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Common fixed point results for set-valued integral type contractions on metric spaces with directed graph

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Abstract. The aim of this paper is to establish new common fixed point results for multivalued integral type contraction mappings on a family of sets endowed with a graph. The obtained results generalize several recent ones. An example of application is included to illustrate the main existence theorem.

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1. Introduction

The Banach contraction principle is one of the most important tools in nonlinear analysis and is considered as the main source of inspiration in metric fixed point theory. Since its proof by S. Banach in 1922, this existence principle has been generalized in many

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directions over various spaces by several authors. Among these generalizations, the multi-valued version was established by Covitz and Nadler [14] in 1969 using Hausdorff-Pompeiu metric H in complete metric spaces. Recall

$$H(A,B)=\max(\sup_{x\in A} d(x,B),\sup_{x\in B} d(x,A)),$$

for bounded subsets A, B. A large amount of research works have followed these theorems. We briefly describe most recent papers and results. In 2002, Branciari [8] obtained a fixed point theorem for single valued maps satisfying an analogue of Banach contraction principle for integral type inequality. This result was further extended by many authors (we refer the reader to [4], [5], [10], [15], [18], and references therein).

In 2008, Jachymski [11] introduced the concept of G-contraction, that is a single-valued contraction mapping defined on a metric space with a graph structure (it preserves the edges and decreases weights of edges of the graph). Then Banach's contraction principle in ordered metric spaces was generalized in this new class of metric spaces.

Recently, Abbas *et al.* [1] obtained the existence of some fixed points for set valued mappings satisfying certain graphic contraction conditions on a domain of sets endowed with a directed graph.

The concept of a multivalued mappings has been used more recently by Nazir *et al.* [3] and Abbas *et al.* [2] in order to prove some common fixed point theorems on a domain of sets endowed with a directed graph.

Based essentially on works [1], [2], [7], [12] and [15], we will introduce in this paper the concept of graph (ψ,ϕ)-weak contraction which allows us to derive some new common fixed point results on the domain of sets endowed with a graph for this class of mappings.

First of all, we collect some basic notions and primary results we need to develop our results.

Let (X,d) be a metric space and denote by $P(X)$ the family of all nonempty subsets of X and by $CB(X)$ the family of all nonempty, closed, and bounded subsets of X. We need to consider two classes of functions:

Definition 1. *The class Ψ consists of nondecreasing continuous functions $\psi:[0,+\infty)\to[0,+\infty)$ such that $\psi(0)=0$ and ψ is sub-additive, i.e., for every $t_1,t_2\in\mathbb{R}^+$, $\psi(t_1+t_2)\le\psi(t_1)+\psi(t_2)$.*

Definition 2. *The class Φ is the set of functions $\varphi:[0,+\infty)\to[0,+\infty)$ which satisfy the following conditions:*
(i) φ is Lebesgue integrable and summable on each compact subset of $[0,+\infty)$,
(ii) $\int_0^\varepsilon\varphi(t)dt>0$, for each $\varepsilon>0$.

We now recall some lemmas that will be used in the sequel.

Lemma 1. *[13] Let $(r_n)_n$ be a nonnegative sequence and $\varphi\in\Phi$. Then*

$$\lim_{n\to+\infty}\int_0^{r_n}\varphi(t)dt=0$$

if and only if $\lim_{n\to+\infty} r_n=0$.

Lemma 2. *[19] For every $\varphi \in \Phi$, we have*

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^a \varphi(t)dt + \int_0^b \varphi(t)dt, \ \forall\, a, b \geq 0.$$

Lemma 3. *[14] If $A, B \in CB(X)$ with $H(A, B) < \epsilon$, then for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \epsilon$.*

In 2010, Ojha *et al.* obtained the following fixed point result for a multivalued mapping satisfying an analogue of Banach's contraction principle for an integral type inequality.

Theorem 1. *[15] Let (X, d) be a complete metric space. Suppose $T : X \to CB(X)$ is a multivalued contraction mapping such that for some $0 \leq \alpha < 1$,*

$$\int_0^{H(T(x),T(y))} \varphi(t)dt \leq \alpha \int_0^{M(x,y)} \varphi(t)dt,$$

where φ is lower semi-continuous, $\varphi(0) = 0$, and $\varphi(t) > 0$, $\forall\, t > 0$ and for all $x, y \in X$

$$M(x, y) = \max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\}.$$

Then T has a fixed point in X.

The second part of this introduction is devoted to graph and fixed point theories. First, by $\Delta = \Delta(X)$ it meant throughout the diagonal of the metric space X. A graph G is an ordered pair (V, E), where V is a set and $E \subset V \times V$ is a binary relation on V. Elements of E are called edges and are denoted by $E(G)$ while elements of V, denoted $V(G)$, are called vertices. If the direction is imposed in E, that is the edges are directed, then we get a digraph (directed graph). We assume that G has no parallel edges, i.e. two vertices cannot be connected by more than one edge. Then G can be identified with the pair $(V(G), E(G))$. If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_n$, $n \in \{0, 1, 2, \ldots k\}$ of vertices such that $x = x_0, \ldots, x_k = y$ and $(x_{n-1}, x_n) \in E(G)$ for $n \in \{1, 2, \ldots, k\}$. A graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Let G^{-1} be the graph obtained from G by reversing the direction of edges (the conversion of the graph G). We have

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of edges is symmetric, then

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Definition 3. *[1] Let A and B be two nonempty subsets of X. Then*
(a) "there is an edge between A and B", means there is an edge between some $a \in A$ and $b \in B$ which we denote by $(A, B) \subset E(G)$.

