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The Structure of Pseudo-BF/BF^*-algebra

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Abstract. In this paper, we study the structure of pseudo-BF/BF^*-algebra as a generalization of BF-algebra. We show how pseudo-BF/BF^*-algebra and pseudo-BCK-algebra are related. We study some elementary properties related to pseudo-BF-algebra and pseudo-BF^*-algebra.

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1. Introduction

Through the work of the Japanese mathematicians Imai and Iseki the notions of BCK/BCI-algebra were introduced (see [7] and [8]). Neggers and Sik introduced the concept of B-algebra, and obtained several results (we refer the reader to [13] for more details). In [17], Walendziak introduced a generalization of B-algebra named BF-algebra and investigated some properties of ideals and normal-ideals in BF-algebra and gave some characterization of them. In [6], Georgescu and Iorgulescu introduced an extension of BCK-algebra called pseudo-BCK-algebra. Moreover, they gave the connection of pseudo-BCK-algebra with pseudo-MV-algebra and with pseudo-BL-algebra. Dudek and Jun introduced the notion pseudo-BCI-algebra as a natural generalization of BCI-algebra and of pseudo-BCK-algebra and investigated some of their properties. They gave some conditions for a pseudo-BCI-algebra to be a pseudo-BCK-algebra (see [4] for more details). In [10], Jun, Kim and Neggers studied pseudo-atoms, pseudo-ideals and pseudo-homomorphisms in pseudo-BCI-algebra. In [12], Kim and So discussed minimality on elements in pseudo-BCI-algebra and concluded some of the properties in B-algebra. Walendziak in [18] introduced the notion of pseudo-BCH-algebra and investigated some properties and gave conditions to when a pseudo-BCH-algebra be a pseudo-BCI-algebra. The authors G. Georgescu and A. Iorgulescu in [5], and independently Rachunek in [15],

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studied a non-commutative generalization of the MV-algebra named pseudo-MV-algebra. In [16], pseudo-BL-algebra was introduced as a generalization of BL-algebra and pseudo-MV-algebra and basic properties, filters, normal-filters and congruences were given. Di Nola, Georgescu and Iorgulescu, in [14], investigated pseudo-BL-algebra including definition, basic properties, filters, normal-filters and congruences. Moreover, they gave some important classes of pseudo-BL-algebra and some results concerning the pseudo-BL-chains. In [11], Jun, Kim and Neggers introduced the notion of pseudo-d-algebra as an extension of d-algebra and they showed that the class of pseudo-d-algebra can be included in the class of coupled d-algebra. In [1], the authors, introduced the concept of pseudo-BE-algebra. They studied the concepts of pseudo-subalgebra, pseudo-filter and pseudo-upper-set and proved that every pseudo-filter is a union of pseudo-upper-sets. In [9], Jun and Ahn studied some properties of pseudo-BH-algebra. Furthermore, they introduced the concept of pseudo-complicated-BH-algebra and got some related properties. In [3], Ciungu introduced and investigated pointed-pseudo-BE-algebra and commutative-pseudo-BE-algebra and proved that the class of commutative-pseudo-BE-algebra and the class of commutative-pseudo-BCK-algebra are equivalent.

In this paper, we study the structure of pseudo-BF/BF^*-algebra. We introduce,in the second section, the notion of pseudo-BF/BF^*-algebra and find the relation between pseudo-BF/BF^*-algebra with pseudo-BCK-algebra. In the third section, we study pseudo-subalgebra, pseudo-ideal and pseudo-normal-ideal of pseudo-BF-algebra. We study pseudo-atoms of pseudo-BF/BF^*-algebra in the last section.

We start by recalling the definitions and elementary properties related to the paper.

Definition 1. *[17, Definition 2.1] An algebra $(E;\bullet,0)$ of type $(2,0)$ is called a BF-algebra if the following axioms are satisfies the following axiom, for all $a,b\in E$:*

(BF(1)) $a\bullet a=0,$

(BF(2)) $a\bullet 0=a,$

(BF(3)) $0\bullet(a\bullet b)=b\bullet a.$

Definition 2. *[2, Definition 2.3] In BF-algebra $(E;\bullet,0)$, we can define a binary relation "$\leq$" on E as follows:*

$$a\leq b \text{ if and only if } a\bullet b=0 \quad \text{for all } a,b\in E.$$

Any BF-algebra, satisfies the properties given in the following Proposition.

Proposition 1. *[17, Proposition 2.5] Let $(E;\bullet,0)$ be a BF-algebra, then,*

(1) $0\bullet(0\bullet a)=a$ *for all* $a\in E$,

(2) *if* $0\bullet a=0\bullet b$, *then* $a=b$ *for all* $a,b\in E$,

(3) *if* $a\bullet b=0$, *then* $b\bullet a=0$ *for all* $a,b\in E$.

We give next the definition of pseudo-BCK-algebra.

Definition 3. *[6, Definition 3] An algebra $(E;\leq,\bullet,\star,0)$ of type $(2,2,0)$, where "$\leq$" is a binary relation on a set E, "$\bullet$" and "$\star$" are binary operations on E and "0" is a constant of E, is called a pseudo-BCK-algebra if the following are satisfied: $\forall a,b,c\in E$,*

$(pBCK(1))$ $(a\bullet b)\star(a\bullet c)\leq c\bullet b$ *and* $(a\star b)\bullet(a\star c)\leq c\star b$,

$(pBCK(2))$ $a\star(a\bullet b)\leq b$ *and* $a\bullet(a\star b)\leq b$,

$(pBCK(3))$ $a\leq a$,

$(pBCK(4))$ $0\leq a$,

$(pBCK(5))$ $a\leq b$ *and* $b\leq a$ *then* $a=b$,

$(pBCK(6))$ $a\leq b\Leftrightarrow a\bullet b=0$ *if and only if* $a\star b=0$.

Theorem 1. *[6, Theorem 7] In a pseudo-BCK-algebra $(E;\leq,\bullet,\star,0)$, for all $a,b,c\in E$ we have*

$$(a\bullet b)\star c=(a\star c)\bullet b.$$

Theorem 2. *[6, Theorem 8] In any pseudo-BCK-algebra $(E;\leq,\bullet,\star,0)$ we have, for all $a,b,c\in E$:*

(1) $a\bullet b\leq c$ *if and only if* $a\star c\leq b$,

(2) $a\bullet b\leq a$ *and* $a\star b\leq a$.

2. Pseudo-BF/BF^*-algebra

In this section, we give a generalization of BF-algebra named pseudo-BF-algebra and study its structure. Also, we will introduce pseudo-BF^*-algebra and find the relation between pseudo-BF/BF^*-algebra and pseudo-BCK-algebra.

