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Solution of Delay Differential equation via N_1^v iteration algorithm

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Abstract. The aim of this paper is to define a new iteration scheme N_1^v which converges to a fixed point faster than some previously existing methods such as Picard, Mann, Ishikawa, Noor, SP, CR, S, Picard-S, Garodia, K and K^* methods etc. The effectiveness and efficiency of our algorithm is confirmed by numerical example and some strong convergence, weak convergence, T-stability and data dependence results for contraction mapping are also proven. Moreover, it is shown that differential equation with retarted argument is solved using N_1^v iteration process.

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Key Words and Phrases: Iteration schemes, N_1^v iteration scheme, Convergence analysis, $T-$*stability*, Data Dependency

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1. Introduction

Throughout in this paper, we will denote set of natural numbers by $\mathbb{N}$ and set of real numbers by $\mathbb{R}$. A mapping T on a subset C of a Banach space E is said to be nonexpansive if

$$||Tx - Ty|| \leq ||x - y||, \text{ for all } x, y \in C.$$

An element $q \in C$ is said to be a fixed point of T if $q = T(q)$. From now on, we will denote set of all fixed points of T by T_f. A mapping $T : C \to C$ is said to be quasi-nonexpansive mappings if $T_f \neq \emptyset$ and $||Tx - Tq|| \leq ||x - q||$ for all $x \in C$ and $q \in T_f$. The existence of fixed points for nonexpansive mappings in the setting of Banach spaces was studied independently by Browder [3], Gohde [6] and Kirk [8]. They proved that, if C is nonempty closed bounded and convex subset of a uniformly convex Banach space, then every nonexpansive mapping $T : C \to C$ has at-least one fixed point. A numbers of generalization of nonexpansive mappings have been considered by some authors in recent years.

It is natural to study the computation of fixed points for the known existence results, which is not an easy task. The Banach contraction mapping principle uses Picard iteration process $x_{n+1} = Tx_n$ for approximation of the unique fixed point. Some other well-known iteration schemes are Mann [9], Ishikawa [7], S [13], Noor [10], Abbas [1], Thakur et. al. [4] and so on. Speed of convergence plays an important role for an iteration process to be preferred on another iteration process. Rhoades [12] mentioned that the Mann iteration process for decreasing function converge faster than the Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann iteration process. More details can be found in [16], [18], [15], [19], [5].

The most popular and simplest iteration method is formulated by

$$\begin{cases} p_0 \in C \\ p_{n+1} = Tp_n, \qquad n \in \mathbb{N} \end{cases} \tag{1.3}$$

and is known as Picard iteration method, which is communally used to approximate fixed point of contraction mappings satisfying

$$\|Tx - Ty\| \leq \mu \|x - y\|, \qquad \mu \in (0, 1), \tag{1.4}$$

for all $x, y \in C$. The subsequent iteration methods are mention to as Mann, Ishikawa, Noor, SP, S, CR, Picard-S, Garodia's, K and K^* iteration methods, respectively:

$$\begin{cases} v_0 \in C, \\ \zeta_{n+1} = (1 - \sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

$$\begin{cases} v_0 \in C, \\ v_{n+1} = (1 - \sigma_n^0)v_n + \sigma_n^0 Tw_n, \\ w_n = (1 - \sigma_n^1)v_n + \sigma_n^1 Tv_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.6}$$

$$\begin{cases} w_0 \in C, \\ w_{n+1} = (1-\sigma_n^0)w_n + \sigma_n^0 T w_n, \\ v_n = (1-\sigma_n^1)w_n + \sigma_n^1 T u_n, \\ u_n = (1-\sigma_n^2)w_n + \sigma_n^2 T w_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.7}$$

$$\begin{cases} q_0 \in C, \\ q_{n+1} = (1-\sigma_n^0)r_n + \sigma_n^0 T r_n, \\ r_n = (1-\sigma_n^1)s_n + \sigma_n^1 T s_n, \\ s_n = (1-\sigma_n^2)q_n + \sigma_n^2 T q_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.8}$$

$$\begin{cases} t_0 \in C, \\ t_{n+1} = (1-\sigma_n^0)T t_n + \sigma_n^0 T u_n, \\ u_n = (1-\sigma_n^1)t_n + \sigma_n^1 T t_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.9}$$

$$\begin{cases} u_0 \in C, \\ u_{n+1} = (1-\sigma_n^0)v_n + \sigma_n^0 T v_n, \\ v_n = (1-\sigma_n^1)T u_n + \sigma_n^1 T w_n, \\ u_n = (1-\sigma_n^2)u_n + \sigma_n^2 T u_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.10}$$

$$\begin{cases} \jmath_0 \in C, \\ \jmath_{n+1} = T k_n, \\ k_n = (1-\sigma_n^0)T \jmath_n + \sigma_n^0 T \ell_n, \\ \ell_n = (1-\sigma_n^1)\jmath_n + \sigma_n^1 T \jmath_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.11}$$

$$\begin{cases} x_0'' \in C, \\ x_{n+1}'' = T y_n'', \\ y_n'' = (1-\sigma_n^0)z_n'' + \sigma_n^0 T z_n'', \\ z_n'' = T x_n'', \qquad\qquad n \in \mathbb{N}, \end{cases} \tag{1.12}$$

$$\begin{cases} \zeta_0 \in C, \\ \zeta_{n+1} = T \eta_n, \\ \eta_n = T((1-\sigma_n^0)T \zeta_n + \sigma_n^0 T \theta_n), \\ \theta_n = (1-\sigma_n^1)\zeta_n + \sigma_n^1 T \zeta_n, \qquad n \in \mathbb{N} \end{cases} \tag{1.13}$$

$$\begin{cases} x'_0 \in C, \\ x'_{n+1} = Ty'_n, \\ y'_n = T((1-\sigma_n^0)z'_n + \sigma_n^0 T z'_n), \\ z'_n = (1-\sigma_n^1)x'_n + \sigma_n^1 T x'_n, \qquad n \in \mathbb{N} \end{cases} \tag{1.14}$$

where $\alpha_n, \beta_n \in (0,1)$.

2. Preliminaries

The following definitions about the rate of convergence are due to Berinde [2].