(b) "there is a path between A and B", means that there is a path between some $a \in A$ and $b \in B$.
In $CB(X)$, we define a relation R in the following way:

for $A, B \in CB(X)$, ARB if and only if there is a path between A and B.

We say that the relation R on $CB(X)$ is transitive if there is a path between A and B and there is a path between B and C, then there is a path between A and C.

Definition 4. *Let $S : CB(X) \to CB(X)$ be a multivalued mapping. The set $A \in CB(X)$ is said to be a fixed point of S if $S(A) = A$. The set of all fixed points of S is denoted by $F(S)$.*

Consider the set:

$$X_S := \{U \in CB(X) : (U, S(U)) \subset E(G)\}.$$

A subset A of $CB(X)$ is said to be complete if for any set $X, Y \in A$, there is an edge between X and Y.

Abbas *et al.* [1] have used the following property:

Definition 5. *A graph G is said to have property $(P^\star)$ if for any sequence $(X_n)_n$ in $CB(X)$ with $X_n \to X$, as $n \to \infty$, the existence of an edge between X_n and X_{n+1} for $n \in \mathbb{N}$ implies the existence of a subsequence $(X_{n_k})_k$ of (X_n) with an edge between X_{n_k} and X, for $k \in \mathbb{N}$.*

Then the authors of [1] obtained some fixed point results for multivalued self mappings on $CB(X)$ satisfying certain graph contraction conditions according to the following definition.

Definition 6. *Let $T : CB(X) \to CB(X)$ be a set-valued mapping. The mapping T is said to be a graph ϕ-contraction if the following conditions hold:*
(i) There is an edge between A and B implies there is an edge between $T(A)$ and $T(B)$ for all $A, B \in CB(X)$.
(ii) There is a path between A and B implies there is a path between $T(A)$ and $T(B)$ for all $A, B \in CB(X)$
(iii) There exists an upper semi-continuous and nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(t) < t$ for each $t > 0$ such that there is an edge between A and B implies

$$H(T(A), T(B)) \leq \phi(H(A, B)), \quad \text{for all } A, B \in CB(X).$$

Then they have established

Theorem 2. *Let (X, d) be a complete metric space endowed with a directed graph G such that $V(G) = X$ and $\Delta \subset E(G)$. If $T : CB(X) \to CB(X)$ is a graph ϕ-contraction mapping such that the relation R on $CB(X)$ is transitive, then the following statements hold:*
(i) If $F(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in F(T)$ is 0.
(ii) $X_T \neq \emptyset$ provided $F(T) \neq \emptyset$.
(iii) If $X_T \neq \emptyset$ and the weakly connected graph G satisfies the property $(P^\star)$, then T has a fixed point.
(iv) $F(T)$ is complete if and only if $F(T)$ is a singleton.

2. Main Existence Result

We first introduce

Definition 7. *Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\Delta \subset E(G)$. Let $S, T : CB(X) \to CB(X)$ be two multivalued mappings. The pair (S, T) of maps is said to be graph (ψ, ϕ)-weak contraction pair if*
(i) for every U in $CB(X)$, $(U, S(U)) \subset E(G)$ and $(U, T(U)) \subset E(G)$,
(ii) there exists an nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, $\varphi \in \Phi$, $\psi \in \Psi$, and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq T(B)$, then

$$\psi\left(\int_0^{H(S(A),T(B))} \varphi(t)dt\right) \leq \phi\left(\psi\left(\int_0^{M_{S,T}(A,B)} \varphi(t)dt\right)\right) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt,$$

where

$$M_{S,T}(A, B) = \max\Big\{H(A, B), H(A, S(A)), H(B, T(B)), \frac{H(A, T(B)) + H(B, S(A))}{2}\Big\}$$

and

$$N_{S,T}(A, B) = \min\{H(A, S(A)), H(B, T(B)), H(A, T(B)), H(B, S(A))\}.$$

Remark 1. *([16], [17]) It is obvious that for each nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, the following statements are satisfied:*
(i) $\lim_{i\to\infty} \phi^i(t) = 0$ for all $t > 0$,
(ii) $\phi(t) < t$ for all $t > 0$,
(iii) $\phi(0) = 0$.

Remark 2. *It is obvious that if a pair (S, T) of multivalued mappings on $CB(X)$ is a graph (ψ, ϕ)-weak contraction for a graph G, then the pair (S, T) is also a graph (ψ, ϕ)-weak contraction for the graphs G^{-1}, $\tilde{G}$, and G_0. Here the graph G_0 is defined by $E(G_0) = X \times X$.*

We are in position to state and prove an existence result of common fixed point results for multivalued self maps on $CB(X)$ satisfying graph (ψ, ϕ)-weak contraction conditions on a metric space endowed with a graph.

