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Schur Geometric Convexity of Related Function for Holders Inequality with application

Sreenivasa Reddy Perla[1,*], S Padmanabhan[2], V Lokesha[3]

[1] *Department of Mathematics, The Oxford College of Engineering, Bangalore, Karnataka, India*
[2] *Department of Mathematics, RNS Institute of Technology, Bangalore, Karnataka, India*
[3] *Department of Mathematics, VSK University, Bellary, Karnataka, India*

Abstract. In this paper, we investigated the Schur geometric convexity of related function for Holders Inequality by using majorization inequality theory, giving a complete critical condition of Schur Geometrically convex function for Holders Inequality related function and some applications are established.
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1. Introduction

Throughout this paper, we assume that the set of n-dimensional row vector on the real number field by R^n. Let

$$R^n_+ = \{x = (x_1, x_2, ...x_n) : x_i \geq 0, i = 1, 2...n\},$$

By Holders inequality [2], we have

$$\sum_{l=1}^{n} r_l s_l \leq \left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} \tag{1}$$

*Corresponding author.
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Email addresses: `srireddy_sri@yahoo.co.in` (S. R. Perla), `padmanabhanrnsit@gmail.com` (S. Padmanabhan), `v.lokesha@gmail.com` (V. Lokesha)

$$\int_r^s \phi(x)\psi(x)dx \leq \left(\int_r^s \left(\phi(x)\right)^u dx\right)^{\frac{1}{u}} \left(\int_r^s \left(\psi(x)\right)^v dx\right)^{\frac{1}{v}} \tag{2}$$

Here $r_l \geq 0, s_l \geq 0, u > 1, \frac{1}{u} + \frac{1}{v} = 1$.

The Schur convexity of functions relating to special means is a very significant research subject and has attracted the interest of many mathematicians.There are numerous articles written on this topic in recent years; (see [3] , [6]) and the references therein. As supplements to the Schur convexity of functions, the Schur geometrically convex functions and Schur harmonically convex functions were investigated by Zhang and Yang ([15], [13]), Chu, Zhang and Wang [14], Shi and Zhang ([8], [7]) , Meng, Chu and Tang [4] , Zheng, Zhang and Zhang [17]. These properties of functions have been found to be useful in discovering and proving the inequalities for special means (see [1] - [2], [11],[12]).

Dong-Sheng Wang, Chun - Ru Fu and Huan-Nan Sh [10] investigated the Schur convexity about related function of Holders inequality by using majorization inequality theory . This result gives a full essential condition of Schur convexity for Holders inequality related function, reached sharpen type of Holders inequality Under certain conditions and new inequalities for Stolarsky mean estabilsihed. This paper motivates us to investigate Schur geometric convexity about related function of Holders inequality by using majorization inequality theory.

2. Preliminaries

To estabilish our main results, we need the following definitions and lemmas.

Definition 1. [[3], [9]] *.Consider two arbitrary n-tuple elements* $\lambda, \mu \in R^n$ $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ *and* $\mu = (\mu_1, \mu_2, ..., \mu_n) \in R^n$.

(i) For the arrangements of λ *and* μ *in descending order of the form if*

$$\sum_{p=1}^{t} \lambda_{[p]} \leq \sum_{p=1}^{t} \mu_{[p]}$$

for $1 \leq t \leq n-1$, λ *is said to by majorized by* μ, *(in icon* $\lambda \prec \mu$*) and*

$$\sum_{p=1}^{n} \lambda_{[p]} = \sum_{p=1}^{n} \mu_{[p]},$$

where $\lambda_{[1]} \geq \cdots \geq \lambda_{[n]}$ *and* $\mu_{[1]} \geq \cdots \geq \mu_{[n]}$

(ii) Let $\Psi \subseteq R^n$ $(n \geq 2)$ $p = 1, 2, \cdots, n$ $\lambda \geq \mu$ *means* $\lambda_p \geq \mu_p$.
The function $\omega : \Psi \to R$ *is declining if and just if* $-\omega$ *is escalating.*

(iii) *For* $\zeta, \eta \in [0,1]$ *with* $\zeta+\eta = 1$, $\Psi \subseteq R^n$ *is a convex set, if* $(\zeta\lambda_1 + \eta\mu_1, \cdots, \zeta\lambda_n + \eta\mu_n) \in \Psi$ *for all* λ *and* μ.