Definition 4. *An algebra $(E;\bullet,\star,0)$ of type $(2,2,0)$ is said to be a pseudo-BF-algebra, if the following axioms are satisfied for all $a,b\in E$:*

$(pBF(1))$ $a\bullet a=0$ *and* $a\star a=0$,

$(pBF(2))$ $a\bullet 0=a$ *and* $a\star 0=a$,

$(pBF(3))$ $0\bullet(a\star b)=b\star a$ *and* $0\star(a\bullet b)=b\bullet a$.

The following examples illustrates the definition.

Example 1. *Consider the group $(G;+,0)$, where "$+$" is the usual addition. Define the operations "$\bullet$" and "$\star$" on G by:*

$$a \bullet b = (-b) + a \text{ and } a \star b = (-b) + a \text{ for all } a, b \in G$$

then $(G; \bullet, \star, 0)$ *is a pseudo-BF-algebra.*

Note: It is obvious that in any pseudo-BF-algebra E if $a \bullet b = a \star b$ for all $a, b \in E$ then E is a BF-algebra.

Example 2. *Define the operations "$\bullet$" and "$\star$" on* $E = \{0, 1, 2, 3\}$*, by the following Cayley tables:*

***Table* 1**

$\bullet$	0	1	2	3
0	0	1	2	3
1	1	0	3	0
2	2	3	0	2
3	3	0	2	0

***Table* 2**

$\star$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then $(E; \bullet, 0)$ *and* $(E; \star, 0)$ *are BF-algebras (shown in [17]). It is obvious that* $a \bullet a = 0$ *and* $a \star a = 0$*. Moreover,* $a \bullet 0 = a$ *and* $a \star 0 = a$*. It is direct to check that* $0 \bullet (a \star b) = b \star a$ *and* $0 \star (a \bullet b) = b \bullet a$ *is satisfied for all* $a, b \in E$*. Thus* $(E; \bullet, \star, 0)$ *is a pseudo-BF-algebra.*

Corollary 1. *Any two BF-algebras does not necessarily construct a pseudo-BF-algebra. Moreover, if* $(\mathbb{R}; \bullet, \star, 0)$ *is a pseudo-BF-algebra then it is not necessary for both* $(\mathbb{R}; \bullet, 0)$ *and* $(\mathbb{R}; \star, 0)$ *to be a BF-algebra. The following two examples proves the Corollary.*

Example 3. *Define the operations "$\bullet$" and "$\star$" on* $E = \{0, 1, 2, 3, 4, 5\}$*, by the following Cayley tables:*

***Table* 3**

$\bullet$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

***Table* 4**

$\star$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	1	0
2	2	3	0	0	0	2
3	3	2	0	0	3	1
4	4	1	0	3	0	0
5	5	0	2	1	0	0

Then $(E; \bullet, 0)$,$(E; \star, 0)$ *are BF-algebras but* $(E; \bullet, \star, 0)$ *is not since* $0 \bullet (0 \star 1) = 0 \bullet 1 = 2 \neq 1 \star 0 = 1$.

Example 4. *Let* $\mathbb{R}$ *be the set of real numbers. Define the operations "$\bullet$" and "$\star$" on* $\mathbb{R}$ *for all* $a, b \in \mathbb{R}$ *by:*

$$a \bullet b = \begin{cases} a & \text{if } b = 0, \\ b & \text{if } a = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad a \star b = \begin{cases} a & \text{if } b = 0, \\ 0 & \text{if } a = 0, a = b, \\ b \star a & \text{otherwise.} \end{cases}$$

Then $(\mathbb{R}; \bullet, \star, 0)$ is a pseudo-BF-algebra. The algebra $(\mathbb{R}; \bullet, 0)$ is BF-algebra [17], but the algebra $(\mathbb{R}; \star, 0)$ is not.

Proposition 2. *If $(E; \bullet, \star, 0)$ is a pseudo-BF-algebra for all $a, b \in E$ then*

(1) $0 \bullet (0 \bullet a) = a$ *and* $0 \star (0 \star a) = a$,

(2) $0 \star (0 \bullet a) = a$ *and* $0 \bullet (0 \star a) = a$,

(3) $0 \bullet a = 0 \star b$, *implies* $a = b$.

Proof.

(1) By $(pBF(2))$, $(pBF(3))$ and let $a \in E$ then $0 \bullet (0 \bullet a) = 0 \bullet [0 \star (a \bullet 0)] = 0 \bullet (0 \star a) = a \star 0 = a$ and $0 \star (0 \star a) = 0 \star [0 \bullet (a \star 0)] = 0 \star (0 \bullet a) = a \bullet 0 = a$.

(2) Let $a \in E$. By $(pBF(2))$ and $(pBF(3))$ we obtain $0 \star (0 \bullet a) = a \bullet 0 = a$ and $0 \bullet (0 \star a) = a \star 0 = a$, that is (2) holds.

(3) Let $0 \bullet a = 0 \star b$, then it follows from (1) and (2) that $a = 0 \star (0 \bullet a) = 0 \star (0 \star b) = b$.

Corollary 2. *In a pseudo-BF-algebra $(E; \bullet, \star, 0)$, $a \bullet b = 0$ does not imply $b \star a = 0$ and similarly $a \star b = 0$ does not imply $b \bullet a = 0$.* $\quad \forall a, b \in E$.

Proof. Let $a, b \in E$ and $a \bullet b = 0$. Then $0 = 0 \star 0 = 0 \star (a \bullet b) = b \bullet a$. Then it is not necessary that $b \star a = 0$. Similarly, if $a \star b = 0$ then it is not necessary that $b \bullet a = 0$.

Note: From the proof of (Corollary 2) we see that if $a \bullet b = 0$, then $b \bullet a = 0$ and if $a \star b = 0$, then $b \star a = 0$, for all $a, b \in E$.

As in BF-algebra, a binary relation "$\leq$" could be defined in pseudo-BF-algebra as follows:

$$a \leq b \Leftrightarrow a \bullet b = 0 \Leftrightarrow a \star b = 0 \quad \forall a, b \in E.$$

Therefore we can rewrite the definition of a pseudo-BF-algebra with a binary relation "$\leq$" as follows:

Definition 5. *The algebra $(E; \leq, \bullet, \star, 0)$ where "$\leq$" is a binary relation on a set E, "$\bullet$" and "$\star$" are binary operations on E and "0" is an element of E, is said to be a pseudo-BF-algebra if for all $a, b, c \in E$ the following axioms are satisfied:*

$(pBF$**(1')**$)$ $a \leq a$,

$(pBF$**(2')**$)$ $a \bullet 0 \leq a$ *and* $a \star 0 \leq a$,

$(pBF$**(3')**$)$ $0 \bullet (a \star b) \leq b \star a$ *and* $0 \star (a \bullet b) \leq b \bullet a$,

$(pBF$**(4')**$)$ $a \leq b \Leftrightarrow a \bullet b = 0 \Leftrightarrow a \star b = 0$.

