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A Private Case of Sendov's Conjecture

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Abstract. In this paper, we prove Sendov's conjecture, when a polynomial is with real coefficients and the conjecture is relevant to the zeros, which belong to the set $M = \overline{D}(0,1) \cap [\overline{D}(1,1) \cup \overline{D}(-1,1)]$. We can see it in Figure 1. The conjecture is true for the filled areas.

2020 Mathematics Subject Classifications: 30D20, 30A10

Key Words and Phrases: Zeros, Complex Polynomial, Real Polynomial, Disk, Derivative, Integral

1. Introduction

The localization of the zeros of the complex polynomials is very important area of the mathematics. The impossibility to find the zeros of any polynomials using the coefficients makes every statement here very significant. There exist many conjectures which are not proved, like Sendov's conjecture and Obreshkoff's conjecture. They localize the zeros of the derivative of many complex polynomial in some areas. Here we present some new results about the zeros of the derivative of real polynomials.

In the second part "Preliminaries" we define some sets, including the set M. Here we formulate Sendov's conjecture.

In the third part "Related results", we present three statements which are similar to our theorems. The proofs of Statement 1 and Statement 2, we can see in [3]. If we put $m = 1$ in the condition of Statement 2 we obtain Statement 1, which is the famous Obreshkoff theorem, that can be regarded as a 'complex version' of a well-known theorem due to Laguerre. The proof of Statement 3 we can see in [1].

The main theorems are in the fourth part "Main results". Theorem 1 is relevant to the real zeros of the real polynomial $r(z)$. Theorem 2 is relevant to the complex zeros of the real polynomial $r(z)$. All the results could be connected with the results of [4].

The proofs of these theorems are independent of references. Only [4] is relevant to them. Many of these results could be applied in [2] and [5].

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2. Preliminaries

We note:
$D(a,r) = \{z \in C : |z-a| < r\}$ is the open disk with center a and radius r.
$\overline{D}(a,r) = \{z \in C : |z-a| \leq r\}$ is the closed disk with center a and radius r.
$C(a,r) = \{z \in C : |z-a| = r\}$ is the circle with center a and radius r.
$M = \overline{D}(0,1) \cap [\overline{D}(1,1) \cup \overline{D}(-1,1)]$.

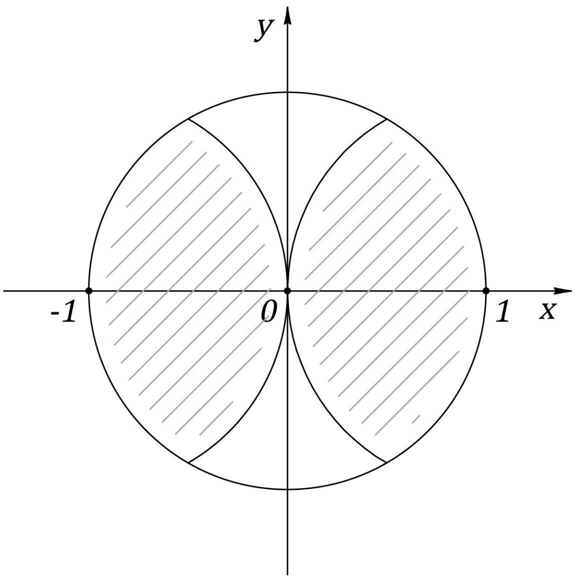

Figure 1:

Sendov‘s conjecture: *Let us put for* $n \geq 2$, $p(z) = \prod_{k=1}^{n}(z-z_k)$, *where* $z_k \in \overline{D}(0,1)$, $k = 1,2,\ldots,n$. *Then* $p'(z)$ *has at least one zero in each of the disks* $\overline{D}(z_k,1)$, $k = 1,2,\ldots n$.

3. Related Results

Statement 1(Obreshkoff). *Let the zeros* z_k, $k = 1,2,\ldots,n$ *of a polynomial* $p(z) \in C[z]$ *satisfy* $z_k \in \overline{D}(0,1)$. *Then the zeros* z *of the polynomial* $q(z) = n\gamma p(z) + zp'(z)$, *where Re* $\gamma \geq -\frac{1}{2}$, *satisfy* $z \in \overline{D}(0,1)$.

Statement 2(Stoyanov). *Let the zeros* z_k, $k = 1,2,\ldots,n$ *of a polynomial* $p(z) \in C[z]$ *satisfy* $z_k \in \overline{D}(0,1)$. *Then if Re* $\gamma \geq -\frac{m}{2}$, $m = 1,2,\ldots,n$, *the zeros* z *of the polynomial* $q(z) = \gamma p(z) + \sum_{k=1}^{m} \frac{(n-k)!}{n!} z^k p^{(k)}(z)$, *satisfy* $z \in \overline{D}(0,\theta)$, *where*

$$\theta = \left(2^{\frac{1}{m}} - 1\right)^{-1}.$$

Statement 3(Bojanov). *If all the zeros* z_k, $k = 1,2,\ldots n$; *of a polynomial* $p(z) \in C[z]$ *satisfy* $z_k \in \overline{D}(0,1)$ *and a is a zero of* $p(z)$ *of modulus 1, then the derivative* $p'(z)$ *has at least one zero in* $\overline{D}(\frac{a}{2},\frac{1}{2})$.

