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On approximation of signals in the generalized Zygmund class using $(E,r)(N,q_n)$ mean of conjugate derived Fourier series

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Abstract. In the present article, we have established a result on degree of approximation of function in the generalized Zygmund class $Z_l^{(m)}, (l \geq 1)$ by $(E,r)(N,q_n)$- mean of conjugate derived Fourier series.

2020 Mathematics Subject Classifications: 42A10, 41A10, 42B05, 42B08

Key Words and Phrases: Degree of approximation, Generalized Zygmund class, Fourier series, Conjugate Fourier Series, Conjugate Derived Fourier series, (E,r)-Summability mean, (N,q_n) - summability mean, $(E,r)(N,q_n)$-summability mean

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1. Introduction

Signal Analysis describes the field of study whose objective is to collect, understand and deduce information and intelligence from various signals. Now-a-days the analysis of signals is a fundamental problem for many engineers and scientists. In the recent past, we have seen the applications of mathematical methods such as Probability theory, Mathematical statistics etc. in the analysis of signals. Very recently, approximation

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theory has got a large popularity as it has given a new dimension in approximating the signals. The estimation of error functions in Lipschitz and Zygmund space using different summability techniques of Fourier series and conjugate Fourier series have been of great interest among the researchers in the last decades. For details see [3, 7, 9, 12, 13] and [15] to [16]. Also, the generalized Zygmund class ${Z_l}^{(m)}$,$(l \geq 1)$ was investigated by Leindler [8], Moricz [4], Moricz and Nemeth [6] etc. Very recently Das et al.[1], Nigam [7], Pradhan et al.[11, 14] and Singh et al.[10] proved approximation of functions in the generalized Zygmund class by using different summability means. In the present paper, we investigate on the degree of approximation of a function in the generalized Zygmund class ${Z_l}^{(m)}$,$(l \geq 1)$ by $(E, r)(N, q_n)$ product mean of the conjugate derived Fourier series.

2. Definitions and Notations

Let h be a function, which is periodic in $[0, 2\pi]$ such that $\int_0^{2\pi} |h(x)|^l dx < \infty$.
Let us denote

$$L_l[0, 2\pi] = \Big\{h : [0, 2\pi] \to R : \int_0^{2\pi} |h(x)|^l dx < \infty\Big\}, l \geq 1.$$

The Fourier series of $h(x)$ is given by

$$\sum_{n=0}^{\infty} u_n(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n cosnx + b_n sinnx\Big) \tag{1}$$

Also,the conjugate Fourier series and derived conjugate Fourier series of $h(x)$ are respectively

$$\sum_{n=1}^{\infty} \Big(b_n cosnx - a_n sinnx\Big)$$

and

$$-\sum n\ u_n(x).$$

Let us define

$$\|h\|_l = \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(x)|^l dx\Big)^{\frac{1}{l}}, 1 \leq l < \infty$$

and

$$\|h\|_l = ess \sup_{0\leq x\leq 2\pi} |h(x)|, l = \infty$$

Let $\overline{S'_p}\Big(h; x\Big)$ denotes the p-th partial sum of conjugate derived Fourier series and is given by

$$\overline{S'_p}\Big(h; x\Big) - \overline{h'}(x) - = -\frac{2}{\pi}\int_0^{\pi} \frac{\Psi(x; v)}{4sin\frac{v}{2}}\Big(k + \frac{1}{2}\Big)sin\Big(k + \frac{1}{2}\Big)v\ dv - \frac{1}{\pi}\int_0^{\pi} \frac{\Psi(x; v)}{4sin\frac{v}{2}} \frac{sin\Big(k + \frac{1}{2}\Big)v}{tan\frac{v}{2}}\ dv$$

Where $\overline{h'}$ is the conjugate derived function of 2π periodic function 'h', which is given by

$$\overline{h'}(x) = -\frac{1}{\pi}\int_0^{\pi} \Psi(x;v)\ cosec^2\frac{v}{2}\ dv$$

