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On (μ_1, μ_2, μ_3)-Weakly Generalized Closed Sets

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Abstract. This paper defines a new generalization of closed sets in a tri-generalized topological space called (μ_1, μ_2, μ_3)-weakly generalized closed set (or briefly (μ_1, μ_2, μ_3)-wg closed set) which is defined as follows: A subset A of X is (μ_1, μ_2, μ_3)-weakly generalized closed set if $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X. At least fifteen defined closed sets found in literature are considered special cases of (μ_1, μ_2, μ_3)-weakly generalized closed set under some conditions. Furthermore, some properties of (μ_1, μ_2, μ_3)-weakly generalized closed sets are obtained.

2020 Mathematics Subject Classifications: 54A05, 54A10

Key Words and Phrases: Tri-generalized topological space, generalized topological space, weakly generalized closed set

1. Introduction

Studies concerning generalized topologies have been in literature since its introduction in 2002 by Csaszar as cited in [1]. Topological properties of generalized topologies have also been explored by some researchers including those by Tabadkan and Taghavi in 2011 [2], and those by Khayyeri and Mohamadian also in 2011 [3]. Other authors named a generalized topology as a supra topology and derived some important definitions and properties such as those of Al-Shami in 2016 and 2018 [11,12], and El-Shafie, et al. in 2020 [13]. New developments of researches pertaining to generalized topologies have been extended to bi-generalized topologies wherein two generalized topologies were considered in the study. Two of which include the researches of Dungthaisong, et al. in 2011 [4] and of Rara and Baculta in 2015 [5]. Several researches involving closed sets, generalized closed sets and many more have been available in literature. In the paper of Mishra, et al. [6], sixteen (16) definitions of closed sets were enumerated. Seven (7) definitions of closed sets were also listed in the paper of Cao, et al.[7] in 1999. In this paper, the researcher defines a new generalization of closed sets called (μ_1, μ_2, μ_3)-weakly generalized closed set containing fifteen literature-defined closed sets. Furthermore, the corresponding (μ_1, μ_2, μ_3)-weakly generalized open sets are characterized. Moreover, some properties of (μ_1, μ_2, μ_3)-weakly generalized closed sets are obtained.

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2. Preliminaries

Definition 1. [1] Let X be a nonempty set. A collection μ of subsets of X is a generalized topology (or briefly GT) in X if it satisfies: *i.* $\emptyset \in \mu$, and *ii.* if $\{M_i : i \in I\} \subseteq \mu$, then $\bigcup_{i\in I} M_i \in \mu$.
If μ is a GT in X, then (X, μ) is called a generalized topological space (or briefly GT space), and the elements of μ are called μ-open sets in X. If μ_1 and μ_2 are GTs in X, then (X, μ_1, μ_2) is called a bi-generalized topological space.If μ_1 μ_2 and μ_3 are GTs in X, then (X, μ_1, μ_2, μ_3) is called a tri-generalized topological space.

Definition 2. [1,3] Let μ be a GT in X. A subset F of X is said to be a μ-closed set if the complement of F (F^c) is μ-open. The μ-closure of a subset A of X, denoted by $cl_\mu(A)$, is the intersection of all μ-closed sets in X containing A while the μ-interior of a subset A of X, denoted by $int_\mu(A)$, is the union of all μ-open subsets of A in X.

The succeeding Theorems 1 to 5 are fundamental properties of any GT μ and can be easily proven.

Theorem 1. Let $X \neq \emptyset$ and μ be a GT in X. If $\{A_i : i \in I\}$ is a collection of μ-closed sets, then $\bigcap_{i\in I} A_i$ is a μ-closed set.

Theorem 2. Let $X \neq \emptyset$ and μ be a GT in X. Suppose also that A and B are subsets of X. Then,

i. If $A \subseteq X$, then $int_\mu(A) \subseteq A$.

ii. $int_\mu(A)$ is the largest open subset of A.

iii. A is μ-open if and only if $int_\mu(A) = A$.

iv. If $A \subseteq B$, then $int_\mu(A) \subseteq int_\mu(B)$.

