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Professor Hari M. Srivastava  
On the Occasion of his 80th Birthday*

**Analysis of The Picard's Iteration Method and Stability  
for Ecological Initial Value Problems of Single Species  
Models with Harvesting Factor**

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**Abstract.** In the recent decades, biology and ecology area and also computer and network sciences are marched on at a rapid pace toward perfection by help of mathematical concepts such as stability, bifurcation, chaos and etc. Because of no existing any interspecific interaction in the single species, one is able to see that this is the simplest model. Meanwhile by adding some assumptions, we see that it has so many practical applications in the nature and any branch of sciences. In this article, some dynamical models of single species are studied. First, Picard's iteration method for exponential growth rate is analyzed. In continuation, some logistic models for both cases without harvesting and having harvested factor which are constant or variable are studied. Indeed, the solution and stability of equilibria for the said models are analyzed. Finally, in the section of simulation analysis by help of *Matlab software*, we give some numerical simulations to support of our mathematical conclusions which show the stability of the equilibria for I.V.Ps. of the logistic equation developed.

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## 1. Introduction

Population ecologists use methods to model the population dynamics. An accurate model should be able to describe the changes occurring in a population and predict future changes. Population growth is the most topic in ecology and biology area. Two simple models of population growth use deterministic equations to describe the rate of change in the size of a population over time. The first one, exponential growth, describes theoretical populations that increase in numbers without any limits to their growth. While the other one, logistic growth, introduces limits to reproductive growth that become more intense as the population size increases. Carrying capacity of the environment is so important to study of logistic dynamics. Neither model adequately describes natural populations, but they provide points of comparison.

As time goes, any researcher may see the application of mathematics in interdisciplinary sciences such as biomathematics and bioscience especially in the last century. One of most important mathematical topics using in biology, ecology etc. is the solution finding for system of differential equations. Most of time finding of solution for a differential equation is impossible unfortunately. On other hand having an initial population for a system of differential equations, an initial value problem may be formed. We should find out the solution or analyze its behavior. Anybody knows the importance and effect of single species in community and nature, and so we present a summary and brief literature of mathematical models such as single species, exponential and logistic modeling used to describe population dynamics, which may help us future model development and highlights the importance of population growth rate modeling in biology, ecology, and another area even computer and network sciences.

The study of population change started by Leonardo of Pisa, who was only given the nickname Fibonacci. He introduced in his arithmetic book of 1202 set, a modeling exercise involving a hypothetical growing rabbit population [9]. Environmental variation in ecological communities and inferences from single species data is studied by Abbott et al [2]. They present a method for estimating dimension of community on a single species. A dynamics of single species population growth is studied by Mueller et al in [8]. The stability at carrying capacity of environment for 25 different populations of *Drosophila melanogaster* are examined by them. The spatial dynamics single species can contribute to long term rarity and commonness. A spatial population dynamics model and regional commonness and regional extinction are investigated in [6]. Anybody is able to find out the answer of question "How great an effect does self-generated spatial structure have on logistic population growth?" which is studied by Law et al in [7]. They showed that population growth given by logistic model may differ greatly from that of the non-spatial logistic equation. They moreover proved that populations may achieve asymptotic densities greater than or less than the carrying capacity of logistic model where it is not spatial, and can even tend towards extinction. Accounting for the local spatial processes indeed brings the theory of single species population growth a step closer to the growth of real spatially structured populations. The population growth of infection of virus and worm in computer networks is analyzed by help of logistic modeling [1]. Pollutant and virus

induced disease effect of on single species animal population and its essential mathematical features by help of the logistic modeling. Moreover, It is shown that the susceptible population does not vanish when it is only under the effect of infection meanwhile in the polluted environment, it can extinct [5]. An application of exponential and logistic growths for single-cell models that are incapable or misleading for inferring population dynamics as there is no any interactions between cells via metabolites or physical contact, nor competition for limited resources such as nutrients or space is studied in [4]. Some more text on applications of single species with exponential or logistic growth rate having harvesting factor can be seen in [3], [9], [10], [11] and [13].

Exponential growth is associated with the name of Thomas Robert Malthus (1766-1834) who first realized that any species can potentially increase in numbers according to a geometric series [12]. He published his book in 1798 stating that populations with abundant natural resources grow very rapidly; however, they limit further growth by depleting their resources. The early pattern of accelerating population size is called exponential growth. Charles Darwin, in developing his theory of natural selection, was influenced by Malthus.

Generally, in open population or open system, there is the following equality

$$\text{Population Change} = \text{Births} - \text{Deaths} + \text{Immigrations} - \text{Emigrations}$$

By assuming there is no any immigrations or emigrations in our population, we consider a closed population or closed system. That is the population has changed only by the occurrence of birth and death. Moreover, By considering  $x(t)$  denotes the size of population at time  $t$ ;  $b$  and  $d$  denote the birth rate and death rate respectively, in the time interval  $[t, t + \Delta t]$ , then we get

$$x(t + \Delta t) - x(t) = bx(t)\Delta t - dx(t)\Delta t.$$

$$\begin{aligned} \Rightarrow \frac{dx}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{bx(t)\Delta t}{\Delta t} - \lim_{\Delta t \rightarrow 0} \frac{dx(t)\Delta t}{\Delta t} \\ &= bx(t) - dx(t), \end{aligned}$$

and so, by taking  $r = b - d$ , we have

$$\frac{dx}{dt} = rx(t). \tag{1}$$

The value of parameter  $r$  as intrinsic growth rate of population can be positive, meaning the population is increasing in size (the rate of change is positive); or negative, meaning the population is decreasing in size; or zero, in which case the population size is unchanging i.e. the population is constant. We now present an example of exponential growth for bacteria population in the following:

**Example 1.** *The famous example of exponential growth in organisms indeed is seen in bacteria. Bacteria are prokaryotes that reproduce largely by binary fission. This division takes about an hour for many bacterial species. If 1000 bacteria are placed in a large flask with an abundant supply of nutrients, the number of bacteria will have doubled from 1000 to 2000 after just an hour. In another hour, each of the 2000 bacteria will divide, producing 4000 bacteria. After the third hour, there should be 8000 bacteria in the flask. The important concept of exponential growth is that the growth rate-the number of organisms added in each reproductive generation-is itself increasing; that is, the population size is increasing at a greater and greater rate. After 24 of these cycles, the population would have increased from 1000 to more than 16 billion bacteria. When the population size  $x(t)$  is plotted over time, a J-shaped growth curve is produced.*

The bacteria-in-a-flask example indeed is not truly representative of the real world where resources are usually limited. However, when a species is introduced into a new habitat that it finds suitable, it may show exponential growth for a while. In the case of the bacteria in the flask, some bacteria will die during the experiment and thus not reproduce; therefore, the growth rate is lowered from a maximal rate in which there is no mortality. The growth rate of a population is largely determined by subtracting the death rate (d) from the birth rate (b). The growth rate can be expressed in a simple equation that combines the birth and death rates into a single factor  $r$  which is shown in the previous formula.