Let the Zygmund modulus of continuity of $h(x)$ be:

$$m(h;r) = \sup_{0\leq r, x\in R} |h(x+v)+h(x-v)| \texttt{(see [2]}$$

Let $\mathbf{B}$ represents the Banach space of all 2π periodic functions which are continuous and defined over $[0,2\pi]$ under the supremum norm. Clearly,

$$Z_{(\alpha)} = \Big\{h \in \mathbf{B} : |h(x+v)+h(x-v)| = O\Big(|v|^{\alpha}\Big), 0<\alpha\leq 1\Big\}$$

is a Banach space under the norm $\|.\|_{(\alpha)}$ defined by

$$\|h\|_{(\alpha)} = \sup_{0\leq x\leq 2\pi} |h(x)| + \sup_{x,t\neq 0} \frac{|h(x+v)+h(x-v)|}{|v|^{\alpha}}$$

For $h \in L_l[0,2\pi], (l \geq 1)$, the integral Zygmund modulus of continuity is defined by

$$m_l(h;r) = \sup_{0<v\leq r}\left\{\frac{1}{2\pi}\int_0^{2\pi} |h(x+v)+h(x-v)|^l dx\right\}^{\frac{1}{l}}$$

and for $h \in \mathbf{B}, l = \infty$,

$$m_{\infty}(h;r) = \sup_{0<v\leq r}\max_{x} |h(x+v)+h(x-v)|.$$

Clearly,

$$m_l(h;r) \to 0 \quad \texttt{as} \quad l \to 0.$$

Let us define the space

$$Z_{(\alpha),\ l} = \Big\{h \in L_l[0,2\pi] : \Big(\int_0^{2\pi} |h(x+v)+h(x-v)|^l dx\Big)^{\frac{1}{l}}$$

which is a Banach space under the norm $\|.\|_{(\alpha),\ l}$ for $0<\alpha\leq 1$ and $l \geq 1$. Clearly,

$$\|h\|_{(\alpha),\ l} = \|h\|_l + \sup_{v\neq 0}\frac{\|h(.+v)+h(.-v)\|_l}{|v|^{\alpha}}$$

Let

$$Z^{(m)} = \Big\{h \in \mathbf{B} : |h(x+v)+h(x-v)| = O\Big(m(v)\Big)\Big\}$$

where m is a Zygmund modulus of continuity satisfying

$$\text{(a)}\, m(0) = 0$$
$$\text{(b)}\, m(v_1 + v_2) \leq m(v_1) + m(v_2).$$

Let $m : [0, 2\pi] \to \mathbf{R}$ a function with $m(v) > 0$ for $0 \leq v < 2\pi$ and

$$\lim_{v \to 0^+} m(v) = m(0) = 0.$$

Define

$$Z_l^{(m)} = \left\{ h \in L_l : 1 \leq l < \infty, \sup_{v \neq 0} \frac{\|h(.+v) + h(.-v)\|_l}{m(v)} < \infty \right\}$$

where

$$\|h\|_l^{(m)} = \|h\|_l + \sup_{v \neq 0} \frac{\|h(.+v) + h(.-v)\|_l}{m(v)}, l \geq 1.$$

Clearly, $\|.\|_l^{(m)}$ is a norm $Z_l^{(m)}$.
Also, $Z_l^{(m)}$ is complete since $L_l, (l \geq 1)$ is complete.
So, $Z_l^{(m)}$ is a Banach space under $\|.\|_l^{(m)}$.
Let $m(v)$ and $\mu(v)$ represents the Zygmund moduli of continuity such that $\left(\frac{m(v)}{\mu(v)}\right)$ is positive and non-decreasing then

$$\|h\|_l^{(\mu)} \leq \max .\left(1, \frac{m(2\pi)}{\mu(2\pi)}\right) \|h\|_l^{(m)} \leq \infty \tag{2}$$

Clearly,

$$Z_l^{(m)} \subseteq Z_l^{(\mu)} \subseteq L_l, (l \geq 1).$$

Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let $\{q_k\}$ represents the sequence of non-negative integers such that

$$Q_n = \sum_{k=0}^{n} q_k \to \infty \quad \text{as} \quad n \to \infty. \tag{3}$$

Let

$$\tau_n^N = \frac{1}{Q_n} \sum_{k=0}^{n} q_{n-k} s_k, \quad n = 0, 1, 2, ... \tag{4}$$

represents the (N, q_n) mean of $\{s_n\}$ generated by the sequence $\{q_n\}$.
By (N, q_n) method, the series $\sum u_n$ is said to be summable to $'s'$ if

$$\lim_{n \to \infty} \tau_n^N \to s.$$

We know, (N, q_n) method is regular [5].
The (E, r) transform of $\{s_n\}$ is given by

$$E_n^r = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) \quad r^{n-k} s_k \tag{5}$$