Definition 1. *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits a and C, respectively. Assume that there exists*

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = \ell, \tag{1.15}$$

(i) If $\ell = 0$, then we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to C.
(ii) If $0 < \ell < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 2. *Suppose that for two fixed point iteration processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ both converging to the same fixed point p, the following error estimates*

$$\|u_n - p\| \le a_n \tag{1.16}$$

for all $n \in \mathbb{N}$

$$\|v_n - p\| \le b_n \tag{1.17}$$

for all $n \in \mathbb{N}$, are available where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers (converging to zero). If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to p.

Recent study of Ullah and Muhammad (1.14), Hussin et. al. (1.13) proved that their iterative methods converges faster than all the above mentioned iterative methods for a different class of mappings which include the aforementioned class of contraction operators. Now, the question arises whether it is possible to find scheme which is faster than K^*.
Inspired by the works mentioned above, we introduce the following iteration method

namely N_1^v iteration:

$$\begin{cases} \zeta_0 \in C, \\ \theta_n = T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n), \\ \eta_n = T\theta_n, \\ \zeta_{n+1} = T\eta_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.18}$$

Let E be a Banach space and C be a nonempty closed convex subset of E. Let $\{x_n\}$ be a bounded sequence in C. For $x \in E$, set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} ||x - x_n||.$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

The asymptotic centre of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well-known that in a uniformly convex Banach spaces, $A(C, x_n)$ consists of exactly one point. Also, $A(C, x_n)$ is nonempty and convex in the case when C is weakly compact and convex, see e.g., [14, 17]. Following are some basic definitions and results.

Definition 3. *A Banach space E is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there is a $\lambda > 0$ such that for every $x, y \in E$,*

$$\left.\begin{array}{r} ||x|| \leq 1 \\ ||y|| \leq 1 \\ ||x-y|| > \varepsilon \end{array}\right\} \Longrightarrow \tfrac{1}{2}||x+y|| \leq (1-\lambda).$$

Definition 4. *[11] A Banach space E is said to have Opial's property if for each sequence $\{x_n\}$ in E which weakly converges to $x \in E$ and for every $y \in E$, it follows the following*

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||.$$

Definition 5. *Let E and E' be two Banach spaces and let $T : E \to E'$. Then the mapping T is said to be demiclosed if $x \rightharpoonup x \in E$ and $Tx_n \rightharpoonup y$ in E' imply $Tx = y$.*

Lemma 1. *Let C be a non-empty closed convex subset of a uniformly convex Banach space E and T be a non-expansive on C. Then $I - T$ is demiclosed at 0.*

3. Convergence Analysis

Theorem 1. Suppose that there is a Banach space E, having subset C, which is nonempty closed and convex. Also, let there be a contraction mapping $T : C \to C$. Let $\{\zeta_n\}_{n=0}^{\infty}$ be an iterative sequence generated by N_1^v and with real sequences $\{\sigma_n^0\}_{n=0}^{\infty}$ and $\{\sigma_n^1\}_{n=0}^{\infty}$ $\in [0, 1]$ such that $\sum_{n=0}^{\infty} \sigma_n^0 \sigma_n^1 = \infty$. Then, $\{\zeta_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .

Proof. It is obvious from Banach contraction theorem that existence and uniqueness of fixed point x_δ is guaranteed. Now, it is to show that $\zeta_n \to x_\delta$ for $n \to \infty$. From N_1^v iteration scheme it follows that,

$$\begin{aligned}
\|\theta_n - x_\delta\| &= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - x_\delta\| \\
&= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - Tx_\delta\| \\
&\leq \xi\|(1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n - x_\delta\| \\
&\leq \xi(1-\sigma_n^0)\|\zeta_n - x_\delta\| + \xi\sigma_n^0\|T\zeta_n - Tx_\delta\| \\
&\leq \xi(1-\sigma_n^0)\|\zeta_n - x_\delta\| + \sigma_n^0\xi^2\|\zeta_n - x_\delta\| \\
&= \xi(1-(1-\xi)\sigma_n^0)\|\zeta_n - x_\delta\|
\end{aligned}$$

also,

$$\begin{aligned}
\|\eta_n - x_\delta\| &\leq \|T\theta_n - x_\delta\| \\
&= \|T\theta_n - Tx_\delta\| \\
&\leq \xi\|\theta_n - x_\delta\|
\end{aligned}$$

using the value of $\|\theta_n - x_\delta\|$, we have

$$\|\eta_n - x_\delta\| \leq \xi^2(1-(1-\xi)\sigma_n^0)\|\zeta_n - x_\delta\|$$

similarly, we have

$$\begin{aligned}
\|\zeta_{n+1} - x_\delta\| &= \|T\eta_n - x_\delta\| \\
&= \|T\eta_n - Tx_\delta\| \\
&= \xi\|\eta_n - x_\delta\|
\end{aligned}$$

using the value of $\|\eta_n - x_\delta\|$ and $\|\zeta_n - x_\delta\|$, we have

$$\|\zeta_{n+1} - x_\delta\| \leq \xi^3(1-(1-\xi)\sigma_n^0)\|\zeta_n - x_\delta\|$$

Now, inductively using the behaviour of sequence, we have

$$\|\zeta_n - x_\delta\| \leq \xi^3(1-(1-\xi)\sigma_{n-1}^0)\|x_{n-1} - x_\delta\|$$

$$\|x_{n-1} - x_\delta\| \le \xi^3(1 - (1-\xi)\sigma^0_{n-2})\|x_{n-2} - x_\delta\|$$

the repetition results

$$\|x_1 - x_\delta\| \le \xi^3(1 - (1-\xi)\sigma^0_0)\|\zeta_0 - x_\delta\|$$

Proceeding in the same manner, we have

$$\|\zeta_{n+1} - x_\delta\| \le \|\zeta_0 - x_\delta\|\xi^{3(n+1)}\prod_{k=0}^{n}(1 - (1-\xi)\sigma^0_k)$$

where $(1 - \sigma^0_n(1-\xi)) \in (0,1)$ because $\xi \in (0,1)$ and $\sigma^0_n \in [0,1]$, for all $n \in \mathbb{N}$, Since we know that $1 - x \le e^{-x}$ for all $x \in [0,1]$, so from the above inequality

$$\|\zeta_{n+1} - x_\delta\| \le \frac{\|\zeta_0 - x_\delta\|\xi^{3(n+1)}}{e^{(1-\xi)\sum_{k=0}^{n}\sigma^0_k}}.$$

Taking the limit both sides of this inequality, it yields

$$\lim_{n\to\infty}\|\zeta_{n+1} - x_\delta\| = 0,$$

which implies that $\zeta_n \to x_\delta$ for $n \to \infty$, as required.