It is clear that Malthus’s model is applicable only if the number of species is small. Otherwise, the limited resources such as food and place limit the growth of population. The above I.V.P. is valid for a short period and can’t go on forever. However, the exponential growth applications in the nature is so much. Indeed, there is no an area of science that no need to help of exponential growth. Some of them are microbiology (growth of bacteria), conservation biology (restoration of disturbed populations), insect rearing (prediction of yield), plant or insect quarantine (population growth of introduced species), fishery (prediction of fish dynamics), biochemical(radioactivity).

## 2. Modeling and Discussion

We now in a position to construct and study some ecological I.V.Ps. of single species models based on P. Verhulst(1838-1845) idea which is published for human population growth rate [12]. The larger population means fewer resources such as food, place etc. which are limited. And so, this implies a smaller rate of growth rate. In the simplest case, he considered that the rate decreases linearly as a function of  $x$ .

### 2.1. The Exponential Model; Having Constant Harvesting Factor

Having considered  $x(0) = x_0$  initial population for equation (1), we obtain the following the initial value problem named I.V.P. :

$$\begin{cases} \frac{dx}{dt} = rx \\ x(0) = x_0 \end{cases} \quad (2)$$

where its solution as follows:

$$x(t) = x_0 \exp(rt). \tag{3}$$

Adding constant harvesting factor  $h$  in I.V.P exponential growth rate (2), we obtain the following I.V.P.:

$$\begin{cases} \frac{dx}{dt} = rx - h \\ x(0) = x_0 \end{cases} \tag{4}$$

Where the above I.V.P. solution is given by

$$x(t) = (x_0 - \frac{h}{r}) \exp(rt) + \frac{h}{r}. \tag{5}$$

We are now going to analyze the solution of I.V.Ps. (2) and (4) by Picard’s iteration method.

**Theorem 1.** *The following statements for I.V.Ps. of exponential growth models are true:*

(i) *The series solution of I.V.P. (2) is*

$$x(t) = x_0 \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!} \tag{6}$$

(ii) *The series solution of I.V.P. (4) is*

$$x(t) = \frac{h}{r} + (x_0 - \frac{h}{r}) \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!} \tag{7}$$

*Proof.* (i) First, we should consider the following iteration scheme:

$$x_n(t) = x_0 + \int_{s=0}^{s=t} r x_{n-1}(s) ds. \tag{8}$$

At the first step, we calculate  $x_1(t)$

$$x_1(t) = x_0 + \int_{s=0}^{s=t} r x_0 ds = x_0 + r x_0 t = x_0(1 + rt),$$

and so,

$$x_1(t) = x_0(1 + rt). \tag{9}$$

At the second step, we calculate the  $x_2(t)$

$$\begin{aligned} x_2(t) &= x_0 + \int_{s=0}^{s=t} r x_1(s) ds \\ &= x_0 + \int_{s=0}^{s=t} r x_0(1 + rs) ds \end{aligned}$$

$$\begin{aligned}
 &= x_0 + x_0 \left[ rs + \frac{(rs)^2}{2} \right]_{s=0}^{s=t} \\
 &= x_0 + x_0 t \left( r + \frac{(rt)^2}{2} \right).
 \end{aligned}$$

Thus,

$$x_2(t) = x_0 \left( 1 + rt + \frac{(rt)^2}{2} \right). \tag{10}$$

At the third step, we calculate the  $x_3(t)$

$$\begin{aligned}
 x_3(t) &= x_0 + \int_{s=0}^{s=t} rx_2(s) ds \\
 &= x_0 + \int_{s=0}^{s=t} rx_0 \left( 1 + rs + \frac{(rs)^2}{2} \right) ds \\
 &= x_0 + x_0 \left[ rs + \frac{(rs)^2}{2} + \frac{(rs)^3}{2 \times 3} \right]_{s=0}^{s=t} \\
 &= x_0 \left( 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3} \right).
 \end{aligned}$$

Hence,

$$x_3(t) = x_0 \left( 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3} \right). \tag{11}$$

At the fourth step, we calculate the  $x_4(t)$

$$\begin{aligned}
 x_4(t) &= x_0 + \int_{s=0}^{s=t} rx_3(s) ds \\
 &= x_0 + \int_{s=0}^{s=t} rx_0 \left( 1 + rs + \frac{(rs)^2}{2} + \frac{(rs)^3}{2 \times 3} \right) ds \\
 &= x_0 + x_0 \left[ rs + \frac{(rs)^2}{2} + \frac{(rs)^3}{2 \times 3} \right]_{s=0}^{s=t} \\
 &= x_0 \left( 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3} + \frac{(rs)^4}{2 \times 3 \times 4} \right).
 \end{aligned}$$

Then,

$$x_4(t) = x_0 \left( 1 + rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3} + \frac{(rt)^4}{2 \times 3 \times 4} \right). \tag{12}$$

Paying attention to relations (9),(10),(11) and (12), one may guess the following series:

$$x_n(t) = x_0 \sum_{k=0}^{k=n} \frac{(rt)^k}{k!}. \tag{13}$$

Approaching  $n$  to infinity, we see that the series solution of I.V.P. (2) may be obtained as follows:

$$x(t) = \lim_{n \rightarrow \infty} x_n(t) = x_0 \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!} \tag{14}$$

Therefore, the truth of relation (6) is proved.

(ii) Now, consider the following iteration scheme:

$$x_n(t) = x_0 + \int_{s=0}^{s=t} (rx_{(n-1)}(s) - h) ds \tag{15}$$

At the first, step we calculate  $x_1(t)$

$$x_1(t) = x_0 + \int_{s=0}^{s=t} (rx_0 - h) ds = x_0 + (x_0 - \frac{h}{r})rt.$$

For simplifying take

$$A = x_0 - \frac{h}{r}, \tag{16}$$

which it implies that

$$x_1(t) = x_0 + Art. \tag{17}$$

At the second step, we calculate the  $x_2(t)$

$$\begin{aligned} x_2(t) &= x_0 + \int_{s=0}^{s=t} (rx_1(s) - h) ds \\ &= x_0 + \int_{s=0}^{s=t} (r(x_0 + Ars) - h) ds \\ &= x_0 + [rx_0s + A\frac{(rs)^2}{2} - hs]_{s=0}^{s=t} \\ &= x_0 + rx_0t + A\frac{(rt)^2}{2} - ht. \end{aligned}$$

Then,

$$x_2(t) = x_0 + A(rt + \frac{(rt)^2}{2}). \tag{18}$$

At the third step, we calculate the  $x_3(t)$

$$\begin{aligned} x_3(t) &= x_0 + \int_{s=0}^{s=t} (rx_2(s) - h) ds \\ &= x_0 + \int_{s=0}^{s=t} (r(x_0 + Ars + A\frac{(rs)^2}{2}) - h) ds \end{aligned}$$

$$\begin{aligned}
 &= x_0 + [rx_0 + Ar^2s + A\frac{r^3}{2}s^2 - hs]_{s=0}^{s=t} \\
 &= x_0 + rx_0t - ht + A\frac{(rt)^2}{2} + A\frac{(rt)^3}{2}.
 \end{aligned}$$