If $E_n^r \to s$ as $n \to \infty$ then $\sum u_n$ is summable to 's' by (E, r) summability. Also, (E, r) method is regular [5].
The $(E, r)(N, q_n)$ transform of $\{s_n\}$ is given by

$$\tau_n^{E_r,N} = \frac{1}{(1+r)^n} \sum_{k=0}^{n} C(n,k) \Big\{ \frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} s_\nu \Big\} \tag{6}$$

The series $\sum u_n$ is summable to s by the $(E, r)(N, q_n)$ transform if $\tau_n^{E_r,N} \to s$ as $n \to \infty$.
Also we have used the following notation in the rest part of our paper.

$$\Psi(x, v) = h(x+v) + h(x-v)$$

3. Known Result

Using Hausdorff mean, Nigam [7] proved the following theorem:

Theorem 1. *Error approximation of a conjugate derived function $\overline{h'}$ of a 2π periodic function $h \in Z_l^{(m)}, l \geq 1$, using $H = \Big(\theta_{j,\alpha}\Big)$ of conjugate derived Fourier series is given by*

$$\|\overline{M'}_j^H(h;.) - \overline{h'}(.)\|_l^{(m)} = O\Big(\frac{1}{j+1} \int_{\frac{1}{j+1}}^{\pi} \frac{(v+1)m(v)}{v^3 \ \mu(v)} dv\Big),$$

where $m(v)$ and $\mu(v)$ are Zygmund moduli of continuity, provided

$$\int_0^{\pi} \frac{m(v)}{v^2 \mu(v)} dv = O\Big(\frac{m(\eta)}{\eta \ \mu(\eta)}\Big), \ 0 < \eta < \pi.$$

4. Main Theorem

Theorem 2. *The degree of approximation of a conjugate derived function $\overline{h'}$ of a 2π periodic function $h \in Z_l^{(m)}, l \geq 1$, using $(E, r)(N, q_n)$- mean of conjugate derived Fourier series is given by*

$$E_n(h) = \inf_n \|{\chi_n}'(.)\|_l^{\mu} = O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v^2 \ \mu(v)} dv\Big)$$

where $m(v)$ *and* $\mu(v)$ *are the Zygmund moduli of continuity and* $\frac{m(v)}{v\ \mu(v)}$ *is positive and non-decreasing, provided*

$$\int_0^{\eta} \frac{m(v)}{v\ \mu(v)} dv = O\Big(\frac{m(\eta)}{\mu(\eta)}\Big).$$

We require the below mentioned lemmas to prove our main theorem:

5. Lemmas

Lemma 1.

$$|\overline{Y_1}'(v)| = O(n^2) \quad \texttt{for} \ \ 0 < v \leq \pi.$$

Lemma 2.

$$|\overline{Y_2}'(v)| = O\Big(\frac{1}{v^2}\Big) \quad \texttt{for} \ \ 0 < v \leq \pi.$$

Lemma 3. *Let* $h \in Z_l^{(m)}$ *then for* $0 < v \leq \pi$,

(i) $\|\Psi(.,v)|_l = O\Big(m(v)\Big)$

(ii) $\|\Psi(.+y,v) + \Psi(.-y,v)\|_l = O\Big(m(v)\Big)$ *or* $O\Big(m(y)\Big)$

(iii) *If* $m(v)$ *and* $\mu(v)$ *are as defined in the main theorem, then*

$$\|\Psi(.+y,v) + \Psi(.-y,v)\|_l = O\Big(\mu(y)\frac{m(v)}{\mu(v)}\Big)$$

where $\Psi(x,v) = h(x+v) + h(x-v)$.