Theorem 2. Suppose that there is a Banach space E, having subset C, which is nonempty closed and convex and also that there is a contraction mapping T on C with a fixed point x_δ. For given $x'_0 = \zeta_0 \in C$, let $\{\zeta_n\}_{n=0}^{\infty}$ and $\{x'_n\}_{n=0}^{\infty}$ be the iterative sequences generated by N^v_1 and K^* respectively, with real sequences $\{\sigma^0_n\}_{n=0}^{\infty}, \{\sigma^1_n\}_{n=0}^{\infty} \in (0,1)$ such that $\sum_{k=0}^{\infty}\sigma^0_n = \infty$ and for all $n \in \mathbb{N}$. Then N^v_1 converges to x_δ faster than K^* iteration scheme.

Proof. Using The result of Theorem 1 it is clear that

$$\|\zeta_{n+1} - x_\delta\| \le \|\zeta_0 - x_\delta\|\xi^{3(n+1)}\prod_{k=0}^{n}(1 - (1-\xi)\sigma^0_k)$$

Now, for the K^* iteration scheme,

$$\begin{aligned}\|z'_n - x_\delta\| &= \|(1-\sigma^1_n)x'_n + \sigma^0_n Tx'_n - x_\delta\|\\ &\le (1-\sigma^1_n)\|x'_n - x_\delta\| + \sigma^0_n\|Tx'_n - Tx_\delta\|\\ &\le (1-\sigma^1_n)\|x'_n - x_\delta\| + \xi\sigma^0_n\|x'_n - x_\delta\|\\ &\le (1-\sigma^1_n(1-\xi))\|x'_n - x_\delta\|\end{aligned}$$

Similarly

$$\|y'_n - x_\delta\| \le \|T((\sigma^0_n z'_n + (1-\sigma^0_n)Tz'_n) - x_\delta\|$$

$$\begin{aligned}
&\le \xi\|\sigma_n^0 z_n' + (1-\sigma_n^0)Tz_n' - x_\delta\| \\
&\le \xi\sigma_n^0\|z_n' - x_\delta\| + (1-\sigma_n^0)\|Tz_n' - x_\delta\| \\
&\le \xi\sigma_n^0\|z_n' - x_\delta\| + \xi(1-\sigma_n^0)\xi\|z_n' - x_\delta\| \\
&\le \xi(1-(1-\xi)\sigma_n^0)\|z_n' - x_\delta\| \\
&\le \xi(1-(1-\xi)\sigma_n^0)(1-(1-\xi)\sigma_n^1)\|x_n' - x_\delta\|
\end{aligned}$$

similarly,

$$\begin{aligned}
\|x_{n+1}' - x_\delta\| &= \|Ty_n' - x_\delta\| \\
&\le \xi\|y_n' - x_\delta\|
\end{aligned}$$

using the value of $\|y_n' - x_\delta\|$ and by using the fact that $(1-(1-\xi)\sigma_n^1) < 0$ and finally we have

$$\begin{aligned}
\|x_{n+1}' - x_\delta\| &\le \xi^2(1-(1-\xi)\sigma_n^0) \\
\|x_n' - x_\delta\| &\le \xi^2(\xi-(1-\xi)\sigma_{n-1}^0)\|x_{n-1}' - x_\delta\|
\end{aligned}$$

also,

$$\|x_{n-1}' - x_\delta\| \le \xi^2(1-(1-\xi)\sigma_{n-2}^0)\|x_{n-2}' - x_\delta\|$$

continually, we have

$$\|x_1' - x_\delta\| \le \xi^2(1-(1-\xi)\sigma_0^0)\|x_0' - x_\delta\|$$

So, it is quite obvious that the following deduction

$$\|x_{n+1}' - x_\delta\| \le \|x_0' - x_\delta\|\xi^{2(n+1)}\prod_{k=0}^{n}(1-(1-\xi)\sigma_k^0)$$

is correct. Now, let

$$r_n = \|\zeta_0 - x_\delta\|\xi^{3(n+1)}\prod_{k=0}^{n}(1-(1-\xi)\sigma_k^0)$$

and

$$p_n = \|x_0' - x_\delta\|\xi^{2(n+1)}\prod_{k=1}^{n}(1-(1-\xi)\sigma_k^0)$$

Then

$$\frac{p_n}{r_n} = \frac{\|x_0' - x_\delta\|\xi^{2(n+1)}\prod_{k=0}^{n}(1-(1-\xi)\sigma_k^0)}{\|\zeta_0 - x_\delta\|\xi^{3(n+1)}\prod_{k=0}^{n}(1-(1-\xi)\sigma_k^0)}$$

approaches to 0 as n approaches to ∞. Thus, $\{\zeta_n\}$ is a sequence defined in N_1^v iteration defined by (1.14), then $\{\zeta_n\}$ converges faster than the iteration scheme of Ullah and Muhhamad known as K^*.

Theorem 3. Suppose that there is a Banach space E, having subset C, which is nonempty closed and convex and also that there is a contraction mapping T with contraction factor $\xi \in (0,1)$ such that $T_F \neq \emptyset$. if $\{\zeta_n\}$ is a sequence defined in N_1^v iteration defined by (1.14), then $\{x''_n\}$ converges faster than the iteration scheme of Garodia and Uddin defined by (1.12).

Proof. Using the result of Theorem 1 it is clear that

$$\|\zeta_{n+1} - x_\delta\| \leq \|\zeta_0 - x_\delta\|\xi^{3(n+1)} \prod_{k=0}^{n}(1 - \sigma_k^0(1-\xi))$$