Thus,

$$x_3(t) = x_0 + A(rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3}). \tag{19}$$

At the fourth step, we calculate the  $x_4(t)$

$$\begin{aligned}
 x_4(t) &= x_0 + \int_{s=0}^{s=t} (rx_3(s) - h)ds \\
 &= x_0 + \int_{s=0}^{s=t} (r(x_0 + Ars + A\frac{(rs)^2}{2} + A\frac{(rs)^3}{2 \times 3}) - h)ds \\
 &= x_0 + [rx_0 + Ar^2s + A\frac{r^3}{2}s^2 + A\frac{r^4}{2 \times 3}s^3 - hs]_{s=0}^{s=t} \\
 &= x_0 + rx_0t - ht + A\frac{(rt)^2}{2} + A\frac{(rt)^3}{2} + A\frac{(rt)^4}{2 \times 3 \times 4}.
 \end{aligned}$$

Therefore,

$$x_4(t) = x_0 + A(rt + \frac{(rt)^2}{2} + \frac{(rt)^3}{2 \times 3} + \frac{(rt)^4}{2 \times 3 \times 4}). \tag{20}$$

Paying attention to relations (17), (18), (19) and (20), we guess the following series:

$$x_n(t) = x_0 + A \sum_{k=1}^{k=n} \frac{(rt)^k}{k!}. \tag{21}$$

And so,

$$\begin{aligned}
 x_n(t) &= x_0 - A\frac{(rt)^0}{0!} + A \sum_{k=0}^{k=n} \frac{(rt)^k}{k!} \\
 &= x_0 - (x_0 - \frac{h}{r}) + A \sum_{k=0}^{k=n} \frac{(rt)^k}{k!}.
 \end{aligned}$$

Therefore,

$$x_n(t) = \frac{h}{r} + A \sum_{k=0}^{k=n} \frac{(rt)^k}{k!}. \tag{22}$$

Approaching  $n$  to infinity, we see that the series solution of I.V.P. (4) may be obtained as follows:

$$x(t) = \lim_{n \rightarrow \infty} x_n(t) = \frac{h}{r} + A \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!}$$



which implies that

$$x(t) = \frac{h}{r} + A \sum_{k=0}^{k=\infty} \frac{(rt)^k}{k!} \quad (23)$$

Regarding the relation (16), we may work out the relation (7) which completes the proof of (ii). Therefore, the proof of theorem is done.

## 2.2. The Logistic Model without Harvesting Factor

Extended exponential growth is possible only when infinite natural resources are available; this is not the case in the real world. Charles Darwin recognized this fact in his description of the "struggle for existence," which states that individuals will compete (with members of their own or other species) for limited resources. The successful ones are more likely to survive and pass on the traits that made them successful to the next generation at a greater rate (natural selection). To model the reality of limited resources, population ecologists developed the logistic growth model.

In the real world, with its limited resources, exponential growth cannot continue indefinitely. Exponential growth may occur in environments where there are few individuals and plentiful resources, but when the number of individuals gets large enough, resources will be depleted and the growth rate will slow down. Eventually, the growth rate will plateau or level off. This population size, which is determined by the maximum population size that a particular environment can sustain, is called the carrying capacity. In real populations, a growing population often overshoots its carrying capacity, and the death rate increases beyond the birth rate causing the population size to decline back to the carrying capacity or below it. Most populations usually fluctuate around the carrying capacity in an undulating fashion rather than existing right at it.

We are now in a position model the logistic growth. By this mean consider positive parameters  $M$  and  $r$  as the carrying capacity of the environment and the rate of growth for small population numbers, respectively. It is obvious that the factor  $rx$  represents unhampered growth and reduced by the term  $\frac{r}{M}x^2$  that corresponds to competition for limited resources within the population. Thus the form the following equation may be obtained:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right). \quad (24)$$

This equation is known as the logistic equation. In precisely speaking, the above formula used to calculate logistic growth adds the carrying capacity as a moderating force in the growth rate. The expression  $M - x$  is equal to the number of individuals that may be added to a population at a given time, and we saw that  $M - x$  is divided by  $M$  is the fraction of the carrying capacity available for further growth. Thus, the exponential growth model is restricted by this factor to generate the logistic growth equation. Therefore, we can take a result as "carrying capacity is the most and important subject in any logistic modeling". When the population size is equal to the carrying capacity, the quantity in

parentheses is equal to zero and growth is equal to zero. A graph of this equation yields the S-shaped curve. It is a more realistic model of population growth than exponential growth. There are three different sections to an S-shaped curve. Initially, growth is exponential because there are few individuals and ample resources available. Then, as resources begin to become limited, the growth rate decreases. Finally, the growth rate levels off at the carrying capacity of the environment, with little change in population number over time.

Making assumption  $x(t_0) = x_0$  as the initial population for time  $t_0$  in logistic equation (24), we may obtain an ecological I.V.P. For simplifying, we denoted  $t_0$  by 0.

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{M}) \\ x(0) = x_0. \end{cases} \tag{25}$$

By using integration part by part, the solution of above I.V.P. may be found as follows:

$$x(t) = \frac{Mx_0}{x_0 + (M - x_0)exp(-rt)} \tag{26}$$

It is clear that the following properties for the last I.V.P. may be obtained:

- (a) It has two equilibria  $x = 0, M$ . That is the density does not change in these cases. For  $0 < x < M$ , it increases, and for  $x > M$  it decreases;
- (b) If  $x_0 < M$ , then the population grows and approaches to  $M$  asymptotically as  $t \rightarrow \infty$ ;
- (c) If  $x_0 > M$ , then the population decreases, again approaching  $M$  asymptotically as  $t \rightarrow \infty$ ;
- (d) If  $x_0 = M$ , then the population remains in time at  $x = M$ .
- (e) Equilibrium point  $x = M$  is globally stable; i.e.

$$\lim_{t \rightarrow \infty} x(t) = M;$$

- (f) The behavior of solution in the region between 0 and  $M$  is known as logistic growth.

The above discussion is verified by numerical simulation for some different values of  $r$  and  $M$  in section 3 and shown in figure (1).

### 2.3. The Logistic Modeling with Constant Harvesting Factor

Having assumed the positive constant number  $h$  of population removed per each duration, we can extend I.V.P. (25) as follows:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{M}) - h \\ x(0) = x_0. \end{cases} \tag{27}$$

The above I.V.P. exhibits the limited population.

**Theorem 2.** *Ecological I.V.P. (27) has two equilibria  $x_1$  and  $x_2$  as follows:*

$$x_1 = \frac{M}{2} \left(1 - \sqrt{1 - \frac{4h}{rM}}\right), \quad x_2 = \frac{M}{2} \left(1 + \sqrt{1 - \frac{4h}{rM}}\right). \tag{28}$$

*Moreover, the above equilibrium points are unstable and asymptotic stable, respectively.*

*Proof.* It is clear that by setting  $\frac{dx}{dt}$ , we get  $\frac{r}{M}x^2 - rx + h = 0$ . The roots of last equation are equilibria of I.V.P. which are in (28).