Theorem 3. *Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$, $\Delta \subset E(G)$, the relation R on $CB(X)$ is transitive, and $S, T : CB(X) \to CB(X)$ is a graph (ψ, ϕ)-weak contraction pair. Then the following statements hold:*
(i) $F(S)$ or $F(T) \neq \emptyset$ if and only if $F(S) \cap F(T) \neq \emptyset$.
(ii) $F(S) \cap F(T) \neq \emptyset$ provided that G is weakly connected and satisfies the property $(P^\star)$.
(iii) If $F(S) \cap F(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to $U, V \in F(S) \cap F(T)$ is 0.
(iv) $F(S) \cap F(T)$ is complete if and only if $F(S) \cap F(T)$ is a singleton.

Proof. **(1)** Suppose that $F(S) \neq \emptyset$. By assumption, $(U, S(U)) \subset E(G)$. To prove that $U \in F(T)$, assume on contrary that $U \notin F(T)$. Since the pair (S,T) is a graph (ψ, ϕ)-weak contraction and $(U,U) \subset E(G)$, then

$$\begin{aligned}
\psi\left(\int_0^{H(U,T(U))} \varphi(t)dt\right) &= \psi\left(\int_0^{H(S(U),T(U))} \varphi(t)dt\right)\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(U,U)} \varphi(t)dt\right)\right) + L\int_0^{N_{S,T}(U,U)} \varphi(t)dt\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(U,U)} \varphi(t)dt\right)\right) + L\int_0^{H(U,S(U))} \varphi(t)dt\\
&= \phi\left(\psi\left(\int_0^{M_{S,T}(U,U)} \varphi(t)dt\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
M_{S,T}(U,U) &= \max = \Big\{H(U,U), H(U,S(U)), H(U,T(U)),\\
&\quad \frac{H(U,T(U)) + H(U,S(U))}{2}\Big\}\\
&= H(U,T(U)).
\end{aligned}$$

By property of ϕ, we have

$$\begin{aligned}
\psi\left(\int_0^{H(U,T(U))} \varphi(t)dt\right) &\leq \phi\left(\psi\left(\int_0^{H(U,T(U))} \varphi(t)dt\right)\right)\\
&< \psi\left(\int_0^{H(U,T(U))} \varphi(t)dt\right),
\end{aligned}$$

leading to a contradiction.

(2) Let $A_0 \in CB(X)$ be arbitrary. If $A_0 \in F(S)$ or $A_0 \in F(T)$, then from (i), $F(S) \cap F(T) \neq \emptyset$. Now suppose that $A_0 \notin F(S)$ and $A_0 \notin F(T)$. By the definition of a (ψ, ϕ)-weak contraction contraction, we have $(A_0, S(A_0)) \subset E(G)$ which implies that there exists some x_0 in A_0 such that there is an edge between x_0 and some $x_1 \in S(A_0)$. Let $A_1 = S(A_0)$; then by definition, $(A_1, T(A_1)) \subset E(G)$ which implies that there is an edge between x_1 and some $x_2 \in T(A_1)$. Then $A_2 = T(A_1)$. By induction, we thus construct a sequence $(A_n)_n$ such that $A_{2n+1} = S(A_{2n})$, $A_{2n+2} = T(A_{2n+1})$, and $(A_n, A_{n+1}) \subset E(G)$ for $n \in \mathbb{N}$. Observe that we have assumed $A_{2n} \neq A_{2n+1}$, otherwise $A_{2n} = A_{2n+1}$, for some n, $S(A_{2n}) = A_{2n+1} = A_{2n}$, and thus $A_{2n} \in F(S)$. By (i), $A_{2n} \in F(S) \cap F(T)$. Since the pair (S,T) is a graph (ψ, ϕ)-weak contraction and $(A_{2n}, A_{2n+1}) \subset E(G)$, we derive the estimates

$$\begin{aligned}
\psi\left(\int_0^{H(A_{2n+1}A_{2n+2})} \varphi(t)dt\right) &= \psi\left(\int_0^{H(S(A_{2n}),T(A_{2n+1}))} \varphi(t)dt\right)\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n},A_{2n+1})} \varphi(t)dt\right)\right)\\
&\quad + L\int_0^{N_{S,T}(A_{2n},A_{2n+1})} \varphi(t)dt\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n},A_{2n+1})} \varphi(t)dt\right)\right)\\
&\quad + L\int_0^{H(A_{2n+1},S(A_{2n}))} \varphi(t)dt\\
&= \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n},A_{2n+1})} \varphi(t)dt\right)\right),
\end{aligned}$$

where

$$\begin{aligned} & M_{S,T}(A_{2n}, A_{2n+1}) \\ = \; & \max\Big\{H(A_{2n}, A_{2n+1}), H(A_{2n}, S(A_{2n})), H(A_{2n+1}, T(A_{2n+1})), \\ & \frac{H(A_{2n}, T(A_{2n+1})) + H(A_{2n+1}, S(A_{2n}))}{2}\Big\} \\ = \; & \max\Big\{H(A_{2n}, A_{2n+1}), H(A_{2n}, A_{2n+1}), H(A_{2n+1}, A_{2n+2}), \\ & \frac{H(A_{2n}, A_{2n+2}) + H(A_{2n+1}, A_{2n+1})}{2}\Big\} \\ \leq \; & \max\Big\{H(A_{2n}, A_{2n+1}), H(A_{2n+1}, A_{2n+2}), \\ & \frac{H(A_{2n}, A_{2n+1}) + H(A_{2n+1}, A_{2n+2})}{2}\Big\} \\ = \; & \max\Big\{H(A_{2n}, A_{2n+1}), H(A_{2n+1}, A_{2n+2})\Big\}. \end{aligned}$$