Proposition 3. *The following proposition holds in any pseudo-BF-algebra $(E; \leq, \bullet, \star, 0)$,:*

$$0 \leq a \text{ implies } a = 0 \quad \forall a \in E.$$

Proof. Since $0 \leq a$, we have $0 \bullet a = 0 \star a = 0$ from $(pBF(4'))$. Using (Proposition 2 (1)), $(pBF(1'))$ and $(pBF(4'))$ we get $a = 0 \bullet (0 \bullet a) = 0 \bullet 0 = 0$.

Next we introduce pseudo-BF^*-algebra and we find some results.

Definition 6. *A pseudo-BF-algebra $(E; \bullet, \star, 0)$ is called a pseudo-BF^*-algebra, for all $a, b, c \in E$ if it satisfies the following identity:*

$$(pBF^*) \quad (a \bullet b) \star c = (a \star c) \bullet b.$$

We can see that any pseudo-BF^*-algebra is a pseudo-BF-algebra and any pseudo-BF-algebra satisfying (pBF^*) is a pseudo-BF^*-algebra.

Example 5. *In Example 1, it is straight forward to see that $(G; \bullet, \star, 0)$ is a pseudo-BF^*-algebra.*

Example 6. *In Example 2, $(E; \bullet, \star, 0)$ is not a pseudo-BF^*-algebra, as* $(1 \bullet 1) \star 2 = 0 \star 2 = 2 \neq (1 \star 2) \bullet 1 = 1 \bullet 1 = 0$.

Proposition 4. *Let $(E; \leq, \bullet, \star, 0)$ be a pseudo-BF^*-algebra. The following axioms are satisfied for any $a, b, c \in E$:*

(1) $a \leq 0$ *implies* $a = 0$,

(2) $a \bullet (a \star b) \leq b$ *and* $a \star (a \bullet b) \leq b$,

(3) $a \bullet b \leq c$ *if and only if* $a \star c \leq b$,

(4) $0 \bullet (a \bullet b) = (0 \star a) \star (0 \bullet b)$,

(5) $0 \star (a \star b) = (0 \bullet a) \bullet (0 \star b)$,

(6) $0 \bullet a = 0 \star a$.

Proof.

(1) Let $a \leq 0$. Then $a \bullet 0 = a \star 0 = 0$ by $(pBF(4'))$. Multiplying by "a" from the right we have $0 \star a = (a \bullet 0) \star a = (a \star a) \bullet 0 = 0 \bullet 0 = 0$ and $0 \bullet a = (a \star 0) \bullet a = (a \bullet a) \star 0 = 0 \star 0 = 0$, using (pBF^*) and $(pBF(1'))$. Now, using (Proposition 2 (1)) and $(pBF(1'))$, we get $a = 0 \bullet (0 \bullet a) = 0 \bullet 0 = 0$.

(2) From (pBF^*), $(pBF(1'))$ and $(pBF(4'))$, we have $[a \bullet (a \star b)] \star b = (a \star b) \bullet (a \star b) = 0$ and $[a \star (a \bullet b)] \bullet b = (a \bullet b) \star (a \bullet b) = 0$. Thus $a \bullet (a \star b) \leq b$ and $a \star (a \bullet b) \leq b$.

(3) By (pBF^*) and $(pBF(4'))$ we have $a \bullet b \leq c \Leftrightarrow (a \bullet b) \star c = 0 \Leftrightarrow (a \star c) \bullet b = 0 \Leftrightarrow a \star c \leq b$.

(4) Let $a, b \in E$. Then by using $(pBF(1'))$, $(pBF(4'))$ and (pBF^*) when needed we have $(0 \star a) \star (0 \bullet b) = ([(a \bullet b) \bullet (a \bullet b)] \star a) \star (0 \bullet b) = ([(a \bullet b) \star a] \bullet (a \bullet b)) \star (0 \bullet b) = ([(a \star a) \bullet b] \bullet (a \bullet b)) \star (0 \bullet b) = ((0 \bullet b) \bullet (a \bullet b)) \star (0 \bullet b) = ((0 \bullet b) \star (0 \bullet b)) \bullet (a \bullet b) = 0 \bullet (a \bullet b)$.

(5) Can be proved as (4).

(6) Let $a \in E$. From $(pBF(1'))$, $(pBF(4'))$ and (pBF^*) we have $0 \bullet a = (a \star a) \bullet a = (a \bullet a) \star a = 0 \star a$.

Theorem 3. *In a pseudo-BF^*-algebra $(E; \leq, \bullet, \star, 0)$, we have:*

$$a \leq b \text{ and } b \leq a \text{ imply } a = b, \quad \text{for all } a, b \in E.$$

Proof. Let $a \leq b$ and $b \leq a$ then $a \bullet b = 0$, $a \star b = 0$ and $b \bullet a = 0$, $b \star a = 0$. By (Proposition 2 (2)), we have $a = 0 \star (0 \bullet a) = 0 \star [(a \star b) \bullet a]$. By using (pBF^*), $(pBF(1'))$ and $(pBF(4'))$ we get $0 \star [(a \star b) \bullet a] = 0 \star [(a \bullet a) \star b] = 0 \star (0 \star b)$. By (Proposition 2 (1)), we get $0 \star (0 \star b) = b$. The proof is complete.

The relation between pseudo-BCK-algebra and pseudo-BF/BF^*-algebra is given in the following theorems.

Theorem 4. *Any pseudo-BCK-algebra is a pseudo-BF-algebra.*

Proof. Let $(E; \leq, \bullet, \star, 0)$ be a pseudo-BCK-algebra. The axioms $(pBF(1'))$, $(pBF(4'))$ are clearly the axioms $(pBCK(3))$, $(pBCK(6))$. Put $b = 0$ in (Theorem 2 (2)) we get $a \bullet 0 \leq a$ and $a \star 0 \leq a$. Then the axiom $(pBF(2'))$ holds. Now, we will show $(pBF(3'))$. By $(pBCK(4))$ and $(pBCK(6))$ we get $[0 \bullet (a \star b)] \bullet (b \star a) = 0 \bullet (b \star a) = 0$ and $[0 \star (a \bullet b)] \star (b \bullet a) = 0 \star (b \bullet a) = 0$ and so $0 \bullet (a \star b) \leq b \star a$ and $0 \star (a \bullet b) \leq b \bullet a$. Thus E is a pseudo-BF-algebra.