6. Proof of the Lemmas

6.1. Proof of Lemma-1

For $v \in \Big(0, \frac{1}{n+1}\Big]$ and $sin\ nv \leq n\ sin\ v$, we have

$$|\overline{Y_1}'(v)| = \Big|\frac{-2}{4\pi(1+r)^n} \sum_{k=0}^{n} k\ C(n,k) r^{n-k} \Big\{\frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu} \frac{sin(\nu+\frac{1}{2})v}{sin\frac{v}{2}}\Big\}\Big|$$

$$\leq \frac{1}{2\pi(1+r)^n}\Big|\sum_{k=0}^{n} k\ C(n,k)\ \ r^{n-k} \Big\{\frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu}\ \ \frac{(2\nu+1))sin\Big(\nu+\frac{1}{2}\Big)v}{sin\frac{v}{2}}\Big\}\Big|$$

$$\leq \frac{1}{2\pi(1+r)^n}\Big|\sum_{k=0}^{n} k\ C(n,k)\ \ r^{n-k}\ \ (2k+1)\Big\{\frac{1}{Q_k} \sum_{\nu=0}^{k} q_{k-\nu}\Big\}\Big|$$

$$= O(n^2)$$

For $v \in \left[\frac{1}{n+1}, \pi\right], \frac{1}{sin\left(\frac{v}{2}\right)} \leq \frac{\pi}{v}, sinv \leq v$

$$|\overline{Y_1}'(v)| = \Big|\frac{-2}{4\pi(1+r)^n}\sum_{k=0}^{n} k\ C(n,k) r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{sin(\nu+\frac{1}{2})v}{sin\frac{v}{2}}\Big\}\Big|$$

$$\leq \frac{1}{4\pi(1+r)^n}\Big|\sum_{k=0}^{n} k\ C(n,k)\ \ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k}\frac{\pi}{v}(2\nu+1)v\ q_{k-\nu}\}\Big|$$

$$\leq \frac{1}{2\pi(1+r)^n}\Big|\sum_{k=0}^{n} k\ C(n,k)\ \ r^{n-k}\ (2k+1)\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\Big\}\Big|$$

$$= O(n^2)$$

6.2. Proof of Lemma-2

We know, $\frac{1}{sin\left(\frac{v}{2}\right)} \leq \frac{\pi}{v}$, $\Big(0 < v \leq \pi\Big)$; $sin\ v \leq v, v > 0$;
and $|sinv| \leq 1, |cosv| \leq 1$ for all t.
Clearly, for $v \in (0, \pi]$,

$$|\overline{Y_2}'(v)| = \Big|\frac{-1}{4\pi(1+r)^n}\sum_{k=0}^{n} C(n,k)\ \ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\ \ \frac{cos\ \nu v}{sin^2\frac{v}{2}}\Big\}\Big|$$

$$= O\Big(\frac{1}{v^2}\Big)$$

6.3. Proof of Lemma-3

See [12].

7. Proof of the Main Theorem

Let $\overline{S_k}'(h;x)$ denotes the k-th partial sum of the conjugate derived Fourier series, we have

$$\overline{S_k}'(h;x) - \overline{h}'(x) = \frac{-2}{\pi}\int_0^{\pi}\frac{\Psi(x,v)}{4sin\frac{v}{2}}\ \Big(k+\frac{1}{2}\Big)\ sin\Big(k+\frac{1}{2}\Big)v\ dv\ \ -\frac{1}{\pi}\int_0^{\pi}\frac{\Psi(x,v)}{4sin\frac{v}{2}}\ \frac{cos\Big(k+\frac{1}{2}\Big)v}{tan\frac{v}{2}}\ \ dv$$

where $\overline{h'}$ is the conjugate derived function of 2π periodic function h, which is given by

$$\overline{h}'(x) = \frac{1}{4\pi}\int_0^{\pi}\Psi(x;v)\ cosec^2\frac{v}{2}\ dv$$

and the (N, q_n) transform of it is given by

$$\frac{1}{Q_n}\sum_{k=0}^{n} q_{n-k}\{\overline{S_k}'(h;x) - \overline{h}'(x)\} = \frac{-2k}{\pi}\int_0^{\pi}\rho(x,v)\frac{1}{Q_n}\sum_{k=0}^{n} q_{n-k} sin\Big(k+\frac{1}{2}\Big)v\ dv$$
$$-\frac{1}{\pi}\int_0^{\pi}\rho(x,v)\frac{1}{Q_n}\sum_{k=0}^{n} q_{n-k}\frac{coskv}{sin\frac{v}{2}}\ dv$$

where $\rho(x,v) = \frac{\Psi(x;v)}{4sin\frac{v}{2}}$.