Hence

$$\psi\left(\int_0^{H(A_{2n+1},A_{2n+2})} \varphi(t)dt\right) \leq \phi\left(\psi\left(\int_0^{\max\{H(A_{2n},A_{2n+1}),H(A_{2n+1},A_{2n+2})\}} \varphi(t)dt\right)\right).$$

By the property of ϕ, we have for all $n \in \mathbb{N}$

$$\psi\left(\int_0^{H(A_{2n+1},A_{2n+2})} \varphi(t)dt\right) \leq \phi\left(\psi\left(\int_0^{H(A_{2n},A_{2n+1})} \varphi(t)dt\right)\right).$$

Since the pair (S,T) is a graph (ψ,ϕ)-weak contraction and $(A_{2n+2}, A_{2n+1}) \subset E(G)$, we have that

$$\begin{aligned} \psi\left(\int_0^{H(A_{2n+2}A_{2n+3})} \varphi(t)dt\right) &= \psi\left(\int_0^{H(T(A_{2n+1}),S(A_{2n+2}))} \varphi(t)dt\right) \\ &= \psi\left(\int_0^{H(S(A_{2n+2}),T(A_{2n+1}))} \varphi(t)dt\right) \\ &\leq \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n+2},A_{2n+1})} \varphi(t)dt\right)\right) \\ &\quad +L\int_0^{N_{S,T}(A_{2n+2},A_{2n+1})} \varphi(t)dt \\ &\leq \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n+2},A_{2n+1})} \varphi(t)dt\right)\right) \\ &\quad +L\int_0^{H(A_{2n+2},T(A_{2n+1}))} \varphi(t)dt \\ &= \phi\left(\psi\left(\int_0^{M_{S,T}(A_{2n+2},A_{2n+1})} \varphi(t)dt\right)\right), \end{aligned}$$

where

$$\begin{aligned}
&M_{S,T}(A_{2n+2},A_{2n+1})\\
=\ &\max\Big\{H(A_{2n+2},A_{2n+1}),H(A_{2n+2},S(A_{2n+2})),H(A_{2n+1},T(A_{2n+1})),\\
&\frac{H(A_{2n+2},T(A_{2n+1}))+H(A_{2n+1},S(A_{2n+2}))}{2}\Big\}\\
=\ &\max\Big\{H(A_{2n+2},A_{2n+1}),H(A_{2n+2},A_{2n+3}),H(A_{2n+1},A_{2n+2}),\\
&\frac{H(A_{2n+2},A_{2n+2})+H(A_{2n+1},A_{2n+3})}{2}\Big\}\\
\leq\ &\max\Big\{H(A_{2n+2},A_{2n+1}),H(A_{2n+2},A_{2n+3}),\\
&\frac{H(A_{2n+1},A_{2n+2})+H(A_{2n+2},A_{2n+3})}{2}\Big\}\\
=\ &\max\Big\{H(A_{2n+2},A_{2n+1}),H(A_{2n+2},A_{2n+3})\Big\}.
\end{aligned}$$

Then

$$\psi\left(\int_0^{H(A_{2n+2},A_{2n+3})}\varphi(t)dt\right)\leq\phi\left(\psi\left(\int_0^{\max\{H(A_{2n+2},A_{2n+1}),H(A_{2n+2},A_{2n+3})\}}\varphi(t)dt\right)\right).$$

By the property of ϕ, we obtain for all $n\in\mathbb{N}$

$$\psi\left(\int_0^{H(A_{2n+2},A_{2n+3})}\varphi(t)dt\right)\leq\phi\left(\psi\left(\int_0^{H(A_{2n+1},A_{2n+2})}\varphi(t)dt\right)\right).$$

Hence

$$\psi\left(\int_0^{H(A_n,A_{n+1})}\varphi(t)dt\right)\ \leq\ \phi\left(\psi\left(\int_0^{H(A_{n-1},A_n)}\varphi(t)dt\right)\right). \tag{1}$$

(1) guarantees that

$$\begin{aligned}
\psi\left(\int_0^{H(A_n,A_{n+1})}\varphi(t)dt\right)\ &\leq\ \phi\left(\psi\left(\int_0^{H(A_{n-1},A_n)}\varphi(t)dt\right)\right)\\
&\leq\ \phi^2\left(\psi\left(\int_0^{H(A_{n-2},A_{n-1})}\varphi(t)dt\right)\right)\\
&\ \vdots\\
&\leq\ \phi^n\left(\psi\left(\int_0^{H(A_0,A_1)}\varphi(t)dt\right)\right).
\end{aligned}$$