Paying attention to the above equilibria, we are able to see the following properties immediately:

- (a) Regarding  $x(t)$  denotes the density or number of population, we have  $h < \frac{rM}{4}$ ;
- (b)  $x_1$  is positive provided  $\sqrt{1 - \frac{4h}{rM}} < 1$ ;
- (c) Since  $h > 0$ ,  $0 < x_1 < x_2 < M$ .

Regarding relations (28), we get:

$$rx\left(1 - \frac{x}{M}\right) - h = a(x - x_1)(x - x_2). \tag{29}$$

And so. I.V.P. (27) leads to the following I.V.P.:

$$\begin{cases} \frac{dx}{dt} = a(x - x_1)(x - x_2) \\ x(0) = x_0. \end{cases} \tag{30}$$

where its solution is given by:

$$x(t) = \frac{x_2(x_0 - x_1) - x_1(x_0 - x_2)\exp\left(-\frac{r}{M}(x_2 - x_1)t\right)}{(x_0 - x_1) - (x_0 - x_2)\exp\left(-\frac{r}{M}(x_2 - x_1)t\right)} \tag{31}$$

Since the following inequality is true

$$-\frac{r}{M}(x_2 - x_1)t < 0, \quad \text{for all } t > 0;$$

$$\lim_{n \rightarrow \infty} \exp\left(-\frac{r}{M}(x_2 - x_1)t\right) = 0.$$

If  $x_0 > x_1$ , we get  $x(t) = x_2$ .

Thus,  $x(t) = x_2$  is an equilibrium limiting solution. Therefore, the point of  $x = x_2$  is stable point.

We now analyze the solution behavior of equation (31) in case of  $x_0 < x_1$ .

By assuming

$$x_0 - x_1 = (x_0 - x_2)\exp\left(-\frac{r}{M}(x_2 - x_1)t\right),$$

we get

$$t = \frac{1}{-\frac{r}{M}(x_2 - x_1)} \ln \frac{x_0 - x_1}{x_0 - x_2}.$$

Now, let  $t_1$  be as following constant value:

$$t_1 = \frac{1}{-\frac{r}{M}(x_2 - x_1)} \ln \frac{x_0 - x_1}{x_0 - x_2}. \tag{32}$$

By setting the value of  $t_1$  in solution (31) we have:

$$x(t_1) = \frac{x_2(x_0 - x_1) - x_1(x_0 - x_2) \exp\left[-\frac{r}{M}(x_2 - x_1) \frac{1}{-\frac{r}{M}(x_2 - x_1)} \ln \frac{x_0 - x_1}{x_0 - x_2}\right]}{(x_0 - x_1) - (x_0 - x_2) \exp\left[-\frac{r}{M}(x_2 - x_1) \frac{1}{-\frac{r}{M}(x_2 - x_1)} \ln \frac{x_0 - x_1}{x_0 - x_2}\right]}.$$

Since the numerator of the above fraction is negative and it's denominator is zero, we get  $x(t_1) = -\infty$ .

Therefore, the value of the population function  $x(t)$  at  $t = x_1$  is a threshold and consequently, point  $x = x_1$  is a unstable equilibrium point. This means that the proof is completed.

## 2.4. The Logistic Modeling with Variable Harvesting Factor

Now we are going to extend the logistic population harvesting into case of harvesting parameter  $h$  in I.V.P. (27) is not constant. Consider the situation that the population has logistic growth rate and harvesting coefficient is not constant. Therefore, the general case of this model is as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - f(x) \tag{33}$$

where  $f(x)$  is an arbitrary function. There is no general way to solve the above equation; and so we should restrict our attention to a few special cases. In continuation, we study two cases for variable harvesting factor which are cubic and fractional.

### 2.4.1. Case of Fractional Harvesting Factor

Let us make assumption that the harvesting factor be following function:

$$f(x) = h \frac{x}{1 + x}. \tag{34}$$

And so, one of the extension of the logistic population harvesting factor will be appeared as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{M}\right) - h \frac{x}{1 + x} \tag{35}$$

The last equation exhibits that the harvesting coefficient for each term depends on it's population density. By adding initial population  $x(0) = x_0$  in the equation (35), we get the following I.V.P.:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{M}) - h\frac{x}{x+1} \\ x(0) = x_0. \end{cases} \tag{36}$$

Since  $\frac{x}{1+x} < 1$ , the term of harvesting factor has inverse relation respect to density of population. In the other word, the term of harvesting decreases as population increases.

**Theorem 3.** *The following statements for logistic modeling I.V.P. (36) are true:*

(i) *It has three equilibria  $x_1, x_2$  and  $x_3$  as follows:*

$$x_1 = 0, \quad x_{2,3} = \frac{M-1}{2} \pm \frac{M}{2} \sqrt{(\frac{1}{M}-1)^2 + \frac{4}{M}(1-\frac{h}{r})}; \tag{37}$$

(ii) *The equilibria  $x_2$  and  $x_3$  are real number provided*

$$\frac{h}{r} \leq 1 + \frac{M}{4}(1 - \frac{1}{M})^2;$$

(iii) *It is impossible that both equilibria  $x_2$  and  $x_3$  are positive;*

(iv) *The solution of this I.V.P. is as follows:*

$$x^{B_1}(x-x_2)^{B_2}(x-x_3)^{B_3} = x_0^{B_1}(x_0-x_2)^{B_2}(x_0-x_3)^{B_3} \exp(\frac{-rt}{M}) \tag{38}$$

(v) *In case of  $M = r = h = 1$ , I.V.P. has equilibria  $x = 0$  which is stable point.*

*Proof.*

(i)

We first set  $\frac{dx}{dt} = 0$ . by a simple calculation, we find equilibria which are given (37) easily.

(ii)

paying attention to (37), we see that for equilibria  $x_2$  and  $x_3$  are real number provided

$$\frac{h}{r} \leq 1 + \frac{M}{4}(1 - \frac{1}{M})^2;$$

(iii)

Now, let us take they are positive. It implies that

$$1 - \frac{1}{M} < \sqrt{(1 - \frac{1}{M})^2 - \frac{4}{M}(\frac{h}{r} - 1)} < \frac{1}{M} - 1.$$

If we take  $M < 1$ , the following contradiction is followed:

$$\sqrt{(1 - \frac{1}{M})^2 - \frac{4}{M}(\frac{h}{r} - 1)} < 0$$

which implies that the truth of (iii).