4. Main results

Case 1.
In this case we consider a polynomial

$$r(z) = z^n + r_{n-1}z^{n-1} + \cdots + r_1 z + r_0,$$

where $r_k \in R$, $n \geq 2$, $n \in N$, $k = \overline{0,\ n-1}$. The zeros z_k of $r(z)$ satisfy the condition

$$z_k \in \ \overline{D}(-a,1), z_0 \in R, z_0 + a = 0, a \in (0,1].$$

The derivative is

$$\frac{dr}{dz} = n(z+a_1)(z+a_2)\cdots(z+a_l)(z-b_1)\left(z-\overline{b_1}\right)\cdots(z-b_s)\left(z-\overline{b_s}\right),$$

where $l + 2s = n-1$, $a_k, b_m \in \ \overline{D}(-a,1)$, $k = \overline{1,l}$, $m = \overline{1,s}$, $a_k \in R$, $k, m \in N$.

Theorem 1. *If $z_0 = 0 \in R$ is a real zero of $r(z)$, then there exists a zero c of $\frac{dr}{dz}$ which satisfies $c \in \overline{D}(0,1)$.*

Proof.

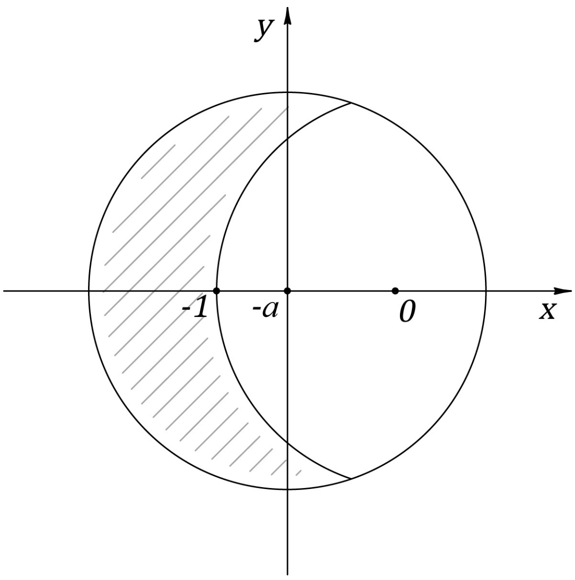

Figure 2:

Without loss of generality, we consider Re $z_0 \geq 0$.

About the root $z_0 = 0$ we assume that in the disk $\overline{D}(0,1)$ there is not exists a root $c \in \ \frac{dr}{dz}$. Then all the zeros of the derivative $a_k, b_m \in \overline{D}(-a,1) \ \setminus \ \overline{D}(0,1)$ and we have

$$r(0) - r(-a) = \int_{-a}^{0} \frac{dr}{dz} dz = -r(-a),$$

i.e.

$$-r(-a) = n\int_{-a}^{0}(z+a_1)(z+a_2)\cdots(z+a_l)(z-b_1)\left(z-\overline{b_1}\right)\cdots(z-b_s)\left(z-\overline{b_s}\right)dz$$

where $a_k > 1$, $b_m = \rho_m e^{i\varphi_m}$, $\rho_m > 1$, $\varphi_m \in \left[\frac{\pi}{2}, \pi\right]$, $k = \overline{1,l}$, $m = \overline{1,s}$,

$$(z-b_m)\left(z-\overline{b_m}\right) = z^2 - 2\rho_m \cos\varphi_m z + \rho_m^2.$$

Then we obtain

$$-r(-a) =$$

$$\begin{aligned}
&= n\int_{-a}^{0}(z+a_1)\cdots(z+a_l)\left(z^2-2\rho_1\cos\varphi_1 z+\rho_1^2\right)\cdots\left(z^2-2\rho_s\cos\varphi_s z+\rho_s^2\right)dz \\
&\geq n\int_{-a}^{0}(z+1)^l\left(z^2+2\rho_1 z+\rho_1^2\right)\cdots\left(z^2+2\rho_s z+\rho_s^2\right)dz \\
&> n\int_{-a}^{0}(z+1)^l(z+1)^{2s}dz = n\int_{-a}^{0}(z+1)^{n-1}dz \\
&= (z+1)^n\,|_{-a}^{0} = 1^n-(1-a)^n > a.
\end{aligned}$$

We obtain that $|r(-a)| > a$. But all the zeros z_k of $r(z)$ belong to $\overline{D}(-a,1)$, i.e. $|r(-a)| \leq a$, because $|z_0+a| = 0$. This contradiction confirms the Theorem 1.