We prove now that $(A_n)_n$ is a Cauchy sequence in $CB(X)$. By Lemma 2 and the property of ψ, we have the estimates

$$\begin{aligned}
\psi\left(\int_0^{H(A_nA_m)}\varphi(t)dt\right)\ &\leq\ \psi\left(\int_0^{\sum_{i=n}^{m-1}H(A_i,A_{i+1})}\varphi(t)dt\right)\\
&\leq\ \psi\left(\sum_{i=n}^{m-1}\int_0^{H(A_i,A_{i+1})}\varphi(t)dt\right)\\
&\leq\ \sum_{i=n}^{m-1}\psi\left(\int_0^{H(A_i,A_{i+1})}\varphi(t)dt\right)\\
&\leq\ \sum_{i=n}^{m-1}\phi^i\left(\psi\left(\int_0^{H(A_0,A_1)}\varphi(t)dt\right)\right),
\end{aligned}$$

for each $m, n \in \mathbb{N}$ with $m > n$. By taking the limit, as $n, m \to \infty$, we find that $\psi\left(\int_0^{H(A_n,A_m)} \varphi(t)dt\right) \to 0$; then $\int_0^{H(A_n,A_m)} \varphi(t)dt \to 0$, as $n, m \to \infty$. By Lemma 1, $H(A_n, A_m) \to 0$, as $n, m \to \infty$. Thus $(A_n)_n$ is a Cauchy sequence in $CB(X)$. Since (X, d) is complete, $(CB(X), H)$ is complete too and we deduce that $A_n \to V$, as $n \to \infty$ for some $V \in CB(X)$.

To prove that $V = S(V) = T(V)$, it is sufficient to show that $V = S(V)$, the result then follows from (i). Suppose that $V \neq S(V)$. Since $(A_{2n+1}, A_{2n+2}) = (A_{2n+1}, T(A_{2n+1})) \subset E(G)$ for all $n \in \mathbb{N}$, by property $(P^\star)$, there exists a subsequence $(A_{2n_k+1})_k$ of $(A_{2n+1})_n$ such that there is an edge between A_{2n_k+1} and V for every $k \in \mathbb{N}$. Since the pair (S, T) is a graph (ψ, ϕ)-weak contraction and $(V, A_{2n_k+1}) \subset E(G)$, we have that

$$\begin{aligned}
\psi\left(\int_0^{H(S(V),A_{2n_k+2})} \varphi(t)dt\right) &= \psi\left(\int^{H(S(V),T(A_{2n_k+1}))} \varphi(t)dt\right)\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(V,A_{2n_k+1})} \varphi(t)dt\right)\right)\\
&\quad + L\int_0^{N_{S,T}(V,A_{2n_k+1})} \varphi(t)dt\\
&\leq \phi\left(\psi\left(\int_0^{M_{S,T}(V,A_{2n_k+1})} \varphi(t)dt\right)\right)\\
&\quad + L\int_0^{H(V,T(A_{2n_k+1}))} \varphi(t)dt\\
&= \phi\left(\psi\left(\int_0^{M_{S,T}(V,A_{2n_k+1})} \varphi(t)dt\right)\right)\\
&\quad + L\int_0^{H(V,A_{2n_k+2})} \varphi(t)dt,
\end{aligned}$$

where

$$\begin{aligned}
&M_{S,T}(V, A_{2n_k+1})\\
= \ &\max\Big\{H(V, A_{2n_k+1}), H(V, S(V)), H(A_{2n_k+1}, T(A_{2n_k+1})),\\
&\frac{H(V, T(A_{2n_k+1})) + H(A_{2n_k+1}, S(V))}{2}\Big\}\\
= \ &\max\Big\{H(V, A_{2n_k+1}), H(V, S(V)), H(A_{2n_k+1}, A_{2n_k+2}),\\
&\frac{H(V, A_{2n_k+2}) + H(A_{2n_k+1}, S(V))}{2}\Big\}.
\end{aligned}$$

Since $\lim\limits_{k\to+\infty} H(V, A_{2n_k+1}) = H(V, V) = 0$, then there exists $k_1 \in \mathbb{N}$ such that

$$H(V, A_{2n_k+1}) \leq \frac{H(V, S(V))}{2}, \quad \forall\, k \geq k_1.$$

Since $\lim\limits_{k\to+\infty} H(A_{2n_k+1}, A_{2n_k+2}) = 0$, then there exists $k_2 \in \mathbb{N}$ such that

$$H(A_{2n_k+1}, A_{2n_k+2}) \leq \frac{H(V, S(V))}{2}, \quad \forall\, k \geq k_2.$$

In addition

$$\lim_{k\to+\infty} \frac{H(V, A_{2n_k+2}) + H(A_{2n_k+1}, S(V))}{2} = \frac{H(V, S(V))}{2}$$

provides the existence of some $k_3 \in \mathbb{N}$ such that

$$\frac{H(V, A_{2n_k+2}) + H(A_{2n_k+1}, S(V))}{2} \le H(V, S(V)), \quad \forall\, k \ge k_3.$$