(iv)

If we take  $M = 1$ , we see that

$$x_{2,3} = \pm \sqrt{1 - \frac{h}{r}}$$

For simplifying we consider

$$\frac{dx}{dt} = \frac{-r}{M} \frac{x}{x+1} (x-x_2)(x-x_3) \tag{39}$$

Then, we get

$$\frac{1+x}{x(x-x_2)(x-x_3)} dx = \frac{-r}{M} dt.$$

By decomposing method, we may find the parameters  $B_1, B_2$  and  $B_3$  which satisfy in the following equation:

$$\frac{1+x}{x(x-x_2)(x-x_3)} = \frac{B_1}{x} + \frac{B_2}{x-x_2} + \frac{B_3}{x-x_3}.$$

By multiply the above equation into the phrases  $x, x-x_2$  and  $x-x_3$ ; and setting  $x = 0, x = x_2$  and  $x = x_3$  respectively in three steps, we have

$$B_1 = \frac{1}{x_2 x_3}, \quad B_2 = \frac{1+x_2}{x_2(x_2-x_3)}, \quad B_3 = \frac{1+x_2}{x_3(x_3-x_2)}.$$

Thus,

$$\int \frac{1+x}{x(x-x_2)(x-x_3)} dx = Ln[x^{B_1}(x-x_2)^{B_2}(x-x_3)^{B_3}]$$

which it implies that

$$x^{B_1}(x-x_2)^{B_2}(x-x_3)^{B_3} = C \exp\left(\frac{-rt}{M}\right), \tag{40}$$

where  $C$  is constant.

By setting  $x(0) = x_0$  in (40), we see that the solution given by formula (38) is right.

(v) Taking  $M = r = h = 1$ , we see equation (35) leads to the following equation:

$$\frac{dx}{dt} = x(1-x) - \frac{x}{1+x}$$

$$\Rightarrow x^3 = 0$$

Thus  $x = 0$  is equilibria of (35). As regarding  $x - x^2 - \frac{x}{x+1} = -\frac{x^3}{x+1} < 0$ ; we have

$$x - x^2 - \frac{x}{x+1} < 0$$

And so,  $x(t)$  is decreasing.

Therefore,

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

The above population function is an implicit function and so, we can't sketch the graph of solution  $x(t)$  respect to  $t$  clearly.

Applying variation table for function  $g(x) = rx(1 - \frac{x}{M}) - h\frac{x}{x+1}$ , one is able to determine the monotonicity (increasing or decreasing) of the above implicit function.

In the next section, we describe the solution behavior by help of method of numerical simulation.

### 2.4.2. Case of Cubic Harvesting Factor

In the second case, we make assumption that the harvesting function be as follows:

$$f(x) = x^3, \tag{41}$$

and so we get:

$$\frac{dx}{dt} = rx(1 - \frac{x}{M}) - hx^3 \tag{42}$$

Like the previous case, by adding initial population  $x(0) = x_0$  in the equation (42), we get the following I.V.P.:

$$\begin{cases} \frac{dx}{dt} = rx(1 - \frac{x}{M}) - hx^3 \\ x(0) = x_0. \end{cases} \tag{43}$$

**Theorem 4.** *The following statements for logistic modeling I.V.P. (43) are true:*

(i) *It has three equilibria  $x_1, x_2$  and  $x_3$  as follows:*

$$x_1 = 0, \quad x_{2,3} = \frac{r \pm \Delta}{2hM} \quad \text{where } \Delta = \sqrt{r^2 + 4hrM^2}; \tag{44}$$

(ii) *The solution of this I.V.P. is the following implicit function:*

$$x^{\frac{h}{r}}(x - x_2)^{\frac{r-\Delta}{-2\Delta}}(x - x_3)^{\frac{r+\Delta}{-2\Delta}} = x_0^{\frac{h}{r}}(x_0 - x_2)^{\frac{r-\Delta}{-2\Delta}}(x_0 - x_3)^{\frac{r+\Delta}{-2\Delta}}; \tag{45}$$

(iii) *In case of  $M = r = h = 1$ , the solution of this I.V.P. is decreasing function. Moreover, the equilibria  $x_1 = 0$  is asymptotically stable.*

*Proof.*

(i)

By setting  $\frac{dx}{dt} = 0$ , then we see that the first equilibrium point is  $x_1 = 0$ .

Another equilibria are the roots of following quadratic polynomial:

$$-hx^2 - \frac{r}{M}x + r,$$

$$\Rightarrow x_{2,3} = \frac{r \pm \sqrt{r^2 + 4hrM}}{2hM},$$

which implies the truth of (44).

(ii)

Then, we get

$$\frac{dx}{dt} = x(x - \frac{r - \Delta}{2hM})(x - \frac{r + \Delta}{2hM}) \tag{46}$$

After calculation, we get:

$$\int dt = \int (\frac{h}{x} + \frac{r-\Delta}{x-x_2} + \frac{r+\Delta}{x-x_3}) dx,$$

We therefore obtain the solution of (42) as follows:

$$x^{\frac{h}{r}}(x - x_2)^{\frac{r-\Delta}{-2\Delta}}(x - x_3)^{\frac{r+\Delta}{-2\Delta}} = Kexp(t). \tag{47}$$

Therefore, considering initial population  $x(0) = x_0$ , we see that the solution of I.V.P. (43) may be found as follows:

$$x^{\frac{h}{r}}(x - x_2)^{\frac{r-\Delta}{-2\Delta}}(x - x_3)^{\frac{r+\Delta}{-2\Delta}} = x_0^{\frac{h}{r}}(x_0 - x_2)^{\frac{r-\Delta}{-2\Delta}}(x_0 - x_3)^{\frac{r+\Delta}{-2\Delta}} exp(t) \tag{48}$$

which is implicit function.

(iii)

Making assumption  $M = r = h = 1$ , implies that the equation (42) leads to the following equation:

$$\frac{dx}{dt} = x(1 - x) - x^3$$

As regarding  $x(t)$  describes the number of population which is positive integer. And so it takes the number one at least. we see that

$$\begin{aligned} x &< x^2 < x^3 \\ \Rightarrow \frac{dx}{dt} &= x - x^2 - x^3 < 0 \end{aligned}$$

Therefore,  $x(t)$  is decreasing function.

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Therefore, the equilibria  $x_1 = 0$  is asymptotically stable. Because of being implicit the solution  $x(t)$ , it is impossible to sketch the graph of this solution. Applying variation table, one is able to determine the monotonicity (increasing or decreasing) of the above implicit function. In the next section, we describe the solution behavior by help of simulation method.

In the next section, we describe the solution behavior by help of method of numerical simulation.



### 3. Numerical Simulation

In this section, we discuss the analytical results by making some simulations on studied I.V.P. describing logistic modeling having and without harvesting factor analyzed in section 2. First, the behavior of solution (26) for I.V.P. (25) is simulated. An numerical simulation for various values to the carrying capacity  $M$  and growth rate  $r$  is formulated as follows:

- (i)  $x_0 = M + i$ , where  $i = 20, 40, 60$  and  $80$ ;
- (ii)  $x_0 = \frac{M}{2} + j$ , where  $i = 10, 20, 30$  and  $40$ ;
- (iii)  $x_0 = \frac{M}{k}$ , where  $i = 3, 4, 5$  and  $6$ .

Indeed, in each column of table (1) by considering carrying capacity ( $M$ ), we worked out formula for finding suitable initial populations. And so, for rows a, b, c, d, e, f; we have:

In case (i): Calculated initial populations is greater than the carrying capacity. In this case, obtained graphs shows the being of solutions decreasing.

