EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 3, 2020, 579-586
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On solvability of p- harmonic type equations in grand Sobolev spaces

Alik M. Najafov[1,2,*], Sain T. Alekberli[3]

[1] *Azerbaijan University of Architecture and Construction, Baku, Azerbaijan*
[2] *Institute of Mathematics and Mechanics, National Academy of Science of Azerbaijan, Baku, Azerbaijan*
[3] *Baku Engineering University, Baku, Azerbaijan*

Abstract. In this paper with the help of variational method existence and uniqueness of solution of p- harmonic type equations in grand Sobolev spaces is studied.
2020 Mathematics Subject Classifications: 35A01,35A02, 35A15, 35D30
Key Words and Phrases: p- harmonic type equations, grand Sobolev space, variational method, Dirichlet problem

1. Introduction and preliminary notes

It is well known that the existence and uniqueness of Dirichlet problem for p-harmonic equations

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \operatorname{div} f, \tag{1}$$

$$u|_{\partial G} = 0 \tag{2}$$

in Sobolev and grand Sobolev spaces were studied, e.g., in [1, 2] see also [4–7, 10–13]. Namely, in these papers the different problems for p-harmonic equations were considered. Similar and various problems of partial differential equations in grand Sobolev, Besov and Morrey type spaces were studied in [8, 9, 14–16, 18–23] and others. Most of these papers were used the variational methods. Evidently, in the above-mentioned papers only p-harmonic equations (1) was considered.

In this paper we consider Dirichlet problem for p-harmonic type equation has a form

$$\operatorname{div}\left(|\nabla u|^{p-q}\nabla u\right) = \operatorname{div} f, \tag{3}$$

$$u|_{\partial G} = \varphi|_{\partial G}\ , \tag{4}$$

where $1 < p < \infty$; $2 \le q < \infty$; $\varphi \in W^1_{p)}(G)$, $f \in L_{(p-\varepsilon)'}(G)$, $(p-\varepsilon)' = \dfrac{p-\varepsilon}{p-\varepsilon-1}$ and G in $\mathbb{R}^n$ is a bounded domain.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i3.3762

Email addresses: aliknajafov@gmail.com (A.M. Najafov), sain.elekberli@bk.ru (S.T. Alekberli)

Definition 1. *([6, 17, 23]) Denote by $W^1_{p)}(G)$ the grand Sobolev space of locally summable functions u on G having the weak partial derivatives $D^1_{x_i}u$ $(i = 1, 2, \ldots, n)$ with the finite norm*

$$\|u\|_{W^1_{p)}(G)} = \|u\|_{L_{p)}(G)} + \|\nabla u\|_{L_{p)}(G)},$$

where

$$\|u\|_{L_{p)}(G)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|G|} \int_G |u(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

and $|G|$ is the Lebesgue measure of G.

We note that the correct choice of space for problem (3)-(4) is the grand Lebesgue space (or grand Sobolev space).

In this paper using the variational method an existence and uniqueness of solution to Dirichlet problem for $p-$ harmonic type equations (3)- (4) in grand Sobolev spaces is studied.

A weak solution for the problem (3)-(4) on G is a function $u(x) \in W^1_{p)}(G)$, if $u - \varphi \in \overset{\circ}{W}{}^1_{p)}(G)$ such that

$$\sum_{i=1}^n \int_G |\nabla u|^{p-q} u_{x_i} \vartheta_{x_i} dx = \sum_{i=1}^n \int_G f \vartheta_{x_i} dx, \tag{5}$$

for every $\vartheta \in \overset{\circ}{W}{}^1_{p)}(G)$.

2. Main results

In this section we prove the existence and uniqueness of weak solution (5) for the problem (3)-(4).

Theorem 1. *Let $G \subset R^n$ is bounded domain, $1 < p < \infty$; $2 \leq q < \infty$; $g, h \in W^1_{p-(q-2)}(G)$, $\varphi \in W^1_{p)}(G)$ and $f \in L^1_{(p-\varepsilon)'}$. Then the Dirichlet problem for pharmonic type equation (3) has a unique weak solutions in $W^1_{p)}(G)$.*