As a consequence for $k \ge k_0 = \max\{k_1, k_2, k_3\}$, we have

$$M_{S,T}(V, A_{2n_k+1}) = H(V, S(V)), \quad \forall\, k \ge k_0.$$

Hence for all $k \ge k_0$,

$$\psi\left(\int_0^{H(S(V),A_{2n_k+1})} \varphi(t)dt\right) \le \phi\left(\psi\left(\int_0^{H(V,S(V))} \varphi(t)dt\right)\right) + L\int_0^{H(V,A_{2n_k+2})} \varphi(t)dt.$$

Taking the limit as $k \to +\infty$ and using properties of ϕ and ψ, we find

$$\begin{aligned} \psi\left(\int_0^{H(S(V),V)} \varphi(t)dt\right) &\le \phi\left(\psi\left(\int_0^{H(S(V),V)} \varphi(t)dt\right)\right) \\ &< \psi\left(\int_0^{H(S(V),V)} \varphi(t)dt\right), \end{aligned}$$

which is a contradiction. Then $S(V) = V$, that is $V \in F(S)$. By (i), $F(S) \cap F(T) \ne \emptyset$.

(3) Suppose that $F(S) \cap F(T)$ is complete. Let $U, V \in F(S) \cap F(T)$ and suppose that $H(U, V) \ne 0$. Since the pair (S, T) is a graph (ψ, ϕ)-weak contraction, we have

$$\begin{aligned} \psi\left(\int_0^{H(U,V)} \varphi(t)dt\right) &= \psi\left(\int_0^{H(S(U),T(V))} \varphi(t)dt\right) \\ &\le \phi\left(\psi\left(\int_0^{M_{S,T}(U,V)} \varphi(t)dt\right)\right) + L\int_0^{N_{S,T}(U,V)} \varphi(t)dt \\ &= \phi\left(\psi\left(\int_0^{M_{S,T}(U,V)} \varphi(t)dt\right)\right), \end{aligned}$$

where

$$\begin{aligned} M_{S,T}(U,V) &= \max = \Big\{H(U,V), H(U,S(U)), H(V,T(V)), \\ &\quad \frac{H(U,T(V)) + H(V,S(U))}{2}\Big\} \\ &= \max\Big\{H(U,V), H(U,U), H(V,V), \\ &\quad \frac{H(U,V) + H(V,U)}{2}\Big\} \\ &= H(U,V). \end{aligned}$$

Again the property of ϕ yields

$$\begin{aligned} \psi\left(\int_0^{H(U,V)} \varphi(t)dt\right) &\le \phi\left(\psi\left(\int_0^{H(U,V)} \varphi(t)dt\right)\right) \\ &< \psi\left(\int_0^{H(U,V)} \varphi(t)dt\right), \end{aligned}$$

leading to a contradiction.

(4) Suppose that $F(S) \cap F(T)$ is complete. Let $U, V \in CB(X)$ be such that $U, V \in F(S) \cap F(T)$. By (iii), we have $H(U, V) = 0$, i.e., $F(S) \cap F(T)$ is singleton. Conversely, suppose that $F(S) \cap F(T)$ is singleton. Since $\Delta \subset E(G)$, then $F(S) \cap F(T)$ is complete.

Example 1. *Let* $X = \{1, 2, 3, \cdots, n\}$ *(with* $n > 3$*) be the set of integers endowed with the metric* $d : X \times X \to [0, +\infty)$ *defined by*

$$d(x, y) = \begin{cases} 0, & \text{if} \quad x = y, \\ \frac{1}{n}, & \text{if} \quad x, y \in \{1, 2, 3, 4\}, x \neq y, \\ \frac{n+2}{n+3}, & \text{if} \quad \text{otherwise}. \end{cases}$$

The Pompeiu-Hausdorff metric is given by

$$H(A, B) = \begin{cases} 0, & \text{if} \quad A = B, \\ \frac{1}{n}, & \text{if} \quad A, B \subseteq \{1, 2, 3, 4\}, A \neq B, \\ \frac{n+2}{n+3}, & \text{if} \quad \text{otherwise}. \end{cases}$$

Define the graph $G = (V(G), E(G))$ *with* $V(G) = X$ *and* $E(G) = \{(i, j) \in X \times X : i \leq j\}$. *The graph* G *for* $n = 4$ *and* $n = 5$ *along with Pompeiu-Hausdorff distance assigned are shown in Figure 1 and 2, respectively.*

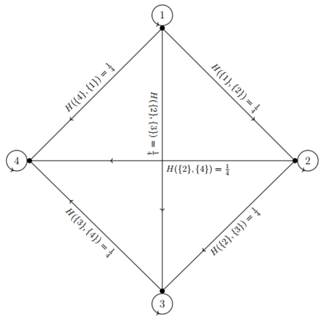

Figure 1: The graph G for $n = 4$.