In case (ii): Calculated initial populations are located between  $\frac{M}{2}$  and  $M$ . That is they are greater than  $\frac{M}{2}$  and less than the carrying capacity  $M$ .

In case (iii): Calculated initial populations is less than  $\frac{M}{2}$ . In this case, obtained graphs shows that all solutions have turning behavior at the point  $x = \frac{M}{2}$ .

In all of above cases, the solutions tend to  $M$  asymptotically. This numerical argument is shown in the table (1).

The graphs related to solution (26) are drawn in figure 1 (a, b, c and d) which verify the presented mathematical discussion. Indeed, green graphs, which are started with value greater than  $M$ , are decreasing and tend to  $M$  asymptotically. Blue graphs, which are located between  $\frac{M}{2}$  and  $M$ , are increasing and tend to  $M$  asymptotically. Red graphs, which are started with value less than the  $\frac{M}{2}$ , are increasing and tend to  $M$  asymptotically. It is clear that it decrease slowly whenever growth rate  $r$  is less than 1, meanwhile it decreases rapidly whenever growth rate  $r$  is greater than 1.

To verify the the results of theorem 2.2, we make simulation for solution (31) of I.V.P.(27) describing logistic modeling with constant harvesting factor. By giving some different values to carrying capacity  $M$ , growth rate  $r$  and harvesting factor  $h$ , we obtain the related equilibria  $x_1$  and  $x_2$ . This argument has been brought in Table 2.

Paying attention to drawn graphs in figure 2, we see that the equilibrium point  $x_1$  is unstable and the equilibrium point  $x_2$  is asymptotically stable which are proved in theorem 2.2. The details may be seen in figure 2. The solution graphs are shown in this figure (a,b,c,d) clearly.

We now make simulation for solution (38) of I.V.P. (36). This I.V.P. describes the logistic modeling variable harvesting factor. Indeed, in this case harvesting factor is a fractional function  $f(x) = \frac{x}{x+1}$ . The details of parameters: carrying capacity( $M$ ), growth

rate( $r$ ), harvesting factor( $h$ ) and initial populations( $x_0$ ) are presented in table (2). We see that for case of  $M = r = h = 1$  all of equilibria for (36) are  $x = 0$ . The related graphs verifying the results of theorem 2.3 are shown in figure (3). These graphs are simulated in two scale  $0 \leq t < 30$  and  $0 \leq t < 1000$ .

For the final case, we make simulation for solution of (45) of I.V.P. (43). This I.V.P. describes logistic modeling having variable harvesting factor. It is assumed that in this model harvesting factor is cubic function  $f(x) = x^3$ . Considering assumption as  $M = r = h = 1$  implies that equilibria are given by

$$x_1 = 0, \quad x_2 = -1.618 \text{ and } x_3 = 6.18 \times 10.$$

If we consider  $M = \times 10^6, r = 1, h = \times 10^{-6}$ , we have equilibria as follows:

$$x_1 = 0, \quad x_2 = -1.0005 \times 10^3 \text{ and } x_3 = 9.995 \times 10^3.$$

In both of the above cases, the equilibria are negative, zero and positive which are proved in theorem 2.4. The another result of solution behavior which are studied in said theorem are shown in figure (4).

Table 1: Parameters, Equilibria and Initial Populations for I.V.P.(25)

	a	b	c	d
$M$	$10^2$	$10^2$	$10^3$	$10^3$
$r$	$5 \times 10$	2	0.5	2
$x_1$	0	0	0	0
$x_2$	$10^2$	$10^2$	$10^3$	$10^3$
$x_{01}$	$1.667 \times 10$	$1.666 \times 10^2$	50	$1.666 \times 10^2$
$x_{02}$	20	20	200	$2 \times 10^2$
$x_{03}$	25	25	250	$2.5 \times 10^2$
$x_{04}$	$3.333 \times 10$	$3.333 \times 10^2$	$3.333 \times 10^3$	$3.333 \times 10^3$
$x_{05}$	$6 \times 10$	$6 \times 10$	$5.1 \times 10^2$	$5.1 \times 10^2$
$x_{06}$	$7 \times 10$	$7 \times 10$	$5.2 \times 10^2$	$5.2 \times 10^2$
$x_{07}$	$8 \times 10$	$8 \times 10$	$5.3 \times 10^2$	$5.3 \times 10^2$
$x_{08}$	$9 \times 10$	$9 \times 10$	$5.4 \times 10^2$	$5.4 \times 10^2$
$x_{09}$	$1.2 \times 10$	$1.2 \times 10^2$	$1.2 \times 10^3$	$1.2 \times 10^3$
$x_{010}$	$1.4 \times 10$	$1.4 \times 10$	$1.4 \times 10^3$	$1.4 \times 10^3$
$x_{011}$	$1.6 \times 10$	$1.6 \times 10$	$1.6 \times 10^3$	$1.6 \times 10^3$
$x_{012}$	$1.8 \times 10$	$1.8 \times 10$	$1.8 \times 10^3$	$1.8 \times 10^3$

Table 2: Parameters, Equilibria and Initial Populations for I.V.P.(27)

	a	b	c	d
$M$	$10^4$	$10^4$	$10^4$	$10^4$
$r$	.5	.5	10	10
$h$	$10^2$	$10^2$	$10^3$	$10^3$
$x_1$	0	0	0	0
$x_2$	$2.041 \times 10^2$	$2.041 \times 10^2$	$1.001 \times 10$	$1.001 \times 10$
$x_3$	$9.795 \times 10^3$	$9.795 \times 10^3$	$9.899 \times 10^3$	$9.899 \times 10^3$
$x_{01}$	$2.2 \times 10^2$	$1.8 \times 10^2$	50	10
$x_{02}$	$5 \times 10^2$	$5 \times 10^2$	$5 \times 10^2$	$5 \times 10^2$
$x_{03}$	$2 \times 10^3$	$2 \times 10^3$	$2 \times 10^3$	$2 \times 10^3$
$x_{04}$	$9 \times 10^3$	$9 \times 10^3$	$9 \times 10^3$	$9 \times 10^3$
$x_{05}$	$1.2 \times 10^3$	$1.2 \times 10^3$	$1.3 \times 10^3$	$1.3 \times 10^3$
$x_{06}$	$1.5 \times 10^3$	$1.5 \times 10^3$	$1.5 \times 10^3$	$1.5 \times 10^3$

Table 3: Parameters, Equilibria and Initial Populations for I.V.P.(36)

	a	b
$M$	1	$10^6$
$r$	1	1
$h$	1	$10^{-6}$
$x_1$	0	0
$x_2$	0	$-9.9999 \times 10^{-1}$
$x_3$	0	$9.9999 \times 10^5$
$x_{01}$	$10^5$	$10^6$
$x_{02}$	$9 \times 10^4$	$9 \times 10^5$
$x_{03}$	$8 \times 10^4$	$8 \times 10^5$
$x_{04}$	$7 \times 10^4$	$7 \times 10^5$
$x_{05}$	$6 \times 10^4$	$6 \times 10^5$
$x_{06}$	$5 \times 10^4$	$5 \times 10^5$
$x_{07}$	$4 \times 10^4$	$4 \times 10^5$
$x_{08}$	$3 \times 10^4$	$3 \times 10^5$
$x_{09}$	$2 \times 10^4$	$2 \times 10^5$
$x_{10}$	$10^4$	$10^5$
$x_{011}$	$6 \times 10^3$	$6 \times 10^4$
$x_{012}$	$4 \times 10^3$	$4 \times 10^4$
$x_{013}$	$2 \times 10^3$	$2 \times 10^4$
$x_{014}$	$10^3$	$10^4$