(iv) *the function* $\omega : \Psi \to R$ *is considered to be Schur-convex whenever* $\lambda \prec \mu$ *on* Ψ *implies* $\omega(\lambda) \leq \omega(\mu)$. ω *is Schur concave on* Ψ *if* $-\omega$ *is Schur convex.*

Lemma 1. [5]. *Let* $\omega : \Psi \to R$ *be differentiable in* Ψ^0 *and continuous on* Ψ *and* $\Psi \subseteq R^n$ *be symmetric with non-empty interior* Ψ^0, *then* ω *is Schur convex on* Ψ *if and only if* ω *is symmetric on* Ψ *and*

$$(p-q)\left(\frac{\partial\omega}{\partial p} - \frac{\partial\omega}{\partial q}\right) \geq 0 (\leq 0) \tag{3}$$

Definition 2. [8]. *If* $(\lambda_1^{\zeta}\mu_1^{\eta}, ..., \lambda_n^{\zeta}\mu_n^{\eta}) \in \Psi$ *for all* λ *and* $\mu \in \Psi$ *and* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ *and* $\mu = (\mu_1, \mu_2, ..., \mu_n) \in R_+^n$, *then* $\Psi \subseteq R^n$ *is identified as geometrically convex set, where* $\zeta, \eta \in [0,1]$ *with* $\zeta + \eta = 1$.

If $(ln\lambda_1, ..., ln\lambda_n) \prec (ln\mu_1, ..., ln\mu_n)$ *on* Ψ *implies* $\omega(\lambda) \leq \omega(\mu)$ *and* $\Psi \subseteq R_+^n$, *then the function* $\omega : \Psi \to R_+$ *is called as Schur geometrically convex function on* Ψ.

Lemma 2. [8]. *Let* $\omega : \Psi \to R$ *be differentiable in* Ψ^0 *and continuous on* Ψ *and* $\Psi \subseteq R^n$ *be symmetric with non-empty interior* Ψ^0, *then* ω *is Schur Schur-geometrically convex (Schur-geometrically concave) function. If* ω *is symmetric on* Ψ *and*

$$(\ln p - \ln q)\left(p\frac{\partial\omega}{\partial p} - q\frac{\partial\omega}{\partial q}\right) \geq 0 (\leq 0). \tag{4}$$

Lemma 3. [16]. *(Chebyshev's inequality) If progressions* $r_n \geq 0$, $s_n \geq 0$ *we have*

(i) *When* r_n, s_n *have opposite monotonicity, then*

$$\sum_{l=1}^{n} r_1 \sum_{l=1}^{n} s_l \geq n \sum_{l=1}^{n} s_l r_l \tag{5}$$

(ii) *When* r_n, s_n *have same monotonicity, then*

$$\sum_{l=1}^{n} r_l \sum_{l=l}^{n} s_1 \leq n \sum_{l=1}^{n} s_l r_l \tag{6}$$

Lemma 4. [16]. *If* $\phi(x)$ *is the convex (concave) function on the interval then*

$$\phi\left(\frac{r+s}{2}\right) \leq (\geq) \frac{1}{s-r}\int_s^r \phi(x)dx \leq (\geq) \left(\frac{\phi(r)+\phi(s)}{2}\right) \tag{7}$$

Lemma 5. [9]. *Let* $x = (x_1, x_2, x_3 ... x_n) \in R^n$ *and* $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$. *then*

$$u = \underbrace{(A_n(x), A_n(x), ..., A_n(x))}_{n} < (x_1, x_2, ... x_n) = x$$

Lemma 6. [2]. *(Young's inequality) Suppose* $r, s \geq 0, u \geq 1, \frac{1}{u} + \frac{1}{v} = 1$ *then*

$$\frac{1}{u}r^u + \frac{1}{v}s^u \geq rs \tag{8}$$

Lemma 7. *Suppose* $r, s \geq 0, u \geq 1, \frac{1}{u} + \frac{1}{v} = 1$ *then*

$$rs \leq \frac{1}{u}(r^u + s^u) + \frac{1}{v}(r^v + s^v) - \frac{r^2 + s^2}{2} \tag{9}$$

Lemma 8. *when* $1 \geq r \geq s \geq 0, u \geq v \geq 1$ *then*

$$\frac{1}{u}r^u + \frac{1}{v}s^v \leq \frac{1}{u}s^u + \frac{1}{v}r^v \tag{10}$$

