphs (multigraphs).

Definition 6. *A graph G is degree colorable if no two adjacent vertices have the same degree.*

Theorem 1. *If all the edges of a graph G are of odd Tosha-degree, then G is a degree colorable graph with even number of vertices.*

Proof. Suppose that G is a graph in which all the edges are of odd Tosha-degree. By the corollaries 2 and 4, it follows that G has an even number of vertices. Let α be an edge in G with end vertices u and v. Then by Eq.(1) and Eq.(4),

$$T(\alpha) = d(u) + d(v) - 2.$$

Since $T(\alpha)$ is odd, $d(u) \neq d(v)$. Thus, no two adjacent vertices in G have the same degree. Therefore G is a degree colorable graph.

By Theorem 1, the following corollary is immediate.

Corollary 7. *An l-Tosha-regular graph, where l is an odd positive integer, is degree colourable.*

Remark 2. *There are degree colorable non-Tosha-regular graphs with odd number of vertices. The following graph is an example for such graphs, in which the edges are indicated by respective Tosha-degrees.*

5. Tosha-even graphs

Definition 7. *A graph G is said to be Tosha-even if all its edges are of even Tosha-degree.*

We recall the following proposition from [4].

Proposition 4. *[4, Proposition 2.15] If G is an Euler graph, then all edges in G are of even Tosha-degree.*

Corollary 8. *Euler graphs are Tosha-even.*

Proof. Follows from the Proposition 4.

Remark 3. *.The converse of the Corollary 8 is not true in general. There are connected graphs with even number of vertices and all vertices are of odd degree, for instance, K_4. Such graphs are not Euler graphs, but are Tosha-even.*

Proposition 5. *There exist degree colorable Tosha-even graphs that are not Euler graphs.*

Proof. The following graph G (see Figure 3) is an example of a degree colorable Tosha-even graph which is not an Euler graph. In G, the vertices and edges are indicated by their degrees and Tosha-degrees, respectively. We see that all vertices of G are of odd degree and hence G is not an Euler graph. But all edges are of Tosha-even, so G is a Tosha-even graph.

6. Tosha-adjacency matrix of a graph

Definition 8. *If G is a graph with n vertices $v_1, \ldots, v_n$ and no parallel edges. The Tosha-adjacency matrix of the graph G is an $n \times n$ matrix $A_T(G) = (t_{ij})$ defined over the ring of integers such that*

$$t_{ij} = \begin{cases} T(v_iv_j), & \text{if } v_iv_j \in E \\ 0, & \text{otherwise.} \end{cases}$$

Observations:

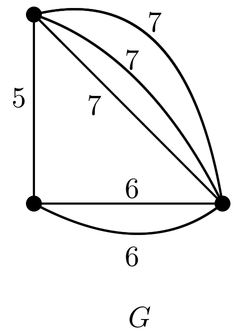

Figure 2: A degree colorable non-Tosha-regular graph with 3 vertices.

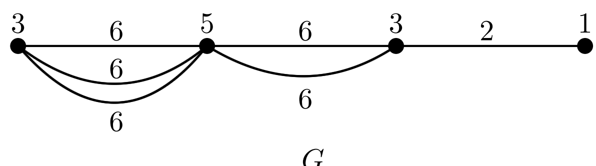

Figure 3: A degree colorable Tosha-even graph which is not an Euler graph.

(i) By the definition of the Tosha-degree of an edge, we have

$$T(v_iv_j) = \begin{cases} d(v_i) + d(v_j) - 2, & \text{if } v_iv_j \in E \text{ and } i \neq j; \\ d(v_i) - 2, & \text{if } v_iv_j \in E \text{ and } i = j; \\ 0, & \text{if } v_iv_j \notin E. \end{cases}$$

Therefore, $t_{ij} = t_{ji}$. Therefore $A_T(G)$ is a real symmetric matrix.

(ii) The entries along the principal diagonal of $A_T(G)$ are all 0s if and only if either G has no self-loops or G has only self loops that are zero edges. Hence if either G has no self-loops or G has only self loops that are zero edges, then $tr(A_T(G)) = 0$. In this case, if $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $A_T(G)$, then

$$\sum_{i=1}^{n} \mu_i = 0.$$

(iii) If G has no zero edges, then the degree of a vertex equals the number of non-zero entries in the corresponding row or column; and the non-zero entry in the ij-th place gives the Tosha-degree of the corresponding edge incident to i-th and j-th vertices.