Theorem 5. *Any pseudo-BCK-algebra is a pseudo-BF^*-algebra.*

Proof. It is obvious from (Theorem 4) above and by using (Theorem 1) that $(a \bullet b) \star c = (a \star c) \bullet b$ (that is (pBF^*)). Therefore every pseudo-BCK-algebra is a pseudo-BF^*-algebra.

3. Pseudo-Ideal of Pseudo-BF-algebra

In this section, we start with the definition of pseudo-subalgebra of pseudo-BF-algebra. Then we study pseudo-ideal and pseudo-normal-ideal. We start with the following definition.

Definition 7. *In a pseudo-BF-algebra $(E; \bullet, \star, 0)$, let $\phi \neq S \subseteq E$. Then S is said to be a pseudo-subalgebra of E if:*

$$a \bullet b \in S \text{ and } a \star b \in S \quad \text{for all } a, b \in S.$$

Note: It is easy to see that if S is a pseudo-subalgebra of E, then $0 \in S$.

Lemma 1. *In a pseudo-BF-algebra* $(E; \bullet, \star, 0)$*, let* S *be a pseudo-subalgebra of* E*. Then for* $a, b \in E$ *we have:*

(1) *If* $a \bullet b \in S$*, then* $b \bullet a \in S$*,*

(2) *If* $a \star b \in S$*, then* $b \star a \in S$*.*

Proof. For $a, b \in S$, let $a \bullet b \in S$ and $a \star b \in S$. By $(pBF(3))$, $b \bullet a = 0 \star (a \bullet b)$. Since $0 \in S$ and $a \bullet b \in S$, we see that $0 \star (a \bullet b) \in S$ and so $b \bullet a \in S$ and $b \star a = 0 \bullet (a \star b)$. Since $0 \in S$ and $a \star b \in S$, we see that $0 \bullet (a \star b) \in S$ and so $b \star a \in S$.

Definition 8. *In a pseudo-BF-algebra* $(E; \bullet, \star, 0)$*, let* $\phi \neq I \subseteq E$*. Then we say that* I *is a pseudo-ideal of* E *if it satisfies for all* $a, b \in E$*:*

(pI1) $0 \in I$*,*

(pI2) $a \bullet b \in I$*,* $a \star b \in I$ *and* $b \in I$ *implies* $a \in I$*.*

Example 7. *In Example 2, let* $C = \{0, 1\}$*,* $A = \{0, 3\}$ *and* $F = \{0, 1, 2\}$ *be subsets of* E*. Then* C *is a pseudo-subalgebra of* E*, whereas* F *is not, as* $1 \bullet 2 = 3 \notin F$*.*
Also, A *is a pseudo-ideal of* E*, but* C *is not, because* $3 \bullet 1 = 0, 3 \star 1 = 1 \in C$*,* $1 \in C$*, but* $3 \notin C$*.*

Definition 9. *In a pseudo-BF-algebra* $(E; \bullet, \star, 0)$*, let* I *be a pseudo-ideal. We say that* I *is a pseudo-normal, if for any* $a, b, c \in E$*:*

$$a \bullet b,\ a \star b \in I \text{ implies } (c \bullet a) \star (c \bullet b) \text{ and } (c \star a) \bullet (c \star b) \in I.$$

Note: $\{0\}$ and E are always pseudo-ideals of E. Whereas if E is a pseudo-normal, $\{0\}$ is not a pseudo-normal in general.

Lemma 2. *Let* I *be a pseudo-normal-ideal of a pseudo-BF-algebra* $(E; \bullet, \star, 0)$ *and* $a, b \in E$*. Then,*

(1) $a \in I \Rightarrow 0 \bullet a \in I$ *and* $0 \star a \in I$*,*

(2) $a \bullet b$ *,* $a \star b \in I \Rightarrow b \bullet a \in I$ *and* $b \star a \in I$*.*

Proof.

(1) Let $a \in I$. Then by $(pBF(2))$ we have $a = a \bullet 0 \in I$ and so $a = a \star 0 \in I$. Since I is a pseudo-normal-ideal, we get $(0 \bullet a) \star (0 \bullet 0)$ and $(0 \star a) \bullet (0 \star 0) \in I$. By $(pBF(1))$ then $(0 \bullet a) \star 0$ and $(0 \star a) \bullet 0 \in I$ and $0 \in I$ from $(pI1)$. By $(pI2)$ we get $(0 \bullet a)$, $(0 \star a) \in I$.

(2) Let $a \bullet b$, $a \star b \in I$. By (1) we get $0 \star (a \bullet b)$, $0 \bullet (a \star b) \in I$. Applying $(pBF(3))$ we have $b \bullet a$, $b \star a \in I$.

Proposition 5. *In a pseudo-BF-algebra* $(E;\bullet,\star,0)$*, let* I *be a pseudo-normal-ideal. Then* I *is a pseudo-subalgebra that satisfies the following condition:*

$$(pNI)\ \text{ If } a \in E \text{ and } b \in I, \text{ then } a \star (a \bullet b), a \bullet (a \star b) \in I.$$

Proof. Let $a \in E$ and $b \in I$. By (Lemma 2 (1)), $0 \bullet b$, $0 \star b \in I$. We have $(a \bullet 0) \star (a \bullet b)$ and $(a \star 0) \bullet (a \star b) \in I$ as I is a pseudo-normal-ideal. By $(pBF(2))$, $a \star (a \bullet b)$ and $a \bullet (a \star b) \in I$. Thus (pNI) holds.
Now let $a, b \in I$. Therefore $a \star (a \bullet b)$, $a \bullet (a \star b) \in I$. By (Lemma 2 (2)), $(a \bullet b) \star a$, $(a \star b) \bullet a \in I$;$a \in I$. From $(pI2)$ we have $(a \bullet b)$, $(a \star b) \in I$. Thus I is a pseudo-subalgebra satisfying (pNI).

Proposition 6. *In a pseudo-BF-algebra* $(E;\bullet,\star,0)$*, let* I *be a pseudo-ideal. Then for* $a, b \in E$ *where* $b \leq a$*, if* $a \in I$ *, we have* $b \in I$.

Proof.
Let $a \in I$ and $b \leq a$. Thus $b \bullet a = 0$, $b \star a = 0$. By $(pI1)$ and $(pI2)$, we have $0 \in I$ and so having $b \bullet a, b \star a \in I$, $a \in I$ we get $b \in I$.