Table 4: Parameters, Equilibria and Initial Populations for I.V.P.(43)

	a	b
$M$	1	$10^6$
$r$	1	1
$h$	1	$10^{-6}$
$x_1$	0	0
$x_2$	-1.618	$-1.0005 \times 10^3$
$x_3$	$6.18 \times 10^{-1}$	$9.9950 \times 10^3$
$x_{01}$	$10^5$	$10^6$
$x_{02}$	$9 \times 10^4$	$9 \times 10^5$
$x_{03}$	$8 \times 10^4$	$8 \times 10^5$
$x_{04}$	$7 \times 10^4$	$7 \times 10^5$
$x_{05}$	$6 \times 10^4$	$6 \times 10^5$
$x_{06}$	$5 \times 10^4$	$5 \times 10^5$
$x_{07}$	$4 \times 10^4$	$4 \times 10^5$
$x_{08}$	$3 \times 10^4$	$3 \times 10^5$
$x_{09}$	$2 \times 10^4$	$2 \times 10^5$
$x_{10}$	$10^4$	$10^5$
$x_{011}$	$6 \times 10^3$	$6 \times 10^4$
$x_{012}$	$4 \times 10^3$	$4 \times 10^4$
$x_{013}$	$2 \times 10^3$	$2 \times 10^4$
$x_{014}$	$10^4$	$10^4$

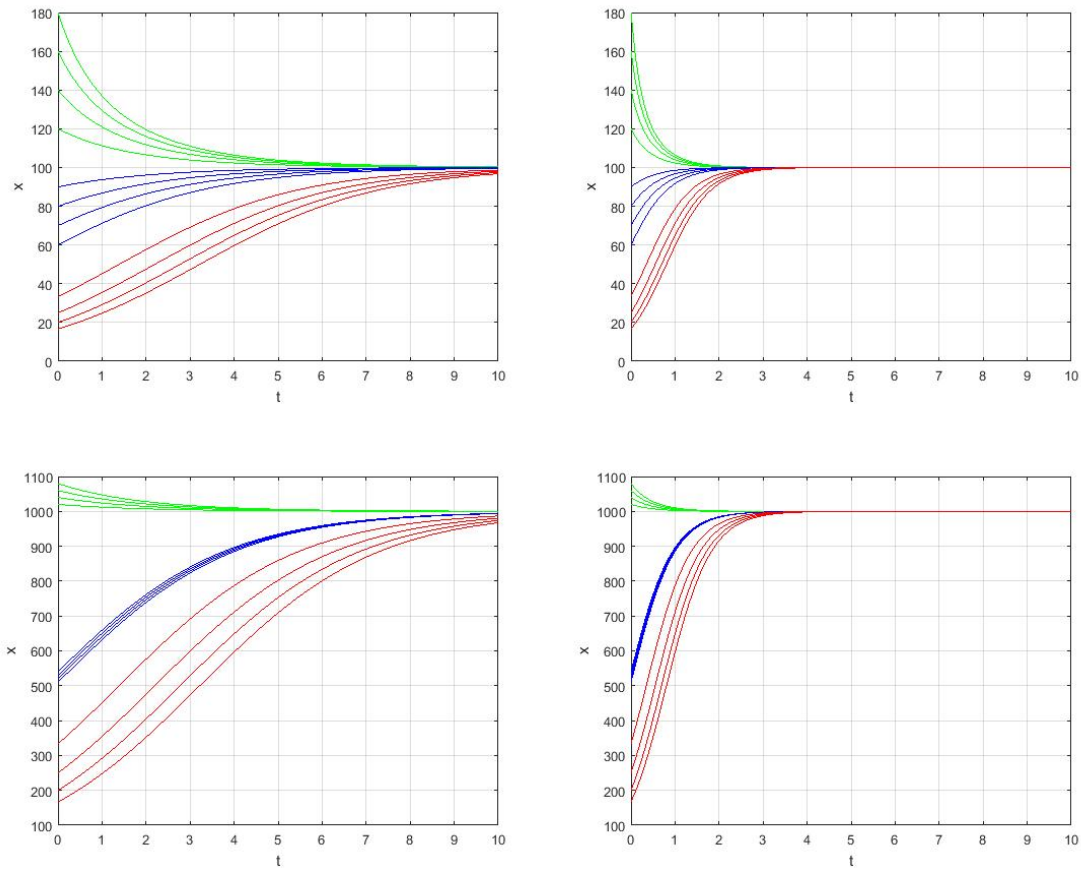


Figure 1: I.V.P. (2.24)