Now, for scheme (1.12),

$$\begin{aligned}
\|z''_n - x_\delta\| &= \|Tx''_n - x_\delta\| \\
&\leq \xi\|x''_n - x_\delta\| \\
\|y''_n - x_\delta\| &= \|(1-\sigma_n^0)z''_n - \sigma_n^0 Tz''_n - x_\delta\| \\
&\leq \|(1-\sigma_n^0)z''_n - \sigma_n^0 Tz''_n - x_\delta\| \\
&\leq (1-\sigma_n^0)\|z''_n - x_\delta\| + \xi\sigma_n^0\|Tz''_n - x_\delta\|] \\
&\leq (1-\sigma_n^0)\|z''_n - x_\delta\| + \xi\sigma_n^0\|z''_n - x_\delta\| \\
&= (1-(1-\xi)\sigma_n^0)\|z''_n - x_\delta\|
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x''_{n+1} - x_\delta\| &= \|Ty''_n - x_\delta\| \\
&\leq \xi\|y''_n - x_\delta\|
\end{aligned}$$

using the value of $\|y''_n - x_\delta\|$ and using the inductive behaviour, we have

$$\begin{aligned}
\|x''_{n+1} - x_\delta\| &= \xi^2(1-(1-\xi)\sigma_n^0)\|x''_n - x_\delta\| \\
\|x''_n - x_\delta\| &= \xi^2(1-(1-\xi)\sigma_{n-1}^0)\|x''_{n-1} - x_\delta\| \\
\|x''_{n-1} - x_\delta\| &= \xi^2(1-(1-\xi)\sigma_{n-2}^0)\|x''_{n-2} - x_\delta\|
\end{aligned}$$

on combining all the inequalities, we have

$$\|x''_{n+1} - x_\delta\| \leq \|x''_0 - x_\delta\|\xi^{2(n+1)} \prod_{k=0}^{n}(1 - \sigma_k^0(1-\xi))$$

Let

$$r_n = \|\zeta_0 - x_\delta\|\xi^{3(n+1)} \prod_{k=0}^{n}(1-(1-\xi)\sigma_k^0)$$

and

$$p_n = \|x_0'' - x_\delta\| \xi^{2(n+1)} \prod_{k=0}^{n} (1 - (1-\xi)\sigma_k^0)$$

Then

$$\frac{p_n}{r_n} = \frac{\|x_0'' - x_\delta\| \xi^{2(n+1)} \prod_{k=0}^{n} (1 - (1-\xi)\sigma_k^0)}{\|\zeta_0 - x_\delta\| \xi^{3(n+1)} \prod_{k=0}^{n} (1 - (1-\xi)\sigma_k^0)}$$

approaches to 0 as n approaches to ∞. Thus $\{\zeta_n\}$ is a sequence defined in N_1^v iteration defined by (1.14), then $\{\zeta_n\}$ converges faster than the iteration scheme of Garodia and Uddin defined by (1.12).

Lemma 2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and a nonexpansive self mapping T on C with $T_f \neq \emptyset$. Let $\{\zeta_n\}$ be an iterative sequence defined as N_1^v. Then $\lim_{n\to\infty} \|\zeta_n - x_\delta\|$ exists for all $x_\delta \in T_f$.

Proof. From N_1^v iteration scheme it follows that,

$$\begin{aligned}
\|\theta_n - x_\delta\| &= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - x_\delta\| \\
&= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - Tx_\delta\| \\
&\leq \|(1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n - x_\delta\| \\
&\leq (1-\sigma_n^0)\|\zeta_n - x_\delta\| + \sigma_n^0 \|T\zeta_n - Tx_\delta\| \\
&\leq (1-\sigma_n^0)\|\zeta_n - x_\delta\| + \sigma_n^0 \|\zeta_n - x_\delta\| \\
&\leq \|\zeta_n - x_\delta\| \\
\|\eta_n - x_\delta\| &\leq \|T\theta_n - x_\delta\| \\
&= \|T\theta_n - Tx_\delta\| \\
&\leq \|\theta_n - x_\delta\|
\end{aligned}$$

using the value of $\|\theta_n - x_\delta\|$, we have

$$\|\eta_n - x_\delta\| \leq \|\zeta_n - x_\delta\|$$

similarly, we have

$$\begin{aligned}
\|\zeta_{n+1} - x_\delta\| &= \|T\eta_n - x_\delta\| \\
&= \|T\eta_n - Tx_\delta\| \\
&\leq \|\eta_n - x_\delta\|
\end{aligned}$$

using the value of $\|\eta_n - x_\delta\|$, we have

$$\|\zeta_{n+1} - x_\delta\| \leq \|\zeta_n - x_\delta\|$$

which confirms the existence of $\lim_{n\to\infty}\|\zeta_n - x_\delta\|$ for all $x_\delta \in T_f$. Since, $\{\|\zeta_n - x_\delta\|\}$ is bounded and non-increasing for all $x_\delta \in T_f$.

Now, we prove the weak convergence of N_1^v iteration process.

Theorem 4. Suppose that there is a uniformly Banach space E, having a nonempty subset C, which is nonempty closed and convex satisfying Opial's condition and also that there is a nonexpansive mapping $T : C \to C$ with $T_f \neq \emptyset$. If $\{\zeta_n\}$ is an iterative sequence defined by N_1^v, then $\{\zeta_n\}$ converges weakly to a fixed point of T.

Proof. Let $x_\delta \in T_f$. Then from Lemma 2 , it is obvious that $\lim_{n\to\infty}\|\zeta_n - x_\delta\|$ exists. To prove weak convergence of N_1^v iterative process, it is to be shown that $\{\zeta_n\}$ has a weak subsequential limit in T_f. Let $\{x_{n_u}\}$ and $\{x_{n_v}\}$ are the subsequences of $\{\zeta_n\}$, converges to u and v respectively. Using Lemma 2 $\lim_{n\to\infty}\|T_n - \zeta_n\| = 0$, $I - T$ is demiclosed at 0. So $u, v \in T_f$.

Next, to show the uniqueness, we assume that $\lim_{n\to\infty}\|\zeta_n - u\|$ and $\lim_{n\to\infty}\|\zeta_n - v\|$ exists. Assuming $u \neq v$. Then using Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty}\|\zeta_n - u\| &= \lim_{n\to\infty}\|x_{n_u} - u\| \\
&< \lim_{n\to\infty}\|x_{n_u} - v\| \\
&= \lim_{n\to\infty}\|\zeta_n - v\| \\
&= \lim_{n\to\infty}\|x_{n_v} - v\| \\
&< \lim_{n\to\infty}\|x_{n_v} - u\| \\
&= \lim_{n\to\infty}\|\zeta_n - u\|
\end{aligned}$$

which is a contradiction, so $u = v$. So, $\{\zeta_n\}$ converges weakly to a fixed point of T.

Now, we prove the strong convergence of N_1^v iteration process.