Denoting the $(E,r)(N,q_n)$ transform of $\overline{S_k}'(h;x)$ by $\overline{\tau_n}'^{E_r,N}$. Then,

$$\overline{\tau_n}'^{E_r,N} - \overline{h}'(x) = \frac{-2k}{\pi(1+r)^n}\int_0^{\pi}\rho(x,v)\sum_{k=0}^{n} C(n,k)\ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu} sin\Big(\nu+\frac{1}{2}\Big)v\ \Big\}dv$$
$$-\frac{1}{\pi(1+r)^n}\int_0^{\pi}\rho(x,v)\sum_{k=0}^{n} C(n,k)\ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{cos\nu v}{sin\frac{v}{2}}\Big\}\ dv$$

$$\overline{\tau_n}'^{E_r,N} - \overline{h}'(x) = \frac{-2k}{4\pi(1+r)^n}\int_0^{\pi}\Psi(x,v)\sum_{k=0}^{n} C(n,k)\ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{sin\Big(\nu+\frac{1}{2}\Big)v}{sin\frac{v}{2}}\ \Big\}dv$$
$$-\frac{1}{4\pi(1+r)^n}\int_0^{\pi}\Psi(x,v)\sum_{k=0}^{n} C(n,k)\ r^{n-k}\Big\{\frac{1}{Q_k}\sum_{\nu=0}^{k} q_{k-\nu}\frac{cos\nu v}{sin^2\frac{v}{2}}\Big\}\ dv$$
$$= \int_0^{\pi}\Psi(x,v)\{\overline{Y_1'}(v) + \overline{Y_2'}(v)\}\ dv$$
$$= {\chi_n}'(x), \quad \text{(say)}$$

Then,

$${\chi_n}'(x+y) + {\chi_n}'(x-y) = \int_0^{\pi}\Big\{\Psi(x+y,v) + \Psi(x-y,v)\Big\}\ \{\overline{Y_1'}(v) + \overline{Y_2'}(v)\}\ dv$$

Using Minkowski's inequality, we have

$$\|{\chi_n}'(.+y) + {\chi_n}'(.-y)\|_l = \Big\{\frac{1}{2\pi}\int_0^{2\pi}|{\chi_n}'(x+y) + {\chi_n}'(x-y)|^l\ dx\Big\}^{\frac{1}{l}}$$
$$\le \int_0^{\pi}\Big\{\frac{1}{2\pi}\int_0^{2\pi}|\Psi(x+y,v) + \Psi(x-y,v)|^l dx\Big\}^{\frac{1}{l}}|\overline{Y_1'}(v) + \overline{Y_2'}(v)|\ dv$$
$$= \int_0^{\pi}\|\Psi(.+y,v) + \Psi(.-y,v)\|_l\ |\overline{Y_1'}(v) + \overline{Y_2'}(v)|\ dv$$
$$= \int_0^{\frac{1}{n+1}}\|\Psi(.+y,v) + \Psi(.-y,v)\|_l\ |\overline{Y_1'}(v) + \overline{Y_2'}(v)|\ dv$$
$$+ \int_{\frac{1}{n+1}}^{\pi}\|\Psi(.+y,v) + \Psi(.-y,v)\|_l\ |\overline{Y_1'}(v) + \overline{Y_2'}(v)|\ dv$$

$$= I_1^{'} + I_2^{'}, \quad \text{(say)} \tag{7}$$

Further,

$$|\Psi(x+y;v) + \Psi(x-y;v)| \leq \|h(x+y+v) + h(x+y-v)\| + \|h(x-y+v) + h(x-y-v)\|$$

By Minkowski's inequality, we have

$$\begin{aligned}|\Psi(.+y;v) + \Psi(.-y;v)|_l &\leq \|h(.+y+v) + h(.+y-v)\|_l + \|h(.-y+v) + h(.-y-v)\|_l \\ &= O(m(v)) \text{ or } O(m(y))\end{aligned}$$

Again, by using lemma-1, lemma-3 and monotonicity of $\frac{m(v)}{\mu(v)}$, we get

$$\begin{aligned}I_1^{'} &= \int_0^{\frac{1}{n+1}} \|\Psi(.+y,v) + \Psi(.-y,v)\|_l |\overline{Y_1^{'}}(v) + \overline{Y_2^{'}}(v)|\, dv \\ &\leq O\Big(\int_0^{\frac{1}{n+1}} \mu(y)\frac{m(v)}{\mu(v)}\, n^2\, dv\Big) + O\Big(\int_0^{\frac{1}{n+1}} \mu(y)\frac{m(v)}{\mu(v)}\, \frac{1}{v^2}\, dv\Big) \\ &= O\Bigg(n^2\mu(y)\, \frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Bigg) + O\Big(\mu(y)\int_0^{\frac{1}{n+1}} \frac{m(v)}{\mu(v)}\, \frac{1}{v^2}\, dv\Big) \end{aligned} \tag{8}$$