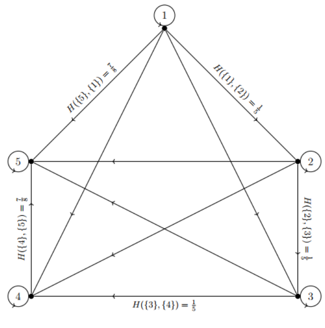

Figure 2: The graph G for $n = 5$.

Let S *and* $T : CB(X) \to CB(X)$ *be defined by*

$$S(U) = \begin{cases} \{1, 2\}, & \text{if} \quad U \subseteq \{1, 2, 3, 4\}, \\ \{3, 4\}, & \text{if} \quad U \subseteq \{5, 6\}, \\ \{1, 2, 3, 4\}, & \text{if} \quad \text{otherwise}. \end{cases}$$

$$T(U) = \begin{cases} \{1,2\}, & \text{if} \quad U \subseteq \{1,2,3,4\}, \\ \{3\}, & \text{if} \quad U \subsetneq \{1,2,3,4\}. \end{cases}$$

Define the mappings $\psi, \varphi, \phi : [0,+\infty) \to [0,+\infty)$ *by* $\psi(t) = \frac{5t}{3t+3}$, $\varphi(t) = t$, *and*

$$\phi(t) = \begin{cases} t^2, & \text{if} \quad t \in [0, \frac{1}{3}], \\ \frac{t}{3}, & \text{if} \quad t \in (\frac{1}{3}, +\infty). \end{cases}$$

Given $L > 0$, *then* S *and* T *form a graph* (ψ, ϕ)*-weak contraction and* $\{1,2\}$ *is the common fixed point of* S *and* T. *We check this*
(i) it clear that for all $U \in CB(X)$, $(U, S(U)) \subset E(G)$ *and* $(U, T(U)) \subset E(G)$,
(ii) for all $(A,B) \subset E(G)$ *and* $S(A) \neq T(B)$, *consider five cases:*

Case 1. *If* $A \subseteq \{1,2,3,4\}$, $B \subsetneq \{1,2,3,4\}$, *then*

$$\begin{aligned} \psi(\int_0^{H(\{1,2\},\{1,3,4\})} \varphi(t)dt) &= \frac{5}{3+6n^2} \\ &\leq \frac{5(n+2)^2}{9(n+2)^2+18(n+3)^2} \\ &\leq \phi(\psi(\int_0^{M_{S,T}(A,B)} \varphi(t)dt)) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt. \end{aligned}$$

Case 2. *If* $A \subseteq \{5,6\}$, $B \subseteq \{1,2,3,4\}$, *then*

$$\begin{aligned} \psi(\int_0^{H(\{3,4\},\{1,2\})} \varphi(t)dt) &= \frac{5}{3+6n^2} \\ &\leq \frac{5(n+2)^2}{9(n+2)^2+18(n+3)^2} \\ &\leq \phi(\psi(\int_0^{M_{S,T}(A,B)} \varphi(t)dt)) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt. \end{aligned}$$

Case 3. *If* $A \subseteq \{5,6\}$, $B \subsetneq \{1,2,3,4\}$, *then*

$$\begin{aligned} \psi(\int_0^{H(\{3,4\},\{1,3,4\})} \varphi(t)dt) &= \frac{5}{3+6n^2} \\ &\leq \frac{5(n+2)^2}{9(n+2)^2+18(n+3)^2} \\ &\leq \phi(\psi(\int_0^{M_{S,T}(A,B)} \varphi(t)dt)) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt. \end{aligned}$$

Case 4. *If* $A \subsetneq \{1,2,3,4,5,6\}$, $B \subseteq \{1,2,3,4\}$, *then*

$$\begin{aligned} \psi(\int_0^{H(\{1,2,3,4\},\{1,2\})} \varphi(t)dt) &= \frac{5}{3+6n^2} \\ &\leq \frac{5(n+2)^2}{9(n+2)^2+18(n+3)^2} \\ &\leq \phi(\psi(\int_0^{M_{S,T}(A,B)} \varphi(t)dt)) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt. \end{aligned}$$

Case 5. *If* $A \subsetneq \{1,2,3,4,5,6\}$, $B \subsetneq \{1,2,3,4\}$, *then*

$$\begin{aligned} \psi(\int_0^{H(\{1,2,3,4\},\{1,3,4\})} \varphi(t)dt) &= \frac{5}{3+6n^2} \\ &\leq \frac{5(n+2)^2}{9(n+2)^2+18(n+3)^2} \\ &\leq \phi(\psi(\int_0^{M_{S,T}(A,B)} \varphi(t)dt)) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt. \end{aligned}$$

3. Consequences

The following results follow from Theorem 3.

Corollary 1. *Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\Delta \subset E(G)$. Suppose that the mappings $S, T : CB(X) \to CB(X)$ satisfy the following conditions:*
(a) for every U in $CB(X)$, $(U, S(U)) \subset E(G)$ and $(U, T(U)) \subset E(G)$,
(b) there exists an nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, $\psi \in \Psi$, and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq T(B)$, then

$$\psi(H(S(A), T(B))) \leq \phi(\psi(M_{S,T}(A, B))) + LN_{S,T}(A, B).$$

If the relation R on $CB(X)$ is transitive, then the following statements hold:
(i) $F(S)$ or $F(T) \neq \emptyset$ if and only if $F(S) \cap F(T) \neq \emptyset$.
(ii) $F(S) \cap F(T) \neq \emptyset$ provided that G is weakly connected and satisfies the property $(P^\star)$.
(iii) If $F(S) \cap F(T)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in F(S) \cap F(T)$ is 0.
(iv) $F(S) \cap F(T)$ is complete if and only if $F(S) \cap F(T)$ is a singleton.