Theorem 5. Suppose that there is a uniformly Banach space E, having a nonempty subset C, which is nonempty closed and convex. Also, there be a nonexpansive mapping $T : C \to C$ with $T_f \neq \phi$. If $\{\zeta_n\}$ is an iterative sequence defined by N_1^v, then $\{\zeta_n\}$ converges strongly to a point of T_f iff $\liminf_{n\to\infty} d(\zeta_n, T_f) = 0$.

Proof. If a sequence $\{\zeta_n\}$ converges to a fixed point $q \in T_f$, then it is obvious that $\liminf_{n\to\infty} d(\zeta_n, T_f) = 0$.

For converse, $\liminf_{n\to\infty} d(\zeta_n, T_f) = 0$. From Lemma 2, we have the existence of $\liminf_{n\to\infty}\|\zeta_n - q\|$ for all $q \in T_f$, we have

$$\|\zeta_{n+1} - q\| \leq \|\zeta_n - q\| \text{ for any } q \in T_f$$

which yields

$$d(\zeta_{n+1}, T_f) \leq d(\zeta_n, T_f)$$

which implies that $\{d(\zeta_n, T_f)\}$ is a decreasing sequence which is bounded below by zero. Results,

$$\lim_{n\to\infty} d(\zeta_n, T_f) = 0$$

Now, to prove that $\{\zeta_n\}$ is a Cauchy sequence in C. Let $\epsilon > 0$ be arbitrarily chosen. Since, $\liminf_{n\to\infty} d(\zeta_n, T_f) = 0$, there is an existence of n_0 in such a manner that $\forall\ n \geq n_0$, we have

$$d(\zeta_n, T_f) < \frac{\epsilon}{4}$$

Particularly,

$$inf\{\|x_{n_0} - q\| : q \in T_f\} < \epsilon$$

so there must be an existence of $\alpha \in T_f$ in such a manner that $\|x_{n_0} - \alpha\| < \epsilon$. Thus, for $m, n \geq n_0$, we have

$$\|x_{n+m} - \zeta_n \leq \|x_{n+m} - \alpha\| + \|\zeta_n - \alpha\| < 2\|x_{n_0} - \alpha\| < 2\frac{\epsilon}{2} = \epsilon$$

which proves the Cauchy behaviour of $\{\zeta_n\}$. Since it is given that C is a closed subset of a Banach space E, therefore the convergence of $\{\zeta_n\}$ in C is confirmed. Let $\lim_{n\to\infty} \zeta_n = \alpha$ for some $\alpha \in B$.

Now using, $\lim_{n\to\infty} \|T\zeta_n - \zeta_n\| = 0$, we get

$$\begin{aligned}\|\alpha - T\alpha\| &\leq \|\alpha - \zeta_n\| + \|\zeta_n - T\zeta_n\| + \|T\zeta_n - T\alpha\| \\ &\leq \|\alpha - \zeta_n\| + \|\zeta_n - T\zeta_n\| + \|\zeta_n - \alpha\|\end{aligned}$$

which proves that $\|\alpha - T\alpha\|$ approaches to 0 as n approaches to ∞. This shows that $\alpha = T\alpha$. This proves our result.

4. Numerical Example

In this section, an example is to be given which confirms the behaviour of N_1^v. In order to support the proof of Theorems 2 and 3, we will use a numerical example as follow

Example. Assuming $E = (-\infty, \infty)$ and $C = [1,\ 50]$. Let $T : C \to C$ be mapping defined as $T(x) = \sqrt{x^2 - 9x + 54}$ for all $x \in C$. Clearly, $x = 5$ is the fixed point of T. Set $\sigma_n^0 = \sigma_n^1 = \sigma_n^2 = 0.75$ for all $n \in \mathbb{N}$. choose initial value as 40. Then, we get the following table and graph. Also, in table N_1^v is represented by n^p of iteration values:

A NEW FASTER METHOD

CR	Picard-S	S	Mann
40	40	40	40
30.79269192254298	29.810580256664	33.73902692017991	36.97915862290742
22.04554167901685	20.221670780422908	27.647788101546006	33.99076370160429
14.213234183814762	11.961302016080737	21.81068402363568	31.040838849624564
8.476620796498484	7.000927795053547	16.382877826904743	28.137119349259297
6.298022369447628	6.049690678581749	11.66859631695794	25.289708311093783
6.02061161491331	6.001825416893833	8.223614190302776	22.51204131218168
6.0013116110571385	6.000065997657282	6.539687645469721	19.8223152821583
6.000082969862263	6.000002384733529	6.090382473599866	17.245590289342086
6.000005246519448	6.000000086167199	6.01340512395956	14.816755272141435
6.000000331750666	6.000000003113463	6.001944962269297	12.584183729729153

Ishikawa	Noor	SP	K
40	40	40	40
34.74597404587743	33.09913487808448	31.040838849624564	25.968740303084196
29.60941701882642	26.436520344771075	22.51204131218168	13.571922184554948
24.63615554168236	20.155969504050155	14.816755272141435	6.615550226137753
19.90092505908047	14.54283644691706	8.977646915366886	6.0068416866188175
15.531660598892563	10.14751401807788	6.453959009081129	6.000061948435167
11.750034102312975	7.576122883221963	6.041927817484542	6.000000559604765
8.888936848096153	6.523734904690029	6.003543275911321	6.000000005055024
7.190457756259381	6.1666335029362465	6.000296944182746	6.000000000045664
6.4304664344781335	6.052332918398102	6.000024867786343	6.000000000000413
6.1472262080237785	6.016370527034029	6.000002082445555	6.000000000000004

Garodia's	K^*	n^p
40	40	40
29.08214360104216	26.209490048705245	25.258002332048
18.89759552091176	13.96352052291395	12.461420120882993
10.484796658965704	6.769336442232295	6.313982447524068
6.452679669681885	6.011327288189042	6.002368918530358
6.014592571160892	6.000135918062143	6.000016205805245
6.000401231816844	6.000001626031984	6.000000110782435
6.000010972856014	6.0000000194520515	6.000000000757303
6.000000300040283	6.000000000232704	6.000000000005177
6.0000000082042275	6.000000000002784	6.0000000000000355
6.000000000224334	6.000000000000033	6.000000000000001

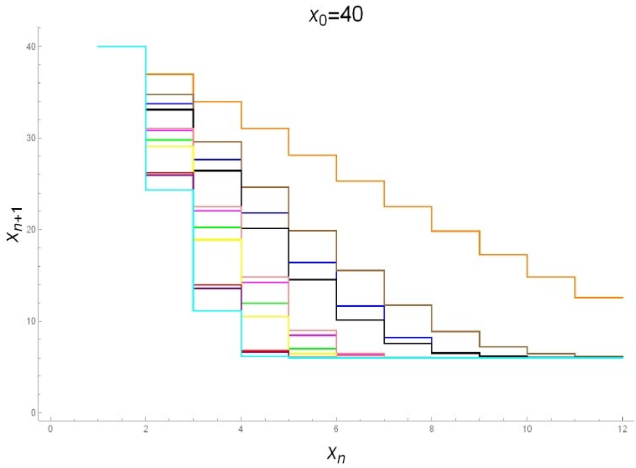

Figure 1: Comparison Graph based on Numerical example to prove the efficiency of N_1^v.