3. Main Results

In this paper, by using the principle of majorization as an example, combined with majorization inequality, the Schur-geometrically convexity of Related Function for Holder's Inequality gives sharpening inequality of the Holders under certain conditions.
Our primary outcome is as follows:

Theorem 1. *Let* $r_n \geq 0$ *and* $s_n \geq 0$ *be any two progressions and let* u *and* v *be two non-zero arbitrary real numbers. Let*

$$H_1(r) = \sum_{l=1}^{n} r_l s_l \leq \left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} \tag{11}$$

If $u \geq 1$, *then* $H_1(r)$ *is Schur-geometric convex on* R_+ *with* $r_1, ..., r_n$ *and if* $u < 1$, *then* $H_1(r)$ *is Schur-geometric concave on* R_+ *with* $r_1, ..., r_n$.

Proof. : Here $H_1(r)$ is obviously symmetric with $r = r_1, ..., r_n$ on R_+.
Let us assume $r_1 > r_2$.
Now by differentiating (11) partially with respect to r_1 and r_2, we get

$$\frac{\partial H_1}{\partial r_1} = \left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}-1} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} r_1^{u-1}$$

and

$$\frac{\partial H_1}{\partial r_2} = \left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}-1} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} r_2^{u-1}$$

Consider,

$$\triangle_1 = (\ln r_1 - \ln r_2)\left(r_1\frac{\partial H_1}{\partial r_1} - r_2\frac{\partial H_1}{\partial r_2}\right)$$

$$\Rightarrow \qquad \triangle_1 = (\ln r_1 - \ln r_2)\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}-1}\left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}}\left(r_1^u - r_2^u\right)$$

It is easy to see that, when $u \geq 1$, then $\triangle_1 \geq 0$ and when $u \leq 1$, then $\triangle_1 \leq 0$.

Hence, by Lemma 2, if $u \geq 1$, then $H_1(r)$ is Schur-geometric convex on R_+ with $r_1, ..., r_n$ and if $u \leq 1$, then $H_1(r)$ is Schur-geometric concave on R_+ with $r_1, ..., r_n$.

This completes proof of Theorem 1.

Theorem 2. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let u and v be two non-zero arbitrary real numbers. Let*

$$H_2(s) = n^{\frac{1}{u}} A_{n,r}\left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} \tag{12}$$

If $v \geq 1$, then $H_2(s)$ is Schur-geometric convex on R_+ with $s_1, ..., s_n$ and if $v \leq 1$, $H_2(s)$ is Schur-geometric concave on R_+ with $s_1, ..., s_n$. Here $A_{n,r} = \frac{1}{n}\sum_{l=1}^{n} r_l$.

Proof. : Here $H_2(r)$ is obviously symmetric with $s = s_1, ..., s_n$ on R_+.
Let us assume $s_1 > s_2$.
Now by differentiating (12) partially with respect to s_1 and s_2, we get

$$\frac{\partial H_2}{\partial s_1} = n^{\frac{1}{u}} A_{n,r}\left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} s_1^{v-1}$$

and

$$\frac{\partial H_2}{\partial s_2} = n^{\frac{1}{u}} A_{n,r}\left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} s_2^{v-1}$$

Consider,

$$\triangle_2 = (\ln s_1 - \ln s_2)\left(s_1\frac{\partial H_1}{\partial s_1} - s_2\frac{\partial H_1}{\partial s_2}\right)$$

$$\Rightarrow \qquad \triangle_2 = (\ln s_1 - \ln s_2) n^{\frac{1}{u}} A_{n,r}\left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} (s_1^v - s_2^v)$$

It is easy to see that,when $v \geq 1$, then $\triangle_2 \geq 0$ and when $v \leq 1$, then $\triangle_2 \leq 0$.

Hence, by Lemma 2, if $v \geq 1$, then $H_2(s)$ is Schur-geometric convex on R_+ with $s_1, ..., s_n$ and if $v \leq 1$, then $H_2(s)$ is Schur-geometric concave on R_+ with $s_1, ..., s_n$.
This completes proof of Theorem 2.