Theorem 3. Let $X \neq \emptyset$ and μ be a GT in X. Suppose also that A and B are subsets of X. Then,

i. If $A \subseteq X$, then $A \subseteq cl_\mu(A)$.

ii. $cl_\mu(A)$ is the smallest closed superset of A.

iii. A is μ-closed if and only if $cl_\mu(A) = A$.

iv. If $A \subseteq B$, then $cl_\mu(A) \subseteq cl_\mu(B)$.

Theorem 4. Let $X \neq \emptyset$ and $\mu_1 \subseteq \mu_2$ where μ_1 and μ_2 are GTs in X. Then A is μ_2-open (μ_2-closed) whenever A is μ_1-open (μ_1-closed).

Theorem 5. Let $X \neq \emptyset$, $A \subseteq X$, and μ be a generalized topology. Then, $[int_\mu(A)]^c = cl_\mu(A^c)$.

Definition 3. [8] Let X be a non empty set, then the collection of all subsets of X is called the discrete topology. We denote this collection as D.

Definition 4. Let Y be a subset of X. A set A is called a μ-open set in Y if $A = Y \bigcap G$ for some μ-open set G in X.

Definition 5. [1,5,6,10] Let X be a non empty set and μ be a generalized topology in X. Then,

i. A subset A of X is called generalized open (briefly g-open) set if $F \subseteq int(A)$ whenever $F \subseteq A$ and F is closed. We denote the collection of g-open sets in X as $G(X)$.

ii. A subset A of X is called semi-open set if $A \subseteq cl(int(A))$. We denote the collection of semi-open sets in X as $SO(X)$.

iii. A subset A of X is called α-open set if $A \subseteq int(cl(int(A)))$. We denote the collection of α-open sets in X as $AO(X)$.

iv. A subset A of X is called semi-preopen set if $A \subseteq cl(int(cl(A)))$. We denote the collection of semi-preopen sets in X as $SPO(X)$.

v. A subset A of X is called b-open set if $A \subseteq cl(int(A)) \bigcup int(cl(A))$. We denote the collection of b-open sets in X as $BO(X)$.

The complements of the above-mentioned open sets are their respective closed sets.

Theorem 6. Let $X \neq \emptyset$. The following can be shown using Definitions 1 and 5, and Theorems 2 and 3.

i. The collection of g-open sets in X is a generalized topology.

ii. The collection of semi-open sets in X is a generalized topology.

iii. The collection of α-open sets in X is a generalized topology.

iv. The collection of semi-preopen sets in X is a generalized topology.

v. The collection of pre-open sets in X is a generalized topology.

vi. The collection of b-open sets in X is a generalized topology.

Definition 6. [1,6,7,9] Let $X \neq \emptyset$ and μ be a topology on X. Then, a subset A of X is:

i. Generalized closed (briefly g-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

ii. Strongly generalized closed (or briefly g^*-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.

iii. Generalized preclosed (or briefly gp-closed) set if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

iv. Semi-generalized closed (or briefly sg-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

v. Generalized semiclosed (or briefly gs-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

vi. Generalized b-closed (or briefly gb-closed) set if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

vii. Generalized α-b-closed (or briefly $g\alpha b$-closed) set if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.

viii. Semi generalized b-closed set (or briefly sbg-closed) set if $bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

ix. Weakly closed (or briefly w-closed) set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

x. Generalized semi-preclosed (or briefly gsp-closed) set if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

xi. Generalized α closed (or briefly g-α-closed) set if $\alpha - cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X.

xii. α-generalized closed (or briefly αg-closed) set if $\alpha - cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

xiii. Weakly generalized closed (or briefly wg-closed) set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X.

xiv. Mildly generalized closed (or briefly mildly g-closed) set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X.

xv. Semi weakly generalized closed (or briefly swg-closed) set if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X.