Thus, it is evident from the above table and graph that the newly defined iteration scheme N_1^v converges much faster and is more efficient than many iteration schemes in exiting literature.

5. T-stability of N_1^v iteration algorithm

Now, we prove the stability of N_1^v.

Theorem 6. Suppose that there is a Banach space E, having subset C, which is nonempty closed and convex. Also, let there be a contraction mapping $T : C \to C$. Let $\{\zeta_n\}_{n=0}^{\infty}$ be an iterative sequence generated by N_1^v and with real sequence $\{\sigma_n^0\}_{n=0}^{\infty} \in [0, 1]$ such that $\sum_{n=0}^{\infty} \sigma_n^0 = \infty$. Then the iteration algorithm defined as N_1^v is $T - stable$.

Proof. Let an arbitrary sequence $\{t_n\}_{n=0}^{\infty} \subset E$ in C, generated by N_1^v and now defined as $\zeta_{n+1} = f(T, \zeta_n)$ converges to a fixed point x_δ (by Theorem 1) and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. We will prove that $\lim_{n\to\infty} t_n = p$.

Let $\lim_{n\to\infty} \epsilon_n = 0$, as from Theorem 1 using the inequality

$$\|\zeta_{n+1} - x_\delta\| \leq \xi^3(\xi - (1 - \xi)\sigma_n^0)\|\zeta_n - x_\delta\| \tag{-14}$$

we have,

$$\begin{aligned}
\|t_{n+1} - x_\delta\| &\leq \|t_{n+1} - f(T, t_n)\| + |f(T, t_n) - x_\delta\| \\
&= \epsilon_n + \left\| T\left(T(T((1 - \sigma_n^0)\zeta_n)) + \sigma_n^0 T\zeta_n \right) - x_\delta \right\|
\end{aligned}$$

$$\leq \xi^3(1-(1-\xi)\sigma_n^0)\|t_n - x_\delta\| + \epsilon_n.$$

Define $\psi_n = \|t_n - x_\delta\|$, $\phi_n = (1-\xi)\sigma_n^0 \in (0,1)$ and $\varphi = \epsilon_n$, which implies that $\frac{\varphi_n}{\phi_n} \to 0$ as $n \to \infty$. Thus all the conditions of Lemma 2 are satisfied by above inequality. Hence, we get $\lim_{n\to\infty} t_n = p$, we have

$$\begin{aligned}
\epsilon_n &= \|t_{n+1} - f(T, t_n)\| \\
&\leq \|t_{n+1} - x_\delta\| - \|f(T, t_n) - x_\delta\| \\
&\leq \xi^3(\xi - (1-\xi)\sigma_n^0)\|t_n - x_\delta\| + \epsilon_n
\end{aligned}$$

This implies that $\lim_{n\to\infty} t_n = 0$. This also implies that N_1^v is $T-stable$ with respect to T.

6. Data Dependence Result

In this section we establish some data dependence result.

Theorem 7. Let $\tilde{T}$ be an approximate operator of a contraction mapping T. Let $\{\zeta_n\}_{n=0}^{\infty}$ be an iterative sequence defines as N_1^v for T and defined an iterative sequence $\{\tilde{\zeta}_n\}_{n=0}^{\infty}$ for $\tilde{T}$, as follows constructed as, for arbitrary $\tilde{\zeta}_0 \in X$ by

$$\begin{cases}
\tilde{\theta}_n = \tilde{T}((1-\sigma_n^0)\tilde{\zeta}_n + \sigma_n^0\tilde{T}\tilde{\zeta}_n) \\
\tilde{\eta}_n = \tilde{T}\tilde{\theta}_n \\
\tilde{\zeta}_{n+1} = \tilde{T}\tilde{\eta}_n \qquad n \in \mathbb{N}
\end{cases}$$

where real sequence $\{\sigma_n^0\}_{n=0}^{\infty}$ in [0,1] satisfying $\frac{1}{2} \leq \sigma_n^0$, for all $n \in \mathbb{N}$ and $\sum \sigma_n^0 = \infty$. Also, if $Tp = p$ and $\tilde{T}\tilde{p} = \tilde{p}$ such that $\lim_{n\to\infty} \tilde{\zeta} = \tilde{p}$, then we have

$$\|p - \tilde{p}\| \leq \frac{11\epsilon}{1-\xi}.$$

Proof. Using $\{\zeta_n\}_{n=0}^{\infty}$ and $\{\tilde{\zeta}_n\}_{n=0}^{\infty}$, we have

$$\begin{aligned}
\|\theta_n - \tilde{\theta}_n\| &= \left\| T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - \tilde{T}((1-\sigma_n^0)\tilde{\zeta}_n + \sigma_n^0\tilde{T}\tilde{\zeta}_n) \right\| \\
&\leq \left\| T\left((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n\right) - T\left((1-\sigma_n^0)\tilde{\zeta}_n + \sigma_n^0\tilde{T}\tilde{\zeta}_n\right)\right. \\
&\quad \left. + T\left((1-\sigma_n^0)\tilde{\zeta}_n + \sigma_n^0\tilde{T}\tilde{\zeta}_n\right) - \tilde{T}\left((1-\sigma_n^0)\tilde{\zeta}_n + \sigma_n^0\tilde{T}\tilde{\zeta}_n\right) \right\| \\
&\leq \xi\left((1-\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0\|T\zeta_n - \tilde{T}\tilde{\zeta}_n)\|\right) + \epsilon
\end{aligned}$$

$$\leq \xi\left((1-\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0\left(\|T\zeta_n - T\tilde{\zeta}_n\| + \|T\tilde{\zeta}_n - \tilde{T}\tilde{\zeta}_n)\|\right)\right) + \epsilon$$
$$\leq \xi\left((1-\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0\left(\xi(\|\zeta_n - \tilde{\zeta}_n\|) + \epsilon + \epsilon\right)\right) + \epsilon$$
$$\leq \xi(1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \xi\sigma_n^0\epsilon + \epsilon$$