Theorem 3. *Let $\phi(x)$ and $\psi(x)$ be two continuous functions with $\phi(x) > 0$, $\psi(x) > 0$ and let $\int_r^s \phi(x)\psi(x)dx \neq 0$, $\int_r^s \big(\phi(x)\big)^u dx \neq 0$, $\int_r^s \big(\psi(x)\big)^v dx \neq 0$,
where u and v are arbitrary real numbers. Let*

$$H_3\big(r,s\big) = \begin{cases} \left[\dfrac{\int_r^s \big(\psi(x)\big)^v dx}{\int_r^s \phi(x)\psi(x)dx}\right]^u \left[\dfrac{\int_r^s \big(\phi(x)\big)^u dx}{\int_r^s \phi(x)\psi(x)dx}\right]^v, & if\ r \neq s \\ \big(\phi(x)\psi(x)\big)^{uv-u-v}, & if\ r = s \end{cases} \tag{13}$$

Then $H_3\big(r,s\big)$ is Schur- geometric concave(convex) with r, s if and only if:

$$\frac{v\big(\phi^u(s) + \phi^u(r)\big)}{\int_r^s \phi^u(x)dx} + \frac{u\big(\psi^v(s) + \psi^v(r)\big)}{\int_r^s \psi^v(x)dx} \leq (\geq) \frac{\Big(\phi(s)\psi(s) + \phi(r)\psi(r)\Big)(u+v)}{\int_r^s \phi(x)\psi(x)dx} \tag{14}$$

Proof. : Here $H_3\big(r,s\big)$ is obviously symmetric with $r = r_1, r_2, ..., r_n$ and $s = s_1, s_2, ..., s_n$ on R_+.
Let us assume $s > r$.
From (13), we have

$$H_3\big(r,s\big) = \left[\frac{\int_r^s \big(\psi(x)\big)^v dx}{\int_r^s \phi(x)\psi(x)dx}\right]^u \left[\frac{\int_r^s \big(\phi(x)\big)^u dx}{\int_r^s \phi(x)\psi(x)dx}\right]^v$$

$$\Rightarrow \quad H_3\big(r,s\big) = \frac{\Big(\int_r^s \phi^u(x)dx\Big)^v \Big(\int_r^s \psi^v(x)dx\Big)^u}{\Big(\int_r^s \phi(x)\psi(x)dx\Big)^{u+v}}$$

Now by differentiating this partially with respect to s and r, we get

$$\frac{\partial H_3}{\partial s} = \frac{v\Big(\int_r^s \phi^u(x)dx\Big)^{v-1} \phi^u(s) \int_r^s \Big(\psi^v(x)dx\Big)^u \int_r^s \Big(\phi(x)\psi(x)dx\Big)^{u+v}}{\Big(\int_r^s \phi(x)\psi(x)dx\Big)^{2(u+v)}}$$

$$+\frac{u\left(\int_r^s \psi^v(x)dx\right)^{u-1}\psi^v(s)\int_r^s\left(\phi^v(x)dx\right)^u\int_r^s\left(\phi(x)\psi(x)dx\right)^{u+v}}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}$$

$$-\frac{(u+v)\left(\int_r^s \phi(x)\psi(x)dx\right)^{(u+v-1)}\phi(s)\psi(s)\left(\int_r^s \phi^u(x)dx\right)^v\left(\int_r^s \psi^v(x)dx\right)^u}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}$$

$$\frac{\partial H_3}{\partial r}=\frac{v\left(\int_r^s \phi^u(x)dx\right)^{v-1}\phi^u(r)\int_r^s\left(\psi^v(x)dx\right)^u\int_r^s\left(\phi(x)\psi(x)dx\right)^{u+v}}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}$$

$$-\frac{u\left(\int_r^s \psi^v(x)dx\right)^{u-1}\psi^v(r)\int_r^s\left(\phi^v(x)dx\right)^v\int_r^s\left(\phi(x)\psi(x)dx\right)^{u+v}}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}$$

$$+\frac{(u+v)\left(\int_r^s \phi(x)\psi(x)dx\right)^{(u+v-1)}\phi(r)\psi(r)\left(\int_r^s \phi^u(x)dx\right)^v\left(\int_r^s \psi^v(x)dx\right)^u}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}$$

Consider,

$$\triangle_3=(\ln s-\ln r)\left(s\frac{\partial H_3}{\partial s}-r\frac{\partial H_3}{\partial r}\right)$$