3. Main Results

3.1. (μ_1, μ_2, μ_3)-Weakly Generalized Closed Sets

Definition 7. Let A be a subset of a nonempty set X and μ_1, μ_2, and μ_3 be generalized topologies in X. We say that A is (μ_1, μ_2, μ_3)-weakly generalized closed (or briefly (μ_1, μ_2, μ_3)-wg closed) set if $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X. We call the complement of every (μ_1, μ_2, μ_3)-weakly generalized closed set as (μ_1, μ_2, μ_3)-weakly generalized open (or briefly (μ_1, μ_2, μ_3)-wg open) set in X.

3.2. Special Cases of (μ_1, μ_2, μ_3)-Weakly Generalized Closed Sets

Let X be a non empty set and $A \subseteq X$. Then the following are special cases of (μ_1, μ_2, μ_3)-weakly generalized closed sets with the corresponding conditions.

i. **Generalized closed (or briefly g-closed) set**
If $\mu_1 = \mu_3$ is a topology for X and μ_2 is the discrete topology, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just a g-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

ii. **Strongly generalized closed (or briefly g*-closed) set**
If μ_1 is a topology in X, μ_2 is the discrete topology in X, and $\mu_3 = G(X)$ where $G(X)$ is the collection of g-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the strongly generalized closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X".

iii. **Generalized preclosed (or briefly gp-closed) set**
If $\mu_1 = PO(X)$ where $PO(X)$ is the collection of pre-open sets sets in X, μ_2 is the discrete topology in X, and μ_3 is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized preclosed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

iv. **Semi-generalized closed (or briefly sg-closed) set**
If $\mu_1 = \mu_3 = SO(X)$ where $SO(X)$ is the collection of semi-open sets in X, and μ_2 is a discrete topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just a semi-generalized closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X".

v. **Generalized b-closed (or briefly gs-closed) set**
If $\mu_1 = SO(X)$ where $SO(X)$ is the collection of semi-open sets in X, μ_2 is the discrete topology in X, and μ_3 is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just a generalized semi-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

vi. **Generalized semi-closed (or briefly gb-closed) set**
If $\mu_1 = BO(X)$ where $BO(X)$ is the collection of b-open sets sets in X, μ_2 is the discrete topology in X, and μ_3 is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized b-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

vii. **Generalized αb-closed (or briefly $g\alpha b$-closed) set**
If $\mu_1 = BO(X)$ where $BO(X)$ is the collection of b-open sets sets in X, μ_2 is the discrete topology in X, and $\mu_3 = AO(X)$ where $AO(X)$ is the collection of α-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized b-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$scl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X".

viii. **Semi generalized b-closed (or briefly sbg-closed) set**
If $\mu_1 = BO(X)$ where $BO(X)$ is the collection of b-open sets sets in X, μ_2 is the discrete topology in X, and $\mu_3 = SO(X)$ where $SO(X)$ is the collection of semi-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the semi-generalized b-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$bcl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X".

ix. **Weakly closed (or briefly w-closed) set**
If μ_1 is a topology in X, μ_2 is the discrete topology in X, and $\mu_3 = SO(X)$ where $SO(X)$ is the collection of semi-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the weakly closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X".

x. **Generalized semi-preclosed (or briefly gsp-closed) set**
If $\mu_1 = SPO(X)$ where $SPO(X)$ is the collection of semi-preopen sets in X, μ_2 is the discrete topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized semi-preclosed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

xi. **Generalized α closed (or briefly $g\alpha$-closed) set**
If $\mu_1 = \mu_3 = AO(X)$ where $AO(X)$ is the collection of α-open sets in X, and μ_2 is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized $g\alpha$-closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$\alpha cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is α-open in X".

xii. **α-generalized closed (or briefly αg-closed) set**
If $\mu_1 = AO(X)$ where $AO(X)$ is the collection of α-open sets in X, μ_2 is the discrete topology in X, and μ_3 is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the generalized α generalized closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$\alpha cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

xiii. **Weakly generalized closed (or briefly wg-closed) set**
If $\mu_1 = \mu_2 = \mu_3$ is a topology in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the weakly generalized closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$

whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is open in X".

xiv. **Mildly generalized closed (or briefly mildly g-closed) set**
If $\mu_1 = \mu_2$ is a topology in X, and $\mu_3 = G(X)$ where $G(X)$ is the collection of g-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the weakly closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open in X".

xv. **Semi-weakly generalized closed (or briefly swg-closed) set**
If $\mu_1 = \mu_2$ in X, and $\mu_3 = SO(X)$ where $SO(X)$ is the collection of semi-open sets in X, then a (μ_1, μ_2, μ_3)-weakly generalized closed set is just the weakly closed set since the condition "$cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in X" becomes "$cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in X".