In similar manner, we have

$$\begin{aligned}
\|\eta_n - \tilde{\eta}_n\| &= \|T\theta_n - \tilde{T}\tilde{\theta}_n\| \\
\|\eta_n - \tilde{\eta}_n\| &= \|T\theta_n - T\tilde{\theta}_n + T\tilde{\theta}_n - \tilde{T}\tilde{\theta}_n\| \\
&\leq \|T\theta_n - T\tilde{\theta}_n\| + \|T\tilde{\theta}_n - \tilde{T}\tilde{\theta}_n\| \\
&\leq \xi\|\theta_n - \tilde{\theta}\| + \epsilon
\end{aligned}$$

on substituting the value of $\|\theta_n - \tilde{\theta}_n\|$, we have

$$\|\eta_n - \tilde{\eta}_n\| \leq \xi\left(\xi(1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \xi\sigma_n^0\epsilon + \epsilon\right) + \epsilon$$

In a similar manner, we have

$$\begin{aligned}
\|\zeta_{n+} - \tilde{\zeta}_{n+1}\| &= \|T\eta_n - \tilde{T}\tilde{\eta}_n\| \\
\|\zeta_n - \tilde{\zeta}_n\| &= \|T\eta_n - T\tilde{\eta}_n + T\tilde{\eta}_n - \tilde{T}\tilde{\eta}_n\| \\
&\leq \|T\eta_n - T\tilde{\eta}_n\| + \|T\tilde{\eta}_n - \tilde{T}\tilde{\eta}_n\| \\
&\leq \xi\|\eta_n - \tilde{\eta}\| + \epsilon
\end{aligned}$$

on substituting the value of $\|\eta_n - \tilde{\eta}_n\|$, we have

$$\|\zeta_n - \tilde{\zeta}_n\| \leq \xi\left(\xi(1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \xi\sigma_n^0\epsilon + \epsilon\right) + \epsilon$$

using $\{\sigma_n^0\}_{n=0}^{\infty}$ in [0,1] and $\xi \in (0,1)$ and combining the above inequalities of same theorem,we have

$$\begin{aligned}
\|\zeta_n - \tilde{\zeta}_n\| &\leq (1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0\epsilon + 5\epsilon \\
&\leq (1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0\epsilon \\
&\quad + 5(1-\sigma_n^0 + \sigma_n^0)\epsilon \\
\|\zeta_n - \tilde{\zeta}_n\| &\leq (1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0(1-\xi)\frac{11\epsilon}{1-\xi}
\end{aligned}$$

Let $\psi_n = \|\zeta_n - \tilde{\zeta}_n\|, \phi_n = (1-\sigma_n^0)(1-\xi), \varphi_n = \frac{11\epsilon}{1-\xi}$, then from the Lemma 2, we have

$$0 \leq \limsup_{n\to\infty} \|\zeta_n - \tilde{\zeta}_n\| \leq \limsup_{n\to\infty} \frac{11\epsilon}{1-\xi}$$

considering the result of Theorem 1 we have $\limsup_{n\to\infty} \zeta_n = p$ and by the assumption we have that $\limsup_{n\to\infty} \tilde{\zeta}_n = \tilde{p}$. Using the results together with

$$\|\zeta_n - \tilde{\zeta}_n\| \leq (1-(1-\xi)\sigma_n^0)\|\zeta_n - \tilde{\zeta}_n\| + \sigma_n^0(1-\xi)\frac{11\epsilon}{1-\xi}$$

we have, $\|p - \tilde{p}_n\| \leq \frac{11\epsilon}{1-\xi}$ as required.

7. An Application

Let a Banach space $(E([a,b]), ||.||_\infty)$ which is space of all continuous real valued functions on a closed interval $[a,b]$ a with endowed chebyshev norm $\|x-y\|_\infty = \max_{t\in[a,b]} |x(t) - y(t)|$. In this section solution of a particular delay differential equation has a solution generated by N_1^v iteration scheme.

$$x'(t) = f(t, x(t), x(t-\tau)), \qquad t \in [t_0, b] \tag{7.1}$$

with initial condition

$$x(t) = \psi(t), \qquad t \in [t_0 - \tau, t_0]. \tag{7.2}$$

We opine that the following conditions are performed

(i) $t_0, b \in \mathbb{R}, \tau > 0$;

(ii) $f \in E([t_0, b] \times \mathbb{R}^2, \mathbb{R})$;

(iii) $\psi \in E([t_0 - \tau, b], \mathbb{R})$;

(iv) if $2L_f(b - t_0) < 1$, there exist $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{n=0}^{2} |u_i - v_i|, \tag{7.3}$$

$\forall u_i, v_i \in \mathbb{R}, i = 1, 2, t \in [t_0, b]$, By a solution of the problem (7.1)-(7.2) we understand function $x \in E([t_0-\tau, b], \mathbb{R}) \bigcap E^1([t_0, b], \mathbb{R})$. The problem (7.1)-(7.2) can be reformulated in the following form of integral

$$x(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b]. \end{cases} \tag{7.4}$$

Theorem 8. *Suppose that conditions (1)-(4) are satisfied. Then the problem (7.1)-(7.4) has a unique solution in* $E([t_0 - \tau, b], \mathbb{R}) \bigcap E^1([t_0, b], \mathbb{R})$.