Theorem 6. *In a pseudo-BF-algebra* $(E;\bullet,\star,0)$*, let* $\phi \neq I \subseteq E$*. Then* I *is a pseudo-ideal of* E *if and only if the following hold:*

(1) *For all* $a, b, c \in E$ *,* $a, b \in I$ *and* $c \bullet b \leq a \Longrightarrow c \in I$.

(2) *For all* $a, b, c \in E$ *,* $a, b \in I$ *and* $c \star b \leq a \Longrightarrow c \in I$.

Proof. Let I be a pseudo-ideal of E. Let $a, b, c \in E$, $a, b \in I$ and $c \bullet b \leq a$ we have $(c \bullet b) \star a = 0 \in I$ from $(pI1)$. Since $a \in I$ then $c \bullet b \in I$ by $(pI2)$. Since $b \in I$ then $c \in I$ by $(pI2)$. Thus (1) is valid. Now, let $a, b, c \in E$, $a, b \in I$ and $c \star b \leq a$ we have $(c \star b) \bullet a = 0 \in I$ from $(pI1)$. Since $a \in I$ then $c \star b \in I$ by $(pI2)$. Since $b \in I$ then $c \in I$ by $(pI2)$. Thus (2) is true.
Conversely, suppose that (1), (2) hold. Suppose that $b \in I$. By using (1), (2) we have $0 \bullet b \leq b$ and $0 \star b \leq b$, then $0 \in I$. Now, let $a \bullet b, a \star b \in I$ and $b \in I$. By using (1), (2) we have $a \bullet b \leq a \bullet b$ and $a \star b \leq a \star b$, then $a \in I$. Therefore I is a pseudo-ideal of E.

Theorem 7. *In a pseudo-BF-algebra* $(E;\bullet,\star,0)$*, let* I *be a pseudo-subalgebra. Then* I *is a pseudo-ideal of* E *if and only if for* $a, b \in E$ *if* $a \in I$ *and* $b \notin I$ *then* $b \bullet a$ *and* $b \star a \notin I$.

Proof. Let $a, b \in E$ and let I be a pseudo-ideal of E where $a \in I$ and $b \in E - I$. We prove by contradiction. Let $b \bullet a$, $b \star a \notin E - I$, we have $b \bullet a$, $b \star a \in I$. Since $a \in I$ then $b \in I$ by $(pI2)$. This contradicts the hypothesis $(b \in E - I)$. Hence $b \bullet a$, $b \star a \in E - I$.
Conversely, let $a \in I$ and $b \in E - I \Rightarrow b \bullet a$, $b \star a \in E - I$. Since I is a pseudo-subalgebra, we have $0 \in I$ (by Definition 7). Now, assume that $a, b \in E$, $a \in I$ and $b \bullet a$, $b \star a \in I$. We prove by contradiction. Let $b \notin I$,i.e. $b \in E - I$. Then $b \bullet a$, $b \star a \in E - I$ by hypothesis. This contradicts the hypothesis $(b \bullet a$, $b \star a \in I)$. Hence $b \in I$. Therefore I is a pseudo-ideal of E.

Proposition 7. *In a pseudo-BF-algebra $(E; \bullet, \star, 0)$, let I be a pseudo-ideal. If J is a pseudo-ideal of I, then J is a pseudo-ideal of E as well.*

Proof. Assume that J is a pseudo-ideal of I, then $0 \in J$. Let $b \in J$ and $a \bullet b$, $a \star b \in J$ for any $a \in E$. If $a \in I$, then $a \in J$ since J is a pseudo-ideal of I. If $a \notin I$,i.e. $a \in E - I$, then b, $a \bullet b$, $a \star b \in J \subseteq I$ and so $a \in I$. Hence $a \in J$. Thus J is a pseudo-ideal.

Proposition 8. *In a pseudo-BF-algebra $(E; \bullet, \star, 0)$, let I be a pseudo-ideal. Then*

$$\forall a \in E \text{ , } a \in I \text{ we have } 0 \bullet (0 \star a), \, 0 \star (0 \bullet a) \in I.$$

Proof. Let $a \in I$ and $0 \star a$, $0 \bullet a \in I$, then $0 \in I$ from $(pI1)$ and $(pI2)$. Since $a \in I$ and $0 \in I$, by using $(pBF(1))$ we have $0 = a \star a$, $0 = a \bullet a \in I$. (By Proposition 2 (2)) we obtain $a \star a = [0 \bullet (0 \star a)] \star a$, $a \bullet a = [0 \star (0 \bullet a)] \bullet a \in I$. Thus $0 \bullet (0 \star a)$, $0 \star (0 \bullet a) \in I$ from $(pI2)$.

4. Pseudo-Atoms of Pseudo-BF/BF^*-algebra

In this section we introduce pseudo-atoms of pseudo-BF/BF^*-algebra and prove related properties. We start with the following definition.

Definition 10. *In a pseudo-BF-algebra $(E; \bullet, \star, 0)$, let τ be an element in E. If $a \leq \tau$ implies $a = \tau \quad \forall a \in E$ then we call τ a pseudo-atom of E and the collection of all pseudo-atoms of E is called the center of E and denoted by $L_p(E)$.*

Theorem 8. *In a pseudo-BF^*-algebra $(E; \bullet, \star, 0)$ the following are equivalent for all $a, b, c, d, \tau \in E$:*

(1) *there exists a pseudo-atom τ,*

(2) $\tau = a \star (a \bullet \tau)$ *and* $\tau = a \bullet (a \star \tau)$*;*

(3) $(a \bullet b) \star (a \bullet \tau) = \tau \bullet b$ *and* $(a \star b) \bullet (a \star \tau) = \tau \star b$*;*

(4) $\tau \bullet (a \star b) = b \star (a \bullet \tau)$ *and* $\tau \star (a \bullet b) = b \bullet (a \star \tau)$*,*

(5) $0 \star (b \bullet \tau) = \tau \bullet b$ *and* $0 \bullet (b \star \tau) = \tau \star b$*,*

(6) $0 \star (0 \bullet \tau) = \tau$ *and* $0 \bullet (0 \star \tau) = \tau$*,*

(7) $0 \star (0 \bullet (\tau \star c)) = \tau \star c$ *and* $0 \bullet (0 \star (\tau \bullet c) = \tau \bullet c$*,*

(8) $c \star (c \bullet (\tau \star d)) = \tau \star d$ *and* $c \bullet (c \star (\tau \bullet d)) = \tau \bullet d$*.*

Proof.

(1) ⇒ (2). Assume that τ is a pseudo-atom of E. As $a \star (a \bullet \tau) \leq \tau$ and $a \bullet (a \star \tau) \leq \tau$ by (Proposition 4 (2)), we have $\tau = a \star (a \bullet \tau)$ and $\tau = a \bullet (a \star \tau)$.