(iv) For a zero edge free graph G, the adjacency matrix $A(G)$ can be obtained from the Tosha-adjacency matrix $A_T(G)$ by replacing all the non-zero entries by 1s. This is possible because, in a zero edge free graph Tosha-degrees of edges are non-zero. Thus, reconstriction of the graph from the Tosha-adjacency matrix is possible if the given graph has no zero edges.

Throughout this section G denotes a graph with no parallel edges.

Theorem 2. *If a graph G with n vertices is l-Tosha-regular, then*

$$A_T(G) = l \cdot A(G).$$

Proof. Suppose that G is l-Tosha-regular. Then $T(\alpha) = l$, for all $\alpha \in E(G)$. Let $A(G) = (a_{ij})$ and $A_T(G) = (t_{ij})$ be the adjacency matrix and the Tosha-adjacency matrix of G, respectively. Then by the definition of the Tosha-adjacency matrix $A_T(G)$, we have

$$\begin{aligned} t_{ij} &= \begin{cases} l, & \text{if } v_iv_j \in E \\ 0, & \text{otherwise} \end{cases} \\ &= l \cdot a_{ij}. \end{aligned}$$

Therefore, $A_T(G) = l \cdot A(G)$.

Corollary 9. *If a graph G with n vertices is r-regular, then*

$$A_T(G) = 2(r-1)A(G).$$

Proof. If a graph G with n vertices is r-regular, then G is $2(r-1)$-Tosha-regular(by [4, Corollary 2.6]) and hence by Theorem 2, $A_T(G) = 2(r-1)A(G)$.

Corollary 10. *A graph G is 1-Tosha-regular if and only if $A_T(G) = A(G)$.*

Proof. ($\Leftarrow$:) Suppose that for a graph G, $A_T(G) = A(G)$. Then by the definitions of $A_T(G)$ and $A(G)$, it follows that, $T(\alpha) = 1$, $\forall \alpha \in E(G)$. Hence, G is 1-Tosha-regular.
($\Rightarrow$:) Follows by Theorem 2.

7. Tosha-energy of a graph

Definition 9. *Let G be graph with n vertices $v_1, \dots, v_n$ and no parallel edges. Let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of the Tosha-adjacency matrix $A_T(G)$ of G. The Tosha-energy of G, denoted by $\mathcal{E}_T(G)$, is defined as*

$$\mathcal{E}_T(G) = \sum_{i=1}^{n} |\mu_i|. \tag{6}$$

Throughout this section G denotes a graph with no parallel edges.

Proposition 6. *The Tosha-energy of an l-Tosha-regular graph G with n vertices is given by*

$$\mathcal{E}_T(G) = l \cdot \mathcal{E}(G) \tag{7}$$

where $\mathcal{E}(G)$ is the energy of G.

Proof. Le G be an l-Tosha-regular graph with n vertices. Then by the Theorem 2, the Tosha-adjacency matrix of G is

$$A_T(G) = l \cdot A(G) \tag{8}$$

where $A(G)$ is the adjacency matrix of G. For brevity we write A for $A(G)$ and A_T for $A_T(G)$. We consider two cases: (i) When $l > 0$ and (i) When $l = 0$.
Case (i): When $l > 0$. Let μ be an eigenvalue of A_T. From Eq.(8) we have,

$$\begin{aligned} \det(A_T - \mu I) = 0 \iff& \det\big(lA - \mu I\big) = 0 \\ \iff& l^n \det\big(A - \frac{\mu}{l} I\big) = 0 \\ \iff& \det\big(A - \frac{\mu}{l} I\big) = 0. \end{aligned}$$

Therefore, μ is an eigenvalue of A_T if and only if $\frac{\mu}{l}$ is an eigenvalue of A. Let μ_1, μ_2, ..., μ_n be the eigenvalues of the A_T. Then $\frac{\mu_1}{l}$, $\frac{\mu_2}{l}$, ..., $\frac{\mu_n}{l}$ are the eigenvalues of A and the Tosha-energy of G is

$$\begin{aligned}\mathcal{E}_T(G) &= \sum_{i=1}^{n} |\mu_i| \\ &= l \cdot \sum_{i=1}^{n} \left|\frac{\mu_i}{l}\right| \\ &= l \cdot \mathcal{E}(G).\end{aligned}$$

Case (ii): When $l = 0$. From Eq.(7), $A_T = 0$ and so zero is the only eigenvalue of A_T of multiplicity n. In this case, $\mathcal{E}_T(G) = 0 = 0 \cdot \mathcal{E}(G)$.