Proof. Let $\{\zeta_n\}_{n=0}^{\infty}$ be an iterative sequence generative by N_K^v iteration method (1.18) for the operator

$$Tx(t) = \begin{cases} \psi(t), & t \in [t_0 - \tau, t_0] \\ \psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds, & t \in [t_0, b]. \end{cases} \tag{7.5}$$

Let x_δ denote the fixed point of T. We will show that $\zeta_n \to x_\delta$ as $n \to \infty$.
For $t \in [t_0 - \tau, t_0]$, it is easy to see that $\zeta_n \to x_\delta$ as $n \to \infty$. For $t \in [t_0, b]$ we obtain

$$\begin{aligned}
\|\theta_n - x_\delta\|_\infty &= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - x_\delta\|_\infty \\
&= \|T((1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n) - Tx_\delta\| \\
&\leq \|(1-\sigma_n^0)\zeta_n + \sigma_n^0 T\zeta_n - x_\delta\| \\
&\leq (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} |T\zeta_n - Tx_\delta| \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \left|\psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds\right. \\
&\left. - \psi(t_0) - \int_{t_0}^{t} f(s, x_\delta(s), x_\delta(s-\tau))ds\right| \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \left|\psi(t_0) + \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds\right. \\
&\left. - \psi(t_0) - \int_{t_0}^{t} f(s, x_\delta(s), x_\delta(s-\tau))ds\right| \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \left| \int_{t_0}^{t} f(s, x(s), x(s-\tau))ds\right. \\
&\left. - \int_{t_0}^{t} f(s, x_\delta(s), x_\delta(s-\tau))ds\right| \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} \left|f(s, x(s), x(s-\tau))\right. \\
&\left. - f(s, x_\delta(s), x_\delta(s-\tau))ds\right|
\end{aligned}$$

$$
\begin{aligned}
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\bigg(|\zeta_n(s) - x_\delta(s)| \\
&+ |\zeta_n(s-\tau) - x_\delta(s-\tau)|\bigg)ds \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 \max_{t\in[t_0-\tau,b]} \int_{t_0}^{t} L_f\bigg(|\zeta_n(s) - x_\delta(s)| + |\zeta_n(s-\tau) \\
&- x_\delta(s-\tau)|\bigg)ds \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + \sigma_n^0 L_f\bigg(\max_{t\in[t_0-\tau,b]}|\zeta_n(s) - x_\delta(s)| + \max_{t\in[t_0-\tau,b]}|\zeta_n(s-\tau) \\
&- x_\delta(s-\tau)|\bigg)\int_{t_0}^{t} ds \\
&= (1-\sigma_n^0)\|\zeta_n - x_\delta\|_\infty + 2\sigma_n^0 L_f(b-t_0)\|\zeta_n - x_\delta\|
\end{aligned}
$$

$$
\|\theta_n - x_\delta\|_\infty = \Big[1 - \sigma_n^0(1 - 2L_f(b-t_0))\Big]\|\zeta_n - x_\delta\| \tag{7.6}
$$

similarly, we have

$$
\begin{aligned}
\|\eta_n - x_\delta\|_\infty &= \|T\theta_n - Tx_\delta\|_\infty \\
&= \max_{t\in[t_0-\tau,b]}\bigg|\int_{t_0}^{t} f(s,\theta_n(s),\theta_n(s-\tau)) - f(s,x_\delta(s),x_\delta(s-\tau))]ds\bigg| \\
&\leq \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t}\bigg|f(s,\theta_n(s),\theta_n(s-\tau)) - f(s,x_\delta(s),x_\delta(s-\tau))\bigg|ds \\
&= \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t} L_f\bigg(|\theta_n(s) - x_\delta(s)| + |\theta_n(s-\tau) - x_\delta(s-\tau)|\bigg)ds
\end{aligned}
$$

$$
\|\eta_n - x_\delta\|_\infty \leq 2L_f(b-t_0)\|\theta_n - x_\delta\|_\infty \tag{7.7}
$$

and hence, we have

$$
\begin{aligned}
\|\zeta_{n+1} - x_\delta\|_\infty &= \|T\eta_n - Tx_\delta\|_\infty \\
&= \max_{t\in[t_0-\tau,b]}\bigg|\int_{t_0}^{t} f(s,\eta_n(s),\eta_n(s-\tau)) - f(s,x_\delta(s),x_\delta(s-\tau))]ds\bigg| \\
&\leq \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t}\bigg|f(s,\eta_n(s),\eta_n(s-\tau)) - f(s,x_\delta(s),x_\delta(s-\tau))\bigg|ds \\
&= \max_{t\in[t_0-\tau,b]}\int_{t_0}^{t} L_f\bigg(|\eta_n(s) - x_\delta(s)| + |\eta_n(s-\tau) - x_\delta(s-\tau)|\bigg)ds
\end{aligned}
$$

$$
\|\zeta_n - x_\delta\|_\infty \leq 2L_f(b-t_0)\|\eta_n - x_\delta\|_\infty \tag{7.7}
$$

using the equations (7.6), (7.7) and (7.8)

$$
\|\zeta_{n+1} - x_\delta\|_\infty \leq 4L_f^2(b-t_0)^2\|\eta_n - x_\delta\|_\infty
$$

$$\|\zeta_{n+1} - x_\delta\|_\infty \leq 4L_f^2(b-t_0)^2\Big[1 - \sigma_n^0(1 - 2L_f(b-t_0))\Big]\|\zeta_n - x_\delta\|$$

Proceeding in the same manner, we have

$$\|\zeta_n - x_\delta\|_\infty \leq \Big[1 - \sigma_{n-1}^0(1 - 2L_f(b-t_0))\Big]\|\zeta_{n-1} - x_\delta\|_\infty$$

and

$$\|\zeta_{n-1} - x_\delta\|_\infty \leq \Big[1 - \sigma_{n-2}^0(1 - 2L_f(b-t_0))\Big]\|\zeta_{n-2} - x_\delta\|_\infty$$

and hence we have

$$\|\zeta_{n+1} - x_\delta\|_\infty \leq \prod_{k=0}^{n}\Big[1 - \sigma_k^0(1 - 2L_f(b-t_0))\Big]\|\zeta_0 - x_\delta\|_\infty \tag{7.9}$$

where $[1-\sigma_k^0(1-2L_f(b-t_0)) \in (0,1)$ because $\sigma_k^0 \in (0,1)$, for all natural numbers n. Also, since $(1-x) \leq e^{-x}$ for all $x \in [0,1]$, from (7.9) we can easily conclude that

$$\|\zeta_{n+1} - x_\delta\|_\infty \leq \frac{\|\zeta_0 - x_\delta\|}{e^{(1-(2L_f(b-t_0)))\sum_{k=0}^{\infty}\sigma_k^0}} \tag{7.10}$$

which led us to $\lim_{n\to\infty}\|\zeta_{n+1} - x_\delta\|_\infty = 0$ when taking limits of both sides of equation (7.10).

8. Conclusion

A whole new iteration scheme namely N_1^v having rate of convergence, faster than almost all pre-existing iteration schemes to find the solution with minimum possible steps is established.

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