(2) ⇒ (3). For all $a \in E$. By (pBF^*) and (2), we have $(a \bullet b) \star (a \bullet \tau) = [a \star (a \bullet \tau)] \bullet b = \tau \bullet b$ and $(a \star b) \bullet (a \star \tau) = [a \bullet (a \star \tau)] \star b = \tau \star b$.

(3) ⇒ (4). Replacing b by $a \star b$ in (3), we get $\tau \bullet (a \star b) = [a \bullet (a \star b)] \star (a \bullet \tau)$. By ($pBF^*$) and (3), we have $[a \bullet (a \star b)] \star (a \bullet \tau) = [a \star (a \bullet \tau)] \bullet (a \star b) = b \star (a \bullet \tau)$. Also, replacing b by $a \bullet b$ in (3), we get $\tau \star (a \bullet b) = [a \star (a \bullet b)] \bullet (a \star \tau)$. By ($pBF^*$) and (3), we have $[a \star (a \bullet b)] \bullet (a \star \tau) = [a \bullet (a \star \tau)] \star (a \bullet b) = b \bullet (a \star \tau)$.

(4) ⇒ (5). Put $b = 0$ and $a = b$ in (4). Hence $\tau \bullet (b \star 0) = 0 \star (b \bullet \tau)$ and $\tau \star (b \bullet 0) = 0 \bullet (b \star \tau)$. From ($pBF(3)$), then $0 \star (b \bullet \tau) = \tau \bullet b$ and $0 \bullet (b \star \tau) = \tau \star b$.

(5) ⇒ (6). Put $b = 0$ in (5). Then it is straightforward that $0 \star (0 \bullet \tau) = \tau \bullet 0 = \tau$ and $0 \bullet (0 \star \tau) = \tau \star 0 = \tau$ by ($pBF(2)$).

(6) ⇒ (7). For any $\tau, c \in E$. By (Proposition 4 (6)), we have $0 \star [0 \bullet (\tau \star c)] = 0 \bullet [0 \bullet (\tau \star c)] = 0 \bullet [0 \star (\tau \star c)]$. By (Proposition 4 (5)), then $0 \bullet [0 \star (\tau \star c)] = 0 \bullet [(0 \bullet \tau) \bullet (0 \star c)]$. By (Proposition 4 (4)), we get $0 \bullet [(0 \bullet \tau) \bullet (0 \star c)] = [0 \star (0 \bullet \tau)] \star [0 \bullet (0 \star c)]$. By (6), then $[0 \star (0 \bullet \tau)] \star [0 \bullet (0 \star c)] = \tau \star c$. Also, by (Proposition 4 (6),(4) and (5), respectively) and (6) we have $0 \bullet [0 \star (\tau \bullet c)] = 0 \star [0 \star (\tau \bullet c)] = 0 \star [0 \bullet (\tau \bullet c)] = 0 \star [(0 \star \tau) \star (0 \bullet c)] = [0 \bullet (0 \star \tau)] \bullet [0 \star (0 \bullet c)] = \tau \bullet c$. Thus (7) holds.

(7) ⇒ (8). For any $c, d, \tau \in E$, we have $\tau \star d = 0 \star [0 \bullet (\tau \star d)] = 0 \star [(c \star c) \bullet (\tau \star d)] = 0 \star ([c \bullet (\tau \star d)] \star c)$ from (7), ($pBF(1)$) and (pBF^*). By (Proposition 4 (5) and (6), respectively) then $0 \star ([c \bullet (\tau \star d)] \star c) = (0 \bullet [c \bullet (\tau \star d)]) \bullet (0 \star c) = (0 \star [c \bullet (\tau \star d)]) \bullet (0 \star c)$. Using ($pBF^*$), $(0 \star [c \bullet (\tau \star d)]) \bullet (0 \star c) = (0 \bullet (0 \star c)) \star [c \bullet (\tau \star d)]$. By (Proposition 4 (6)), we get $(0 \bullet (0 \star c)) \star [c \bullet (\tau \star d)] = (0 \star (0 \star c)) \star [c \bullet (\tau \star d)]$. Using ($pBF(3)$), the hypothesis and ($pBF(2)$), respectively we have $(0 \star (0 \star c)) \star [c \bullet (\tau \star d)] = (0 \star [0 \bullet (c \star 0)]) \star [c \bullet (\tau \star d)] = (c \star 0) \star [c \bullet (\tau \star d)] = c \star [c \bullet (\tau \star d)]$. Similarly $c \bullet [c \star (\tau \bullet d)] = \tau \bullet d$ is proved.

(8) ⇒ (1). Let $c \leq \tau$ we have $c \bullet \tau = c \star \tau = 0$. By ($pBF(2)$) we have $\tau = \tau \bullet 0$. Then by (8) with $d = 0$ we obtain $\tau \bullet 0 = c \bullet [c \star (\tau \bullet 0)]$. Using ($pBF(2)$) we have $c \bullet [c \star (\tau \bullet 0)] = c \bullet [c \star \tau] = c \bullet 0 = c$. Thus τ is a pseudo-atom of E.

Corollary 3. *In a pseudo-BF*-algebra $(E; \bullet, \star, 0)$, let τ be a pseudo-atom of E. Then $\tau \bullet a$ and $\tau \star a$ are pseudo-atoms, for all $a \in E$. Hence $L_p(E)$ is a pseudo-subalgebra of E.*

Proof. For $a, b \in E$, let $b \leq \tau \bullet a$ and $b \leq \tau \star a$ then $b \star (\tau \bullet a) = 0$ and $b \bullet (\tau \star a) = 0$. Multiplying by "$b$" from the right we have $(\tau \bullet a) \star b = 0 \star (0 \bullet [(\tau \bullet a) \star b])$ and $(\tau \star a) \bullet b = 0 \bullet (0 \star [(\tau \star a) \bullet b])$ from (Theorem 8 (7)). By ($pBF(3)$) we get $0 \star (0 \bullet [(\tau \bullet a) \star b]) = 0 \star [b \star (\tau \bullet a)]$ and $0 \bullet (0 \star [(\tau \star a) \bullet b]) = 0 \bullet [b \bullet (\tau \star a)]$. By the hypothesis ($b \star (\tau \bullet a) = 0$ and $b \bullet (\tau \star a) = 0$) and ($BF(1)$) we have $0 \star [b \star (\tau \bullet a)] = 0 \star 0 = 0$ and $0 \bullet [b \bullet (\tau \star a)] = 0 \bullet 0 = 0$. Then $\tau \bullet a \leq b$ and $\tau \star a \leq b$ and so $b = \tau \bullet a$ and $b = \tau \star a$, thus $\tau \bullet a$ and $\tau \star a$ are pseudo-atoms. By (Definition 10) we have $L_p(E)$ is the set of all pseudo-atoms of E then $\tau \bullet a$ and $\tau \star a \in L(E)$. Therefore $L_p(E)$ is a pseudo-subalgebra of E.