Corollary 11. *The Tosha-energy of an r-regular graph G with n vertices is given by*

$$\mathcal{E}_T(G) = 2(r-1)\mathcal{E}(G) \tag{9}$$

where $\mathcal{E}(G)$ is the energy of G.

Proof. Let G be an r-regular graph with n vertices. By [4, Corollary 2.6] G is a $2(r-1)$-Tosha-regular graph. Then by Proposition 6, the proof follows.

Corollary 12. *(i) For the complete graph K_n on $n > 1$ vertices,*

$$\mathcal{E}_T(K_n) = 2(n-2)\mathcal{E}(K_n) = 4(n-1)(n-2).$$

(ii) For the cycle graph C_n on $n > 1$ vertices,

$$\mathcal{E}_T(C_n) = 2\mathcal{E}(C_n) = 4\sum_{i=0}^{n-1}\left|\cos\left(\frac{2\pi i}{n}\right)\right|.$$

(iii) For the complete bipartite graph $K_{m,n}$,

$$\mathcal{E}_T(K_{m,n}) = (m+n-2)\mathcal{E}(K_{m,n}) = 2(m+n-2)\sqrt{mn}.$$

Proof. (i) The eigen values of $A(K_n)$ are given below:

$$\begin{matrix}\text{eigen value} \rightarrow \\ \text{multiplicity} \rightarrow\end{matrix} \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$

Therefore

$$\mathcal{E}(K_n) = |n-1| + (n-1)|-1| = 2(n-1).$$

Since K_n is an $(n-1)$-regular graph, from Eq.(9) we have,

$$\mathcal{E}_T(K_n) = 2(n-2)\mathcal{E}(K_n) = 2(n-2) \cdot 2(n-1) = 4(n-1)(n-2).$$

(ii) The eigen values of $A(C_n)$ are

$$2\cos\left(\frac{2\pi i}{n}\right), \quad i = 0, 1, \ldots, n-1.$$

Therefore

$$\mathcal{E}(C_n) = 2\sum_{i=0}^{n-1}\left|\cos\left(\frac{2\pi i}{n}\right)\right|.$$

Since C_n is an 2-regular graph, from Eq.(9) we have,

$$\mathcal{E}_T(C_n) = 2 \cdot \mathcal{E}(C_n) = 4\sum_{i=0}^{n-1}\left|\cos\left(\frac{2\pi i}{n}\right)\right|.$$

(iii) The eigen values of $A(K_n)$ are given below:

$$\begin{matrix} \text{eigen value } \rightarrow \\ \text{multiplicity } \rightarrow \end{matrix} \begin{pmatrix} -\sqrt{mn} & 0 & \sqrt{mn} \\ 1 & n+m-2 & 1 \end{pmatrix}$$

Therefore

$$\mathcal{E}(K_{m,n}) = 2\sqrt{mn}.$$

Since $K_{m,n}$ is an $(m+n-2)$-Tosha-regular graph, from Eq.(8) we have,

$$\mathcal{E}_T(K_{m,n}) = (m+n-2)\mathcal{E}(K_{m,n}) = 2(m+n-2)\sqrt{mn}.$$

Corollary 13. *(i) For the path P_2 of 2 vertices, $\mathcal{E}_T(P_2) = 0$.*

(ii) For the path P_3 of 3 vertices, $\mathcal{E}_T(P_3) = \mathcal{E}(P_3) = 2\sqrt{2}$.

Proof. Since $P_2 = K_{1,1}$ and $P_3 = K_{2,1}$, (i) and (ii) follow immediately from Corollary 12 (iii).

Theorem 3. *Let G be a simple connected graph with at least one edge. Then*

$$A_T(G) = A(G) \Longleftrightarrow G = P_3.$$

Proof. ($\Leftarrow$:) If $G = P_3$, then it has two edges and each of these are of Tosha-degree 1. Therefore, it is 1-Tosha-regular and hence by Theorem 2, $A_T(G) = A(G)$.
($\Rightarrow$:) Suppose that $A_T(G) = A(G)$. Then G is 1-Tosha-regular and hence

$$T(v_iv_j) = 1, \ \forall \ v_iv_j \in E(G)$$

$$\implies d(v_i) + d(v_j) - 2 = 1, \ \forall \ v_iv_j \in E(G)$$
$$\implies d(v_i) = 3 - d(v_j), \ \forall \ v_iv_j \in E(G)$$