Proof. Since functions g and $h \in W^1_{p-(q-2)}(G)$, then we consider the bilinear functional as the form

$$F(g,h) = \sum_{i=1}^n \int_G |\nabla \mathrm{g}|^{p-q} g_{x_i} h_{x_i} dx - \sum_{i=1}^n \int_G f h_{x_i} dx =$$

$$= I(g,h) - \sum_{i=1}^n \int_G f h_{x_i} dx = I(g,h) - (f,h), \tag{6}$$

since $f \in L_{(p-\varepsilon)'}(G)$, $(p-\varepsilon)' = \frac{p-\varepsilon}{p-\varepsilon-1}$. Consequently, we have

$$|I(g,g)| = |I(g)| = \left|\sum_{i=1}^{n} \int_G |\nabla g|^{p-q}\, g_{x_i}\, g_{x_i}\, dx\right| \leq$$

$$\leq \sum_{i=1}^{n} \int_G |\nabla g|^{p-q}\, |g_{x_i}|\, |g_{x_i}|\, dx = \sum_{i=1}^{n} \int_G |\nabla g|^{p-q} |g_{x_i}|^2 \, dx =$$

$$= \int_G |\nabla g|^{p-(q-2)} \, dx < \infty,$$

$$|I(g)| \leq \|\nabla g\|^{p-(q-2)}_{L_{p-(q-2)}(G)}.$$

Consequently, for every $q-2<\varepsilon<p-1$ function $g \in W^1_{p)}(G)$ and

$$\|g\|_{W^1_{p)}(G)} \leq C_1 \|g\|_{W^1_{p-(q-2)}(G)},$$

and, note that

$$\|\nabla g\|^{p-\varepsilon}_{L_{p)}(G)} \leq C_2 |I(g)|\ , \tag{7}$$

where C_1 and C_2 are constants independents on function g.

The variational problem is stated as follows. Find a function $g \in W^1_{p)}(G)$ such that which gives the minimum value to the integral $F(g)$ and is unique. The Euler-Lagrange equation for the variational problem (6) under consideration is the equation (3). With the help of the inequality (7), we have

$$|F(g,g)| = |F(g)| = \left|I(g) - \sum_{i=1}^{n} \int_G f\, g_{x_i}\, dx\right| \geq |I(g)| -$$

$$-\left|\sum_{i=1}^{n} \int_G f\, g_{x_i}\, dx\right| \geq |I(g)| - \sum_{i=1}^{n} \left|\int_G f g_{x_i}\, dx\right| \geq |I(g)| - \left|\sum_{i=1}^{n} \int_G f\, g_{x_i}\, dx\right| \geq$$

$$\geq |I(g)| - \sum_{i=1}^{n} \int_G |f|\, |g_{x_i}| dx \geq C_3\, \|g\|^{p-\varepsilon}_{W^1_{p)}(G)} - \|g\|^{p-\varepsilon}_{L_{p)}(G)} -$$

$$-\|f\|_{L_{(p-\varepsilon)'}(G)} \|\nabla g\|_{L_{p)}(G)} \geq C_4 \|g\|_{W^1_{p)}(G)} = M_0,$$

C_3 and C_4 are constants independent on the function $g(x)$.

This means that $F(g)$ is lower bounded on $W^1_{p)}(G)$ show that there exists $g_0 \in W^1_{p)}(G)$ such that $F(g_0) = \min_{g \in W^1_{p)}(G)} F(g)$. Fix some sequence $\{g_m\} \in W^1_{p)}(G)$ $(m = 1, 2, \dots)$ such that $\lim_{m\to\infty} F(g_m) = r_0$. Let $\sigma > 0$ choose m_σ so for $m \geq m_\sigma$ and $s = 1, 2, \dots$ it holds $F(g_{m+s}) < r_0 + \sigma$. Then noting that $\frac{1}{2}(g_{m+s} + g_m) \in W^1_{p)}(G)$ we have

$F\left(\frac{g_{m+s}+g_m}{2}\right) \geq r_0$. By direct calculations we show that $I\left(\frac{g_{m+s}-g_m}{2}\right) < 4\sigma$, and we have $\|g_{m+s}+g_m\|_{W^1_{p)}(G)} \leq 2\left(\frac{\varepsilon}{C}\right)^{\frac{1}{p-\varepsilon}}$. This means that the sequence $\{g_m\}$ is fundamental in the spaces $W^1_{p)}(G)$, consequently in view of completeness the spaces $W^1_{p)}(G)$ there exist a function $g_0 \in W^1_{p)}(G)$ such that $\lim\limits_{m\to\infty} \|g_m - g_0\|_{W^1_{p)}(G)} = 0$. By theorem on trace in $W^1_p(G)$, ([3, p.143]), we get

$$W^1_{p)}(G) \to W^1_{p-\varepsilon}(G) \to L_{t-\varepsilon}(G_k),\ G_k = G\bigcap \mathbb{R}^k,\ p < t \leq \infty,\ 1 \leq k \leq n.$$

So

$$|F(g_m) - F(g_0)| \leq C\|g_m - g_0\|_{W^1_{p)}(G)}$$

and hence it follows that $r_0 = \lim\limits_{m\to\infty} F(g_m) = F(g_0)$. Show that the function delivering minimum to the functional $F(g)$ is unique and satisfies equation (3) in the space $W^1_{