Corollary 4. *If pseudo-BF-algebra* $(E; \bullet, \star, 0)$ *is generated by an element* g *then* g *is a pseudo-atom.*

Proof. For $g \in E$, suppose that g generates E and let τ be a pseudo-atom of E. Thus we have $g \leq \tau$. Then $g \bullet \tau = 0$ and $g \star \tau = 0$. By (Corollary 2) we get $\tau \bullet g = 0$ and $\tau \star g = 0$. Therefore $\tau \leq g$ and so $\tau = g$. Hence g is a pseudo-atom.

Proposition 9. *In a pseudo-BF-algebra* $(E; \bullet, \star, 0)$*, let* $\tau \in E$*. If* $\{0, \tau\}$ *is a pseudo-ideal then* $0 \neq \tau$ *is a pseudo-atom.*

Proof. Let $\{0, \tau\}$ be a pseudo-ideal of E and for all $a \in E$ let $a \leq \tau$ we have $a \bullet \tau = a \star \tau = 0 \in \{0, \tau\}$ from $(pI1)$. By $(pI2)$ we have $a \in \{0, \tau\}$, then $a = 0$ or $a = \tau$. Since $\tau \neq 0$ and $(pBF(1))$, we get $a = \tau$. Thus τ is a pseudo-atom of E.

Proposition 10. *In a pseudo-*BF^**-algebra* $(E; \bullet, \star, 0)$*, if a non-zero element is a pseudo-atom of* E*, then any pseudo-subalgebra is a pseudo-ideal.*

Proof. We prove $(pI1)$ and $(pI2)$. Let S be a pseudo-subalgebra of E, then $0 \in S$ from (Definition 7). For $(pI2)$, let $b \bullet a$, $b \star a \in S$ and $a \in S$. By (Theorem 8 (2) and (5), respectively) we have $b = a \bullet (a \star b) = a \bullet [0 \bullet (b \star a)]$. Since 0, $b \star a \in S$ and S is a pseudo-subalgebra of E, we obtain $0 \bullet (b \star a) \in S$. So $a \bullet [0 \bullet (b \star a)] \in S$. Also, similarly we can show it if $b \bullet a \in S$. Then $b \in S$. Hence the proposition is proved.

For any pseudo-BF-algebra $(E; \bullet, \star, 0)$, define the subsets $K(E)$, $V(\tau)$ of E as follows:

$$K(E) = \{a \in E : \ 0 \leq a\} \text{ and } V(\tau) = \{a \in E : \ \tau \leq a\}.$$

Theorem 9. *In a pseudo-*BF^**-algebra* $(E; \bullet, \star, 0)$ *if* τ *and* ω *is pseudo-atoms then the following hold:*

(1) $a \in V(\tau)$*,* $b \in V(\omega)$*, imply* $a \bullet b \in V(\tau \bullet \omega)$ *and* $a \star b \in V(\tau \star \omega)$*,*

(2) $a, b \in V(\tau)$*, implies* $a \star b$, $a \bullet b \in K(E)$*,*

(3) *If* $\tau \neq \omega$*, then we have* $a \bullet b$, $a \star b \in K(E)$*,for all* $a \in V(\tau)$, $b \in V(\omega)$*,*

(4) $a \in V(\omega)$*, implies* $\tau \bullet a = \tau \bullet \omega$ *and* $\tau \star a = \tau \star \omega$*,*

(5) *If* $\tau \neq \omega$*, then* $V(\tau) \cap V(\omega) = \phi$*.*

Proof.

(1) Let $a \in V(\tau)$, $b \in V(\omega)$. Then $\tau \leq a$ we have $\tau \bullet a = \tau \star a = 0$ and $\omega \leq b$ we have $\omega \bullet b = \omega \star b = 0$. From (Theorem 8 (7)) we obtain $(\tau \bullet \omega) \star (a \bullet b) = [0 \bullet (0 \star (\tau \bullet \omega))] \star (a \bullet b)$. Using (pBF^*), $[0 \bullet (0 \star (\tau \bullet \omega))] \star (a \bullet b) = [0 \star (a \bullet b)] \bullet [0 \star (\tau \bullet \omega)]$. By (Proposition 4 (6) and (4), respectively) then $[0 \star (a \bullet b)] \bullet [0 \star (\tau \bullet \omega)] = [0 \bullet (a \bullet b)] \bullet [0 \star (\tau \bullet \omega)] = [(0 \star a) \star (0 \bullet b)] \bullet [0 \star (\tau \bullet \omega)]$. By applying (pBF^*), we get $[(0 \star a) \star (0 \bullet b)] \bullet [0 \star (\tau \bullet \omega)] =$

$[(0 \star a) \bullet [0 \star (\tau \bullet \omega)]] \star (0 \bullet b) = [(0 \bullet [0 \star (\tau \bullet \omega)]) \star a] \star (0 \bullet b)$. By (Theorem 8 (7)) we have $[(0 \bullet [0 \star (\tau \bullet \omega)]) \star a] \star (0 \bullet b) = [(\tau \bullet \omega) \star a] \star (0 \bullet b)$. Using (pBF^*) we get $[(\tau \bullet \omega) \star a] \star (0 \bullet b) = [(\tau \star a) \bullet \omega] \star (0 \bullet b)$. From the hypothesis we have $[(\tau \star a) \bullet \omega] \star (0 \bullet b) = (0 \bullet \omega) \star (0 \bullet b)$. By (Proposition 4 (6) and (4), respectively) we get $(0 \bullet \omega) \star (0 \bullet b) = (0 \star \omega) \star (0 \bullet b) = 0 \bullet (\omega \bullet b)$. Using the hypothesis and $(pBF(1))$, respectively we get $0 \bullet (\omega \bullet b) = 0 \bullet 0 = 0$, and so $\tau \bullet \omega \leq a \bullet b$. Thus $a \bullet b \in V(\tau \bullet \omega)$ and similarly $a \star b \in V(\tau \star \omega)$.

(2) Let $a, b \in V(\tau)$, by (1) we have $a \bullet b \in V(\tau \bullet \tau)$, $a \star b \in V(\tau \star \tau)$. Using $(pBF(1))$ then $a \bullet b \in V(0)$, $a \star b \in V(0)$. We get $0 \leq a \bullet b$, $0 \leq a \star b$. Then $a \bullet b$, $a \star b \in K(E)$.