Theorem 7. Let (X, μ_1, μ_2, μ_3) be a trigeneralized space and $A \subseteq X$. Then, A is (μ_1, μ_2, μ_3)-weakly generalized open set if and only if $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$ whenever $F \subseteq A$ and F is μ_3-closed in X.

Proof. Let A be (μ_1, μ_2, μ_3)-wg open set and $F \subseteq A$ such that F is μ_3-closed. Then A^c is (μ_1, μ_2, μ_3)-wg closed, F^c is μ_3-open, and $A^c \subseteq F^c$. That is $F \subseteq [cl_{\mu_1}(int_{\mu_2})]^c = int_{\mu_1}(cl_{\mu_2}(A))$. Hence, $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$ whenever $F \subseteq A$ and F is μ_3-closed.

Now, let $F \subseteq A$ and F be μ_3-closed set in X such that $F \subseteq int_{\mu_1}(cl_{\mu_2}(A))$. Taking complementation, we have $\left[int_{\mu_1}(cl_{\mu_2}(A))\right]^c \subseteq F^c$ whenever $A^c \subseteq F^c$ and F^c is μ_3-open in X. But

$$\left[int_{\mu_1}(cl_{\mu_2}(A))\right]^c = cl_{\mu_1}\left[(cl_{\mu_2}(A))^c\right] = cl_{\mu_1}(int_{\mu_2}(A^c))$$

So $cl_{\mu_1}(int_{\mu_2}(A^c)) \subseteq F^c$ whenever $A^c \subseteq F^c$ and F^c is μ_3-open in X. This means that A^c is (μ_1, μ_2, μ_3)-wg closed set. Therefore A is (μ_1, μ_2, μ_3)-wg open set.

Theorem 8. If A is μ_1-closed then A is (μ_1, μ_2, μ_3)-wg closed set.

Proof. Let A be μ_1-closed and U be μ_3-open such that $A \subseteq U$. By Theorem 2(i.), $int_{\mu_2}(A) \subseteq A$ and applying Theorem 3(iv.), $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq cl_{\mu_1}(A)$. Since A is μ_1 closed, $cl_{\mu_1}(A) = A$. Thus, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq A \subseteq U$. Therefore, A is (μ_1, μ_2, μ_3)-wg closed set.

Theorem 9. If $F \subseteq A$ and A is μ_1-closed, then F is (μ_1, μ_2, μ_3)-wg closed set.

Proof. Let A be μ_1-closed and $F \subseteq A$. Suppose $A \subseteq U$ and U is μ_3-open. Since $F \subseteq A$, $int_{\mu_2}(F) \subseteq int_{\mu_2}(A)$ using Theorem 2(iv.). Consequently, by Theorem 3(iv.), $cl_{\mu_1}(int_{\mu_2}(F)) \subseteq cl_{\mu_1}(int_{\mu_2}(A))$. Moreover, by Theorem 8, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$, thus $cl_{\mu_1}(int_{\mu_2}(F)) \subseteq U$. Therefore, F is (μ_1, μ_2, μ_3)-wg closed set.

Theorem 10. If A is (μ_1, μ_2, μ_3)-wg closed subset of X and $A \subseteq B \subseteq cl_{\mu_1}(int_{\mu_2}(A))$, then B is (μ_1, μ_2, μ_3)-wg closed set.