Therefore, for any edge α in G with end vertices u and v,

$$d(u) = 3 - d(v) \tag{10}$$

Since G is connected, $d(v) > 0$ and $d(u) > 0$, and from Eq.(10) we have, $d(u) < 3$; which implies

$$d(u) = 1 \text{ or } 2. \tag{11}$$

Let u be an arbitrary vertex in G. Since G is a simple connected graph with at least one edge, u is an end vertex of at least one edge say α. Let v be the other end vertex of α in G. Then by Eq.(10) and Eq.(11), either $d(u) = 1$ and $d(v) = 2$ or $d(u) = 1$ and $d(v) = 2$.

If $d(u) = 1$ and $d(v) = 2$, there is another vertex w adjacent to v and $d(w) = 1$ (by above argument). There are no other vertices adjacent to the vertices u, v and w. So, G is a path with 3 vertices. A similar argument can be used for the case $d(u) = 1$ and $d(v) = 2$, to show that G is P_3.

8. Edge-adjacency matrix and edge-energy of a graph

Definition 10. *We say that two distinct edges α and β in a graph G (where self-loops and parallel edges are allowed) are k-adjacent if they are adjacent and share k end vertices. We consider that an edge in a graph is not adjacent to itself.*

Definition 11. *If G is a graph with m edges $e_1, \ldots, e_m$. The edge-adjacency matrix of the graph G is an $m \times m$ matrix $A_E(G) = (x_{ij})$ defined over the ring of integers such that*

$$x_{ij} = \begin{cases} k, & \text{if } e_i \text{ and } e_j \text{ are } k\text{−adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Observations:

(i) $A_E(G)$ is a $\{0, 1, 2\}$-matrix and it is real symmetric. If G is a simple graph, then $A_E(G)$ is a $\{0, 1\}$-matrix.

(ii) The entries along the principal diagonal of $A_E(G)$ are all 0s. Therefore, $tr(A_E(G)) = 0$. Hence if $\nu_1, \nu_2, \ldots, \nu_m$ are the eigenvalues of $A_E(G)$, then

$$\sum_{i=1}^{m} \nu_i = 0.$$

(iii) If G has no self-loops, then the Tosha-degree of an edge equals the sum of entries in the corresponding row or column of $A_E(G)$.

Proposition 7. *For a multigraph G, the edge-adjacency matrix of G is the adjacency matrix of the T-line graph of G. That is,*

$$A_E(G) = A(TL(G)).$$

Proof. Follows by the definitions 4 and 11.

Corollary 14. *For a simple graph G, the edge-adjacency matrix of G is the adjacency matrix of the line graph of G. That is,*

$$A_E(G) = A(L(G)).$$

Proof. For simple graph G, $TL(G) = L(G)$ and so by Proposition 7 the result follows.

Definition 12. *Let G be graph with m edges $e_1, \ldots, e_m$. Let $\nu_1, \nu_2, \ldots, \nu_m$ be the eigenvalues of the edge-adjacency matrix $A_E(G)$ of G. The edge-energy of G, denoted by $\mathcal{E}_E(G)$, is defined as*

$$\mathcal{E}_E(G) = \sum_{i=1}^{m} |\nu_i|. \tag{12}$$

Corollary 15. *For a multigraph G, the edge-energy of G is the energy of the T-line graph of G. That is,*

$$\mathcal{E}_E(G) = \mathcal{E}(TL(G)).$$

Proof. Follows by Proposition 7 .

Corollary 16. *For a simple graph G, the edge-energy of G is the energy of the line graph of G. That is,*

$$\mathcal{E}_E(G) = \mathcal{E}(L(G)).$$

Proof. Follows by Corollary 14.

Acknowledgements

The authors would like to thank the referees for their invaluable comments and suggestions which led to the improvement of the manuscript.

References

[1] R B Bapat. *Graphs and matrices.* Springer, London, 2010.

[2] F Harary. *Graph theory.* Addison-Wesley Pub. Co., Massachusetts, 1969.

[3] R Rajendra and P Siva Kota Reddy. Tosha-degree equivalence signed graphs. *Vladikavkaz. Mat. Zh.*, 22(2):48–52, 2020.

[4] R Rajendra and P Siva Kota Reddy. Tosha-degree of an edge in a graph. *Southeast Asian Bull. Math.*, accepted for publication.