(3) Let 0 be a pseudo-atom from (Definition 10) we get $a \bullet b \leq 0$ then $a \bullet b = 0$. By (Corollary 2) we get $b \bullet a = 0$. Using $(pBF(3))$ then $0 \star (a \bullet b) = 0$ and so $0 \leq a \bullet b$. Therefore $a \bullet b \in V(0)$ and so $a \bullet b \in K(E)$. Similarly we can show that $a \star b \in K(E)$.

(4) Let $a \in V(\omega)$, then $\omega \leq a$ we have $\omega \bullet a = 0$ and $\omega \star a = 0$. By (Theorem 8 (3)) we get $(\tau \bullet a) \star (\tau \bullet \omega) = \omega \bullet a = 0$. So $\tau \bullet a \leq \tau \bullet \omega$. Moreover, $\tau \bullet \omega$ is a pseudo-atom by (Corollary 3). Therefore $\tau \bullet a = \tau \bullet \omega$. Similarly $\tau \star a = \tau \star \omega$.

(5) We prove by contradiction. Let $\tau \neq \omega$ and let $V(\tau) \cap V(\omega) \neq \phi$ then there exists $c \in V(\tau) \cap V(\omega)$. From (1), we have $c \bullet c \in V(\tau \bullet \omega)$, $c \star c \in V(\tau \star \omega)$. Using $(pBF(1))$ then $c \bullet c = 0 = c \star c$ and so $0 \in V(\tau \bullet \omega), V(\tau \star \omega)$. Hence $\tau \bullet \omega \leq 0$ and $\tau \star \omega \leq 0$. That is $\tau \bullet \omega$, $\tau \star \omega$ are pseudo-atoms from (1), then $\tau \bullet \omega = 0 = \tau \star \omega$ we have $\tau \leq \omega$. That is ω is a pseudo-atom then $\tau = \omega$ this is a contradiction with hypothesis $(\tau \neq \omega)$. Thus $V(\tau) \cap V(\omega) = \phi$.

Proposition 11. *In a pseudo-BF^*-algebra $(E; \bullet, \star, 0)$, let $\tau \in E$. Then τ is a pseudo-atom if and only if there is $a \in E$ such that $\tau = 0 \bullet a$.*

Proof. Let τ be a pseudo-atom of E. Then $\tau = 0 \bullet (0 \star \tau)$, from (Theorem 8 (6)). Set $a = 0 \star \tau$, we get $\tau = 0 \bullet a$.
Conversely, let $\tau = 0 \bullet a$ for some $a \in E$. We use (Proposition 2 (2)) to have $0 \bullet (0 \star \tau) = 0 \bullet (0 \star (0 \bullet a)) = 0 \bullet a = \tau$. By (Theorem 8 (6) and (1)) we conclude that τ is a pseudo-atom.

Proposition 12. *In a pseudo-BF^*-algebra $(E; \bullet, \star, 0)$, the following properties hold for any $a, b, c \in E$:*

(1) *if $a \leq b$ then $c \bullet b \leq c \bullet a$ and $c \star b \leq c \star a$,*

(2) *if $a \leq b$, $b \leq c$ then $a \leq c$,*

(3) *if $a \bullet b = c = a \star b$ then $c \bullet a = c \star a$,*

(4) *$(a \bullet b) \bullet (c \bullet b) \leq a \bullet c$ and $(a \star b) \star (c \star b) \leq a \star c$,*

(5) *if $a \leq b$ then $a \bullet c \leq b \bullet c$ and $a \star c \leq b \star c$.*

Proof.

(1) Let $a, b \in E$, $a \leq b$ then $a \bullet b = 0$ and $a \star b = 0$. By (Theorem 8 (3)) then $(c \bullet b) \star (c \bullet a) = a \bullet b = 0$ and $(c \star b) \bullet (c \star a) = a \star b = 0$ we get $c \bullet b \leq c \bullet a$ and $c \star b \leq c \star a$.

(2) Let $a, b, c \in E$, $a \leq b$ and $b \leq c$ we have $a \bullet b = 0$, $a \star b = 0$ and $b \bullet c = 0$, $b \star c = 0$. Also, by (1) since $b \leq c$ then $a \bullet c \leq a \bullet b \Rightarrow a \bullet c \leq 0$. By (Proposition 4 (1)) we get $a \bullet c = 0$ and so $a \leq c$.

(3) Let $a \bullet b = c = a \star b$. By using $(pBF(1))$ and (pBF^*) we obtain $c \star a = (a \bullet b) \star a = (a \star a) \bullet b = 0 \bullet b$. By (Proposition 4 (6)), $0 \bullet b = 0 \star b$. Using $(pBF(1))$ and (pBF^*) we have $0 \star b = (a \bullet a) \star b = (a \star b) \bullet a = c \bullet a$.

(4) By (pBF^*), (Theorem 8 (3)) and $(pBF(1))$, respectively we have $[(a \bullet b) \bullet (c \bullet b)] \star (a \bullet c) = [(a \bullet b) \star (a \bullet c)] \bullet (c \bullet b) = (c \bullet b) \bullet (c \bullet b) = 0$. Then $(a \bullet b) \bullet (c \bullet b) \leq a \bullet c$. Similarly, $(a \star b) \star (c \star b) \leq a \star c$.

(5) Suppose that $a, b \in E$, $a \leq b$ we have $a \bullet b = 0$, $a \star b = 0$. Using (4), we have $(a \bullet c) \bullet (b \bullet c) \leq a \bullet b$ but $a \bullet b = 0$. By (Proposition 4 (1)) then $(a \bullet c) \bullet (b \bullet c) = 0$ and so $a \bullet c \leq b \bullet c$. By a similar way, we can show that $a \star c \leq b \star c$.

Theorem 10. *In a pseudo-BF^*-algebra $(E; \bullet, \star, 0)$, the set $K(E)$ is a pseudo-subalgebra.*

Proof. For $a, b \in K(E)$, we have $0 \leq a$, $0 \leq b$, then $0 \bullet a = 0$, $0 \star a = 0$ and $0 \bullet b = 0$, $0 \star b = 0$. By using (Proposition 12 (5)) since $0 \leq a$ we get $0 \bullet b \leq a \bullet b$ and $0 \star b \leq a \star b$. Hence $0 \leq a \bullet b$ and $0 \leq a \star b$ and so $a \bullet b$, $a \star b \in K(E)$. Thus $K(E)$ is a pseudo-subalgebra of E.

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