Proof. Suppose that A is (μ_1, μ_2, μ_3)-wg closed subset of X and $A \subseteq B \subseteq cl_{\mu_1}(int_{\mu_2}(A))$. Let U be μ_3-open and $B \subseteq U$. Since $A \subseteq B$, then $A \subseteq U$. Also, since A is (μ_1, μ_2, μ_3)-wg closed set, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$. Now, since $B \subseteq cl_{\mu_1}(int_{\mu_2}(A))$ and using Theorem 2(i.), $int_{\mu_2}(B) \subseteq B \subseteq cl_{\mu_1}(int_{\mu_2}(A))$. Consequently, by Theorem 3(iv.), $cl_{\mu_1}(int_{\mu_2}(B)) \subseteq cl_{\mu_1}(B) \subseteq cl_{\mu_1}(int_{\mu_2}(A))$. In effect, $cl_{\mu_1}(int_{\mu_2}(B)) \subseteq U$. Hence, B is (μ_1, μ_2, μ_3)-wg closed set.

Theorem 11. If A is (μ_1, μ_2, μ_3)-wg closed set, then $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set.

Proof. Let A be (μ_1, μ_2, μ_3)-wg closed set and F be a nonempty μ_3-closed set such that $F \subseteq cl_{\mu_1}(int_{\mu_2}(A) - A$. Then $F \subseteq cl_{\mu_1}(int_{\mu_2}(A)) \bigcap A^c$. This implies that $F \subseteq cl_{\mu_1}(int_{\mu_2}(A))$ and $F \subseteq \bigcap A^c$. Note that F^c is μ_3-open and $A \subseteq F^c$. Now, since A is (μ_1, μ_2, μ_3)-wg closed, then $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq F^c$. Thus, $F \subseteq cl_{\mu_1}(int_{\mu_2}(A)) \subseteq F^c$. This means that $F = F \bigcap F^c = \emptyset$. This is a contradiction. Hence if A is (μ_1, μ_2, μ_3)-wg closed set, then $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set.

Remark 1. The converse of Theorem 11 is not necessarily true.

Example 1. Let $X = \{1, 2, 3\}$, $\mu_1 = \{\emptyset, \{1\}, \{1, 3\}\}$, $\mu_2 = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $\mu_3 = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. The μ_1-closed sets are $X, \{2, 3\}$ and $\{2\}$. Also, the μ_2-open sets are $\emptyset, \{1\}, \{2\}$, and $\{1, 2\}$. The μ_3-open sets on the other hand are $\emptyset, \{2\}, \{3\}$, and $\{2, 3\}$ whose corresponding μ_3-closed sets are $X, \{1, 3\}, \{1, 2\}$, and $\{1\}$. Observe that considering all the possible subsets of X, only the sets: $\emptyset$ and $\{3\}$ are not (μ_1, μ_2, μ_3)-wg closed sets. If $A = \emptyset, int_{\mu_2}(\emptyset) = \emptyset$. Consequently, $cl_{\mu_1}(int_{\mu_2}(\emptyset)) = \{2\}$. Hence, $cl_{\mu_1}(int_{\mu_2}(\emptyset))\backslash\emptyset = \{2\}$ which is not a μ_3-closed set. So " $cl_{\mu_1}(int_{\mu_2}(A))\backslash A$ contains no nonempty μ_3-closed set" is satisfied. Thus, when $A = \emptyset$ the statement " If $cl_{\mu_1}(int_{\mu_2}(A))\backslash A$ contains no nonempty μ_3-closed set, then A is (μ_1, μ_2, μ_3)-wg closed sets." is false.
Moreover, if $A = \{3\}$, $int_{\mu_2}(\{3\}) = \emptyset$. Consequently, $cl_{\mu_1}(int_{\mu_2}(\{3\})) = \{2\}$. Thus, $cl_{\mu_1}(int_{\mu_2}(\{3\}))\backslash\{3\} = \{2\}$ which is not a μ_3-closed set. That is, "$cl_{\mu_1}(int_{\mu_2}(A))\backslash A$ contains no nonempty μ_3-closed" is satisfied. Thus, if $A = \{3\}$ the statement "$cl_{\mu_1}(int_{\mu_2}(A))\backslash A$ contains no nonempty μ_3-closed set, then A is (μ_1, μ_2, μ_3)-wg closed sets." is false.