Similarly, by using lemma-2, lemma-3 and monotonicity of $\frac{m(v)}{\mu(v)}$, we get

$$\begin{aligned}I_2^{'} &= \int_{\frac{1}{n+1}}^{\pi} \|\Psi(.+y,v) + \Psi(.-y,v)\|_l |\overline{Y_1^{'}}(v) + \overline{Y_2^{'}}(v)|\, dv \\ &= O\Big(n^2\mu(y)\, \int_0^{\frac{1}{n+1}} \frac{m(v)\, dv}{\mu(v)}\Big) + O\Bigg((n+1)\mu(y)\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Bigg)\end{aligned} \tag{9}$$

By, (7), (8) and (9)

$$\begin{aligned}&\|{\chi_n}'(.+y) + {\chi_n}'(.-y)|_l \\ &= O\Bigg(n^2\mu(y)\, \frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Bigg) + O\Big(\mu(y)\int_0^{\frac{1}{n+1}} \frac{m(v)}{\mu(v)}\, \frac{1}{v^2}\, dv\Big) \\ &+ O\Big(n^2\mu(y)\, \int_0^{\frac{1}{n+1}} \frac{m(v)\, dv}{\mu(v)}\Big) + O\Bigg((n+1)\mu(y)\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Bigg)\end{aligned}$$

Therefore, we have

$$\sup_{y\neq 0} \frac{\|{\chi_n}'(.+y) + {\chi_n}'(.-y)\|_l}{\mu(y)}$$

$$= O\Big(n^2 \ \frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Big) + O\Big(\int_0^{\frac{1}{n+1}} \frac{m(v)}{\mu(v)} \ \frac{1}{v^2} \ dv\Big)$$

$$+O\Big(n^2 \ \int_0^{\frac{1}{n+1}} \frac{m(v) \ dv}{\mu(v)}\Big) + O\Big((n+1)\frac{m\Big(\frac{1}{n+1}\Big)}{\mu\Big(\frac{1}{n+1}\Big)}\Big) \tag{10}$$

Since, $h \in Z_l^{(m)}$ and $\Psi(x;v) = |h(x+v) + h(x-v)|$, by Minkowski's inequality, we have

$$\|\Psi(x,v)\|_l = \|h(x+v) + h(x-v)\|_l \ = \ O\Big(m(v)\Big)$$

Therefore,

$$\|{\chi_n}'(.)\|_l \leq \Big(\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\Big) \ \|\Psi(.,v)\|_l| \ \ \overline{Y_1}'(v) + \overline{Y_2}'(v)| \ dv$$

$$= O\Big(n^2 \ \int_0^{\frac{1}{n+1}} m(v) \ dv\Big) + O\Big(\int_0^{\frac{1}{n+1}} \frac{m(v)}{v^2} \ dv\Big)$$

$$+O\Big(n^2 \ \int_{\frac{1}{n+1}}^{\pi} m(v) \ dv\Big) + O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v^2} dv\Big) \tag{11}$$

From (10) , (11) and by the monotonicity of $\mu(v)$ we have

$$\|{\chi_n}'(.)\|_l^{\mu} = \|{\chi_n}'(.)\|_l + \sup_{y\neq 0} \frac{\|{\chi_n}'(.+y) + {\chi_n}'(.-y)\|_l}{\mu(y)}$$

$$= O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v^2 \ \mu(v)} dv\Big)$$

provided

$$\int_0^{\eta} \frac{m(v)}{v \ \mu(v)} dv \ = \ O\Big(\frac{m(\eta)}{\mu(\eta)}\Big)$$

Hence,

$$E_n(h) = \inf_n \|{\chi_n}'(.)\|_l^{\mu} = O\Big(\int_{\frac{1}{n+1}}^{\pi} \frac{m(v)}{v^2 \ \mu(v)} dv\Big)$$

This completes the proof of our main theorem.

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