Case 2.

In this case we consider a polynomial $r(z) = z^n + r_{n-1}z^{n-1} + \cdots + r_1 z + r_0$, where $r_k \in R$, $n \geq 2$, $n \in N$, $k = \overline{0, n-1}$. The zeros z_k of $r(z)$ satisfy the condition $z_k \in \overline{D}(0,1)$, $z_0 = ae^{i\theta_0}$, where $a \in (0,1]$, $\theta_0 \in \left[0, \frac{\pi}{2}\right]$. The derivative is

$$\frac{dr}{dz} = n(z-a_1)(z-a_2)\cdots(z-a_l)(z-b_1)\left(z-\overline{b_1}\right)\cdots(z-b_s)\left(z-\overline{b_s}\right),$$

where $l+2s = n-1$, $a_k, b_m \in \overline{D}(0,1)$, $k = \overline{1,l}$, $m = \overline{1,s}$, $k,m \in N$.

Theorem 2. *If $z_0 = ae^{i\theta_0}$ is a zero of $r(z)$ belongs to M (in Preliminaries), then there exists a zero c of $\frac{dr}{dz}$ which satisfies $c \in \overline{D}(z_0,1)$.*

Proof.

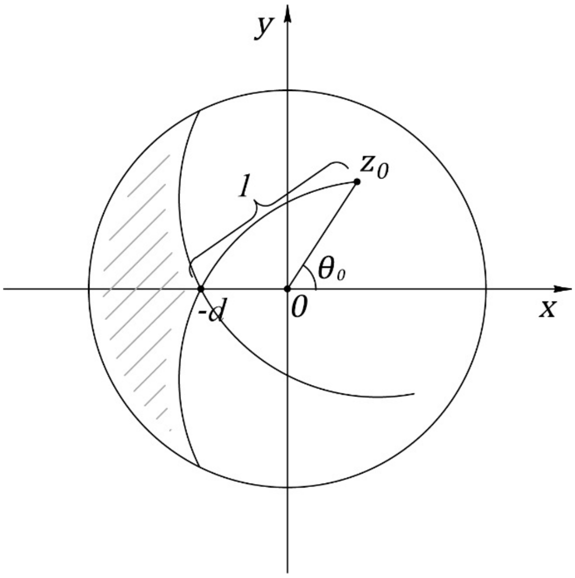

Figure 3:

Without loss of generality, we consider $Re\ z_0 \geq 0$.

Let us assume that in the disk $\overline{D}(z_0, 1)$ there is not exists a root $c \in \frac{dr}{dz}$. Then all the zeros of the derivative $a_k, b_m \in \overline{D}(0,1) \setminus \overline{D}(z_0,1) \setminus \overline{D}(\overline{z_0},1)$, because the zero $z_0 = ae^{i\theta_0}$ of the polynomial must belongs to M. We note with $-d, d > 0 : -d = C(z_0, 1) \bigcap C(\overline{z_0}, 1)$ and let us put $v(\theta) = ae^{i\theta}$, $\theta \in [0, \theta_0]$, $l + 2s = n - 1$,

$$t(z) = \frac{1}{n}\frac{dr(z)}{dz} = \prod_{p=1}^{l}(z + a_p)\prod_{p=1}^{s}(z - b_p)\left(z - \overline{b_p}\right),$$

$l, s \in N$ [one of these factors could be not existing, i.e. $l = 0$ or $s = 0$].

We put

$$f(\theta) = \int_0^{v(\theta)} \frac{dr}{dz}dz = n\int_0^{v(\theta)} t(z)\,dz,$$

$$g(\theta) = f(\theta).\overline{f}(\theta).$$

Let us calculate

$$\frac{dg}{d\theta} = n\left[t(v(\theta))\frac{dv}{d\theta}\overline{f}(\theta) + t(\overline{v}(\theta))\frac{d\overline{v}}{d\theta}f(\theta)\right],$$

$$\frac{dv}{d\theta} = \frac{dae^{i\theta}}{d\theta} = iae^{i\theta},$$

and if we put $U_0 = v(\theta) = ae^{i\theta}$, $U_p = v(\theta) + a_p$, $p = \overline{1, l}$, $U_{l+2p+1} = v(\theta) - b_p$,

$$U_{l+2p+2} = v(\theta) - \overline{b_p}, p = \overline{0, s-1}.$$

Knowing $\frac{df}{d\theta}.\prod_{p=0}^{n-1}\overline{U_p} = \frac{d\overline{f}}{d\theta}.\prod_{p=0}^{n-1} U_p$ we have