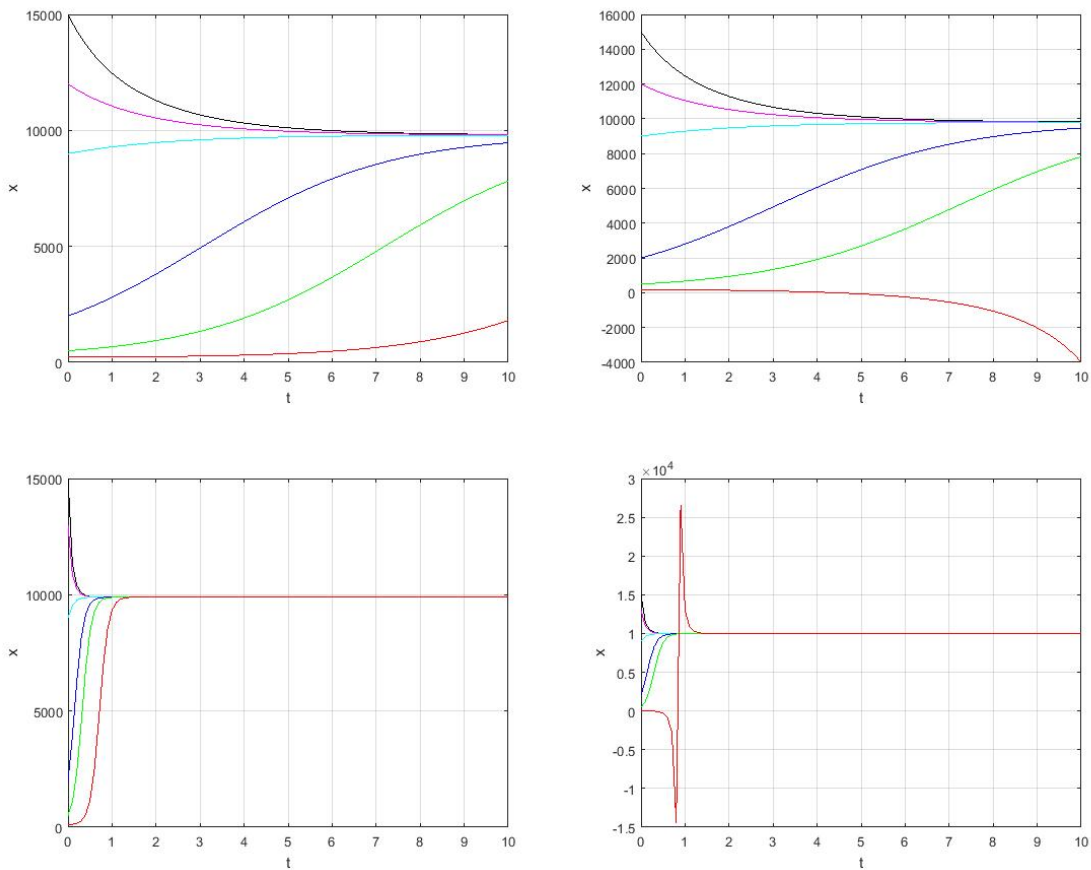


Figure 2: I.V.P. (2.26)

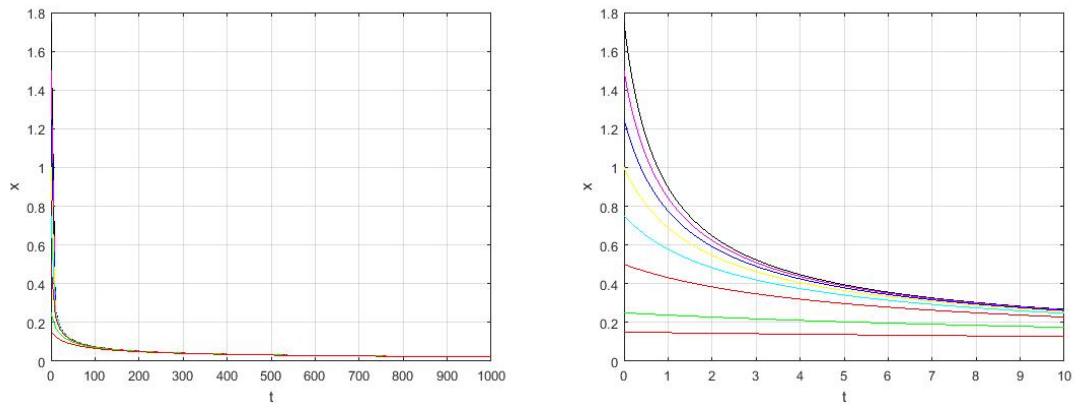


Figure 3: I.V.P. (2.35)

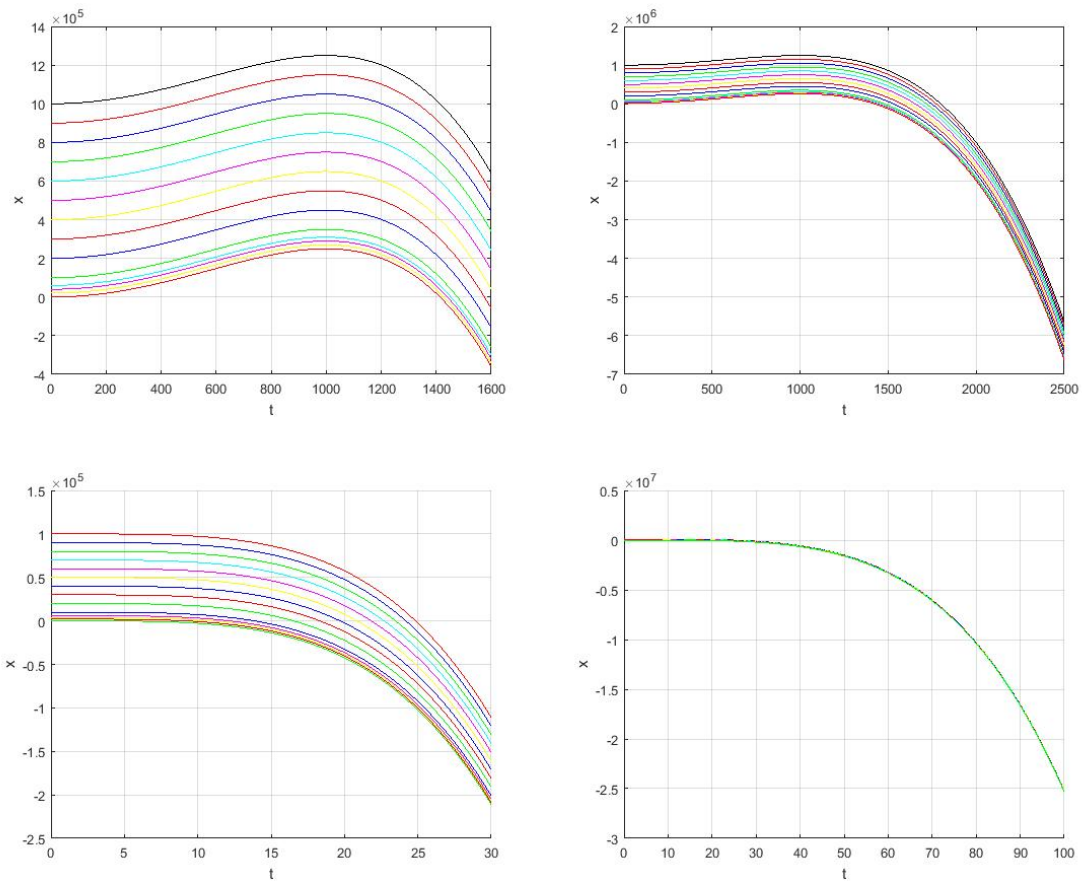


Figure 4: I.V.P. (2.42)

#### 4. Conclusion

The importance and effects of single species in the nature and community is clear for anybody. The discussed models in this work have so many practical application. In deed, by help of these models, one may predict, check and defence to spread of viruses, microbe and bacteria in a community. There are so many dangerous viruses for humanity. Anybody gets involved with microbe viruses and such as Black Death, Spanish Flu, HIV/AIDS, Swine Flu, Ebola virus, Zika virus, Corona viruses such as: SARS-Cov, MERS-Cov, COVID-19. Especially, last one which is the most deadly and disastrous viruses nowadays in the world. This viruses is spread in 2019, and all of the countries in the world get involved by it. Some of them such as Corona viruses: COVID-19 are not only epidemic but also are pandemic. As a consequence in this research, we worked out the series solution for exponential modeling for both cases of having constant harvesting factor or simple model without harvesting factor. Moreover, making some conditions for single species of logistic modeling, we find out they have stable solutions. And also, their asymptotical stability is obtained. Making some various simulations, we observed the obtained results which are proved theorems are true. The important result is: Carrying capacity of the environment, growth rate of population, harvesting factor are so important to stability the equilibria.

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