Remark 1. states that the converse of Theorem 11 is not necessarily true. This is illustrated by Example 1. However, considering $\mu_1 \subseteq \mu_3$ where μ_1 and μ_3 are GTs in X, then we can consider the statement "If $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set, then A is (μ_1, μ_2, μ_3)-wg closed set."

Theorem 12. Let $\mu_1 \subseteq \mu_3$. If $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set, then A is (μ_1, μ_2, μ_3)-wg closed set.

Proof. Let $A \subseteq X$. Suppose $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set and A is not a (μ_1, μ_2, μ_3)-wg closed set. Then there exists μ_3-open set U such that $A \subseteq U$ and $cl_{\mu_1}(int_{\mu_2}(A)) \nsubseteq U$. Now, $cl_{\mu_1}(int_{\mu_2}(A)) \nsubseteq U$ implies that $cl_{\mu_1}(int_{\mu_2}(A)) \bigcap U^c \neq \emptyset$. Let $F = cl_{\mu_1}(int_{\mu_2}(A)) \bigcap U^c$. By Theorem 3(ii.), $cl_{\mu_1}(int_{\mu_2}(A))$ is μ_1-closed. Since $\mu_1 \subseteq \mu_3$, applying Theorem 4, $cl_{\mu_1}(int_{\mu_2}(A))$ is μ_3-closed. Thus since U^c is μ_3-closed and using Theorem 1, $F = cl_{\mu_1}(int_{\mu_2}(A)) \bigcap U^c$ is μ_3-closed. Now, $F \neq \emptyset$ and

$$F = cl_{\mu_1}(int_{\mu_2}(A)) \bigcap U^c \subseteq cl_{\mu_1}(int_{\mu_2}(A)) \bigcap A^c = cl_{\mu_1}(int_{\mu_2}(A)) - A.$$

This is a contradiction. Therefore, if $cl_{\mu_1}(int_{\mu_2}(A)) - A$ contains no nonempty μ_3-closed set, then A is (μ_1, μ_2, μ_3)-wg closed set.

Theorem 13. If A be μ_1-closed and μ_2-open, then A is (μ_1, μ_2, μ_3)-wg closed set.

Proof. Let $A \subseteq U$ such that A is both μ_1-closed and μ_2-open, and U is μ_3-open such that $A \subseteq U$. Since A is μ_2-open and μ_1-closed, then applying Theorem 2(iii.) and Theorem 3(iii.), $int_{\mu_2}(A) = A$ and $cl_{\mu_1}(A) = A$. Thus, $cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(A) = A$. In effect, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$. Hence, A is (μ_1, μ_2, μ_3)-wg closed set.

Definition 8. *Let $A \subseteq Y \subseteq X$. Then A is (μ_1, μ_2, μ_3)-wg closed set in Y if $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq U$ whenever $A \subseteq U$ and U is μ_3-open in Y.*

Theorem 14. Let $X \neq \emptyset$ and $A \subseteq Y \subseteq X$. If A is (μ_1, μ_2, μ_3)-wg closed set in X and Y is μ_1-closed set in X, then A is (μ_1, μ_2, μ_3)-wg closed set in Y.

Proof. Let $A \subseteq U$ such that U is μ_3-open in Y. Then $U = Y \bigcap G$ for some μ_3-open set G in X. Note that $A \subseteq G$. Since A is (μ_1, μ_2, μ_3)-wg closed in X, then $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq G$. Now, by Theorem 2(i.), $int_{\mu_2}(A) \subseteq A$. Since $A \subseteq Y$, $int_{\mu_2}(A) \subseteq Y$. By Theorem 3(iv.), $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq cl_{\mu_1}(Y)$. But Y is μ_1-closed, so $cl_{\mu_1}(Y) = Y$. Thus, $cl_{\mu_1}(int_{\mu_2}(A)) \subseteq Y$. Accordingly, $cl_{\mu_1}(int_{\mu_2}(A)) = cl_{\mu_1}(int_{\mu_2}(A)) \bigcap Y \subseteq G \bigcap Y = U$. Therefore, A is (μ_1, μ_2, μ_3)-wg closed set in Y.

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