This implies that,

$$\triangle_3=\frac{(\ln s-\ln r)}{\left(\int_r^s \phi(x)\psi(x)dx\right)^{2(u+v)}}\Bigg[v\left(\int_r^s \phi^u(x)dx\right)^{v-1}\left(\int_r^s \psi^v(x)dx\right)^u$$

$$\times\left(\int_r^s \phi(x)\psi(x)dx\right)^{u+v}\big(s\phi^u(s)+r\phi^u(x)\big)+u\left(\int_r^s \psi^v(x)dx\right)^{u-1}$$

$$\times\left(\int_r^s \phi^u(x)dx\right)^v\int_r^s\left(\phi(x)\psi(x)dx\right)^{u+v}\left(s\psi^v(s)+r\psi^v(r)\right)$$

$$-(u+v)\int_r^s\Big(\phi(x)\psi(x)dx\Big)^{u+v-1}\Big(\int_r^s\phi^u(x)dx\Big)^v\Big(\int_r^s\psi^v(x)dx\Big)^u\times\Big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\Big)\Big]$$

$$=\frac{(\ln s-\ln r)}{\Big(\int_r^s\phi(x)\psi(x)dx\Big)^{2(u+v)}}\Big[v\int_r^s\Big(\phi(x)\psi(x)dx\Big)^{u+v-1}\Big(\int_r^s\phi^u(x)dx\Big)^{v-1}$$

$$\times\Big(\int_r^s\psi^v(x)dx\Big)^u\Big\{\int_r^s\phi(x)\psi(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)-\Big(\int_r^s\phi^u(x)dx\Big)\times\Big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\Big)\Big\}$$
$$+u\int_r^s\Big(\phi(x)\psi(x)dx\Big)^{u+v-1}\Big(\int_r^s\phi^u(x)dx\Big)^v$$
$$\times\Big(\int_r^s\psi^v(x)dx\Big)^{u-1}\Big\{\int_r^s\phi(x)\psi(x)dx\big(s\psi^v(s)+r\psi^u(r)\big)-\Big(\int_r^s\psi^u(x)dx\Big)\times\Big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\Big)\Big\}\Big]$$

$$=\frac{(\ln s-\ln r)}{\Big(\int_r^s\phi(x)\psi(x)dx\Big)^{2(u+v)}}\Big(\int_r^s\phi(x)\psi(x)dx\Big)^{u+v-1}\Big(\int_r^s\phi^u(x)dx\Big)^{v-1}\Big(\int_r^s\psi^v(x)dx\Big)^{u-1}$$

$$\Big\{v\int_r^s\psi^v(x)dx\Big[\int_r^s\phi(x)\psi(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)-\int_r^s\phi^u(x)dx\big(s\phi(s)\psi(s)\big)+r\phi(r)\psi(r)\Big]$$
$$+u\int_r^s\phi^u(x)dx\Big[\int_r^s\phi(x)\psi(x)dx\big(s\psi^u(s)+r\psi^u(r)\big)-\int_r^s\psi^v(x)dx\big(s\phi(s)\psi(s)\big)+r\phi(r)\psi(r)\Big]\Big\}$$

Since

$$\frac{(\ln s-\ln r)}{\Big(\int_r^s\phi(x)\psi(x)dx\Big)^{2(u+v)}}\Big(\int_r^s\phi(x)\psi(x)dx\Big)^{u+v-1}\Big(\int_r^s\phi^u(x)dx\Big)^{v-1}\Big(\int_r^s\psi^v(x)dx\Big)^{u-1}\geq 0$$

So $\triangle_3$ and

$$v\int_r^s\psi^v(x)dx\Big[\int_r^s\phi(x)\psi(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)-\int_r^s\phi^u(x)dx\big(s\phi(s)\psi(s)\big)+r\phi(r)\psi(r)\Big]$$

$$+\,u\int_r^s \phi^u(x)dx\Big[\int_r^s \phi(x)\psi(x)dx\big(s\psi^u(s)+r\psi^u(r)\big)-\int_r^s \psi^v(x)dx\big(s\phi(s)\psi(s)\big)+r\phi(r)\psi(r)\Big]$$

$$=\int_r^s \phi(x)\psi(x)dx\Big[v\int_r^s \psi^v(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)+u\int_r^s \phi^v(x)dx\big(s\phi^v(s)+r\psi^v(r)\big)\Big]$$

$$-\int_r^s \phi^u(x)dx\int_r^s \psi^v(x)dx\big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\big)\big(u+v\big)$$

have the same symbol.

Hence, we have $H_3\big(r,s\big)$ is Schur- Geometric concave (convex) with r, s, if and only if:

$$\int_r^s \phi(x)\psi(x)dx\Big[v\int_r^s \psi^v(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)+u\int_r^s \phi^v(x)dx\big(s\phi^v(s)+r\psi^v(r)\big)\Big]$$

$$\leq(\geq)\int_r^s \phi^u(x)dx\int_r^s \psi^v(x)dx\big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\big)\big(u+v\big)$$

$$\Leftrightarrow\quad \frac{v\int_r^s \psi^v(x)dx\big(s\phi^u(s)+r\phi^u(r)\big)+u\int_r^s \phi^v(x)dx\big(s\phi^v(s)+r\psi^v(r)\big)}{\int_r^s \phi^u(x)dx\int_r^s \psi^v(x)dx}$$

$$\leq(\geq)\ \frac{\big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\big)\big(u+v\big)}{\int_r^s \phi(x)\psi(x)dx}$$

$$\Leftrightarrow \frac{v\big(s\phi^u(s)+r\phi^u(r)\big)}{\int_r^s \phi^u(x)dx}+\frac{u\big(s\psi^v(s)+r\psi^v(r)\big)}{\int_r^s \psi^v(x)dx}\leq(\geq)\frac{\big(s\phi(s)\psi(s)+r\phi(r)\psi(r)\big)(u+v)}{\int_r^s \phi(x)\psi(x)dx}$$

This completes proof of Theorem 3.

Corollary 1. *Let $\phi(x)$ and $\psi(x)$ be two continuous functions and let their second order derivatives exists with*

$$\phi(x>0,\psi(x)>0),\int_r^s \phi(x)\psi(x)dx\neq 0,\int_s^r (\phi(x))^u dx\neq 0,\int_s^r \psi(x)^u dx\neq 0.$$

If $u,v>1$ and $\phi(x),\psi(x)$ are convex functions of opposite monotonicity and

$$\phi''\psi+\psi''\phi+2\phi'\psi'<0$$

then $H_3(r,s)$ is Schur-geometric convex with $r=r_1,r_2,...,r_n$, and $s=s_1,s_2,...s_n$ on R_+.

Corollary 2. *Let $\phi(x)$ and $\psi(x)$ be two continuous functions and let their second order derivatives exists with*

$$\phi(x > 0), \psi(x) > 0, \int_r^s \phi(x)\psi(x)dx \neq 0, \int_s^r (\phi(x))^u dx \neq 0, \int_s^r \psi(x)^u dx \neq 0.$$

If $u, v < 0$ and $\phi(x), \psi(x)$ are concave functions of opposite monotonicity then $H_3(r, s)$ is Schur-geometric concave with $r = r_1, r_2, ..., r_n,$ and $s = s_1, s_2, ...s_n$ on R_+.

Corollary 3. *Let $\phi(x)$ and $\psi(x)$ be two continuous functions and let their second order derivatives exists with*

$$\phi(x > 0), \psi(x) > 0, \int_r^s \phi(x)\psi(x)dx \neq 0, \int_s^r (\phi(x))^u dx \neq 0, \int_s^r \psi(x)^u dx \neq 0.$$

If $-1 < u < 0, 0 < v < 1, u+v > 0$ and $\phi(x), \psi(x)$ are concave functions of opposite monotonicity then $H_3(r, s)$ is Schur-geometric convex with $r = r_1, r_2, ..., r_n,$ and $s = s_1, s_2, ...s_n$ on R_+.

4. Application

The following applications are established by using our main results.

Theorem 4. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let u and v be two non-zero arbitrary real numbers. Then*

(i) if $u \geq 1, v \geq 1$ then

$$\left(\sum_{i=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{i=1}^{n} r_l^v\right)^{\frac{1}{v}} \geq \left(n^{\frac{1}{u}} + \frac{1}{v}\right) A_{n,r}\, A_{n,s}\,.$$

(ii) if $u \leq 1, v \leq 1$ then

$$\left(\sum_{i=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{i=1}^{n} r_l^v\right)^{\frac{1}{v}} \leq \left(n^{\frac{1}{u}} + \frac{1}{v}\right) A_{n,r}\, A_{n,s}\,.$$

Here

$$A_{n,r} = \frac{\sum_{l=1}^{n}(r_l)}{n},$$

$$A_{n,s} = \frac{\sum_{l=1}^{n}(s_l)}{n}$$

Proof. : (i) By Lemma 7 has a majorization inequality:

$$(r_1, r_2, ..., r_n) \succ \left(\frac{r_1 + r_2 + r_3 + ... + r_n}{n}, ..., \frac{r_1 + r_2 + r_3 + ... + r_n}{n}\right)$$

and by Theorem 1 and by definition 1, we have

if $u \geq 1$, then $H_1(r) \geq H_1(A_n, r)$, that is

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} \geq \left(n(A_{n,r})^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_l^v\right)^{\frac{1}{v}} = n^{\frac{1}{u}} A_{n,r} \left(n(A_{n,s})^v\right)^{\frac{1}{v}}$$

By majorization inequality, we have

$$(s_1, s_2, ..., s_n) \succ \left(\frac{s_1 + s_2 + s_3 + ... + s_n}{n}, ..., \frac{s_1 + s_2 + s_3 + ... + s_n}{n}\right)$$

and by Theorem 2 and Definition 1, we have
if $v \geq 1$, then $H_2(s) \geq H_2(A_n, s)$, that is

$$n^{\frac{1}{u}} A_{n,r} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \geq n^{\frac{1}{u}} A_{n,r} \left(n(A_{n,s})^v\right)^{\frac{1}{v}} = n^{\left(\frac{1}{u}+\frac{1}{v}\right)} A_{n,r}\, A_{n,s}$$

From the above relations, we have

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \geq n^{\left(\frac{1}{u}+\frac{1}{v}\right)} A_{n,r}\, A_{n,s}$$

exactness.
By Similar method the following inequality is also established,

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \leq n^{\left(\frac{1}{u}+\frac{1}{v}\right)} A_{n,r}\, A_{n,s} \tag{15}$$

The proof of Theorem 4 is complete.

Theorem 5. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let u and v be two non-zero arbitrary real numbers . Then*

(i) When $u > 1$, if $\dfrac{1}{u} + \dfrac{1}{v} = 1$ and $\{r_n\}$, $\{s_n\}$ have the opposite of monotonicity, then

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \geq nA_{n,r}\, A_{n,s} \geq \sum_{l=1}^{n} r_1 s_1$$

(ii) When $0 < u < 1$, if $\dfrac{1}{u} + \dfrac{1}{v} = 1$ and $\{r_n\}$,$\{s_n\}$ have the opposite of monotonicity, then

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \leq nA_{n,r}\, A_{n,s} \geq \sum_{l=1}^{n} r_1 s_1$$

Proof. : (i) When $u > 1$, if $\dfrac{1}{u} + \dfrac{1}{v} = 1$ and by Theorem 1, we have

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \geq nA_{n,r}\,A_{n,s} = A_{n,r}\,A_{n,s}$$

and by Lemma 5, we have

$$nA_{n,r}\,A_{n,s} = n\frac{\sum_{l=1}^{n}(r_1)}{n}\frac{\sum_{l=1}^{n}(s_1)}{n} = \frac{\sum_{l=1}^{n}(r_1)\sum_{l=1}^{n}(s_1)}{n} \geq n\frac{\sum_{l=1}^{n} r_1 s_1}{n} = \sum_{l=1}^{n} r_1 s_1$$

From the above relations, we have

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \geq nA_{n,r}\,A_{n,s} \geq \sum_{l=1}^{n} r_1 s_1$$

exactness.
By Similar method the following inequality is also established

$$\left(\sum_{l=1}^{n} r_l^u\right)^{\frac{1}{u}} \left(\sum_{l=1}^{n} s_j^v\right)^{\frac{1}{v}} \leq nA_{n,r}\,A_{n,s} \geq \sum_{l=1}^{n} r_1 s_1$$

The proof of Theorem 5 is complete.

5. Conclusion

In this paper, by using of majorization inequality theory we investigated the Schur geometrically convex about related functions of Holders Inequality, giving a complete critical condition of Schur geometrically convex function to Holders Inequality and some applications were established. Despite of these results, the authors are also interested to investigate the results of Schur harmonically convex and m-power convexity about related functions of Holders inequality in future research work.

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