Corollary 2. *Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$, $\Delta \subset E(G)$. Suppose that the mapping $S : CB(X) \to CB(X)$ satisfy the following conditions:*
(a) for every U in $CB(X)$, $(U, S(U)) \subset E(G)$,
(b) there exists an nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, $\psi \in \Psi$, $\varphi \in \Phi$, and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq S(B)$, then

$$\psi\left(\int_0^{H(S(A),S(B))} \varphi(t)dt\right) \leq \phi\left(\psi\left(\int_0^{M(A,B)} \varphi(t)dt\right)\right) + L\int_0^{N(A,B)} \varphi(t)dt,$$

where

$$\begin{aligned} M(A, B) \quad = \quad & \max\Big\{H(A, B), H(A, S(A)), H(B, S(B)), \\ & \frac{H(A, S(B)) + H(B, S(A))}{2}\Big\} \end{aligned}$$

and

$$N(A, B) = \min\{H(A, S(A)), H(B, S(B)), H(A, S(B)), H(B, S(A))\}.$$

If the relation R on $CB(X)$ is transitive, then the following statements hold
(i) $F(S) \neq \emptyset$ provided that G is weakly connected and satisfies the property $(P^\star)$.
(ii) If $F(S)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in F(S)$ is 0.
(iv) $F(S)$ is complete if and only if $F(S)$ is a singleton.

Corollary 3. *Let (X, d) be a metric space endowed with a directed graph G such that $V(G) = X$ and $\Delta \subset E(G)$. Suppose that the mapping $S : CB(X) \to CB(X)$ satisfies the following conditions:*
(a) for every U in $CB(X)$, $(U, S(U)) \subset E(G)$,
(b) there exists an nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, $\psi \in \Psi$, and $L \geq 0$ such that if there is an edge between A and B with $S(A) \neq S(B)$, then

$$\psi(H(S(A), S(B))) \leq \phi(\psi(M(A, B))) + LN(A, B).$$

If the relation R on $CB(X)$ is transitive, then the following statements hold:
(i) $F(S) \neq \emptyset$ provided that G is weakly connected and satisfies the property $(P^\star)$.
(ii) If $F(S)$ is complete, then the Pompeiu-Hausdorff weight assigned to the $U, V \in F(S)$ is 0.
(iv) $F(S)$ is complete if and only if $F(S)$ is a singleton.

In case of ε-chainable complete metric spaces,we have

Theorem 4. *Let (X, d) be a ε-chainable complete metric space for some $\varepsilon > 0$. Suppose that the mappings $S, T : CB(X) \to CB(X)$ satisfy that for every $A, B \in CB(X)$ with $S(A) \neq T(B)$ and $0 < H(A, B) < \varepsilon$, there exists an nondecreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\sum_{i=0}^{\infty} \phi^i(t)$ is convergent for all $t > 0$, $\psi \in \Psi$, $\varphi \in \Phi$, and $L \geq 0$ with*

$$\psi\left(\int_0^{H(S(A),T(B))} \varphi(t)dt\right) \leq \phi\left(\psi\left(\int_0^{M_{S,T}(A,B)} \varphi(t)dt\right)\right) + L\int_0^{N_{S,T}(A,B)} \varphi(t)dt.$$

Then S and T have a common fixed point.

Proof. Define the graph $G = (V(G), E(G))$ by $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$. It clear that the ε-chainability of (X, d) implies that G is connected. Let $A, B \in CB(X)$ be such that $0 < H(A, B) < \varepsilon$; by Lemma 3, $(A, B) \subset E(G)$. It is easily seen that the pair (S, T) is a graph (ψ, ϕ)-weak contraction and that property $(P^\star)$ also holds true. Therefore Theorem 4 follows directly from Theorem 3.

Remark 3 (Concluding remarks). **(1)** *If in Corollary 2, we take $\psi(t) = t$, $\varphi(t) = 1$, $L = 0$ and $E(G) = X \times X$ then G is connected and Corollary 2 improves and generalizes Theorem 2.1 by Abbas et al. [1], Theorem 3.1 by Beg and Butt [6], and Theorem 3.1 by Jachymski [11].*
(2) *Taking G with $E(G) = X \times X$, $\psi(t) = t$, $\phi(t) = \alpha t$, and $L = 0$ in Theorem 3, we recover the main common fixed point theorem proved in [15].*
(3) *If in Theorem 4 $S = T$, then we obtain an extension and generalization of [9][Theorem 5.1].*
(4) *If in Corollary 2, we take $E(G) = X \times X$, then we obtain a generalization of [8][Theorem 2.1] and [18][Theorem 2].*

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