$$\begin{aligned}
\frac{dg}{d\theta} =& in\left[\overline{f}(\theta)\prod_{p=0}^{n-1} U_p - f(\theta)\prod_{p=0}^{n-1}\overline{U_p}\right],\\
\frac{d^2g}{d\theta^2} =& n\left[2\frac{df}{d\theta}.\prod_{p=0}^{n-1}\overline{U_p} + i\frac{d\prod_{p=0}^{n-1}U_p}{d\theta}\overline{f}(\theta) - i\frac{d\prod_{p=0}^{n-1}\overline{U_p}}{d\theta}f(\theta)\right]\\
\frac{d^2g}{d\theta^2} =& n\left[2n\prod_{p=0}^{n-1}|U_p|^2 - \left(U_0\sum_{p=0}^{n-1}\prod_{j\neq p}U_j\right)\overline{f}(\theta) - \left(\overline{U_0}\sum_{p=0}^{n-1}\prod_{j\neq p}\overline{U_j}\right)f(\theta)\right]\\
\frac{d^2g}{d\theta^2} =& 2n\left[n\prod_{p=0}^{n-1}|U_p|^2 - Re\left(U_0\sum_{p=0}^{n-1}\prod_{j\neq p}U_j\right)\overline{f}(\theta)\right],
\end{aligned}$$

and consequently

$$\frac{d^2g}{d\theta^2} \geq 2n\prod_{p=0}^{n-1}|U_p|\left[n\prod_{p=0}^{n-1}|U_p| - \frac{\left|U_0\sum_{p=0}^{n-1}\prod_{j\neq p}U_j\right|.\left|\overline{f}(\theta)\right|}{\prod_{p=0}^{n-1}U_p}\right].$$

Since $\theta \in [0,\theta_0] \Rightarrow |U_p(\theta)| \geq |U_p(\theta_0)| \geq 1$, $p = \overline{1, n-1}$.
If we assume $|f(\theta)| = \left|\overline{f}(\theta)\right| \leq a^2$, then

$$\begin{aligned}
\frac{d^2g}{d\theta^2} \geq& 2n\prod_{p=0}^{n-1}|U_p|\left[na - \left(1 + \left|\frac{U_0}{U_1}\right| + \cdots + \left|\frac{U_0}{U_{n-1}}\right|\right)a^2\right]\\
&\geq 2na\prod_{p=0}^{n-1}|U_p|\left[n - (1 + a(n-1))a\right] = 2na\prod_{p=0}^{n-1}|U_p|\left[n - a - a^2(n-1)\right]\\
&\geq 2na\prod_{p=0}^{n-1}|U_p|\left[n - a - a(n-1)\right] = 2nan(1-a)\prod_{p=0}^{n-1}|U_p|.
\end{aligned}$$

Then we get $\frac{d^2g}{d\theta^2} \geq 0$.

Hence $\frac{dg}{d\theta}(\theta) \geq \frac{dg}{d\theta}(0) = 0$. Consequently $g(\theta_0) > g(0)$, i.e. $|f(\theta_0)| > a$, according to the proof of Theorem 1.

Therefore $a < |f(\theta_0)| \leq a^2$, which is impossible. The contradiction proves that $|f(\theta_0)| > a^2$, but

$$f(\theta_0) = \int_0^{v(\theta_0)}\frac{dr}{dz}dz = \int_0^{z_0}\frac{dr}{dz}dz = r(z_0) - r(0) = -r(0),$$

i.e. $|r(0)| \leq a^2$, because $z_0,\ \overline{z_0}$ are roots of $r(z)$ and $|z_0|=|\overline{z_0}| = a$. Then we have

$$a^2 < |r(0)| \leq a^2.$$

This contradiction shows that, there exists a zero c of $\frac{dr}{dz}$, which satisfies $c \in \overline{D}(0,1)$.

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References

[1] B Bojanov, Q I Rahman, and J Szynal. On a Conjecture of Sendov About the Critical Points of a Polynomial. *Math. Z*, 190:281–285, 1985.

[2] D M Souroujon and T Zapryanova. On the relation between the number of real and complex zeros of polynomials of a certain kind. In *AIP Conference Proceeding.*, volume 2159, St Constantine and Helena, 2019. Sixth International Conference on New Trends in the Applications of Differential Equations in Sciences.

[3] T Stoyanov. About the Zeros of Some Entire Functions and Their Derivatives. *Journal of the Australian Mathematical Society*, 68:165–169, 2000.

[4] T Stoyanov. Some Estimates Below the Modulus of Integrals of Some Polynomials in the Complex Plane. *European Journal of Pure and Applied Mathematics*, 12(2):649–653, 2019.

[5] T Zapryanova and D Souroujon. On the Iterates of Jackson Type Operator $G_{s,n}$. *Mediterr. J. Math*, 13:5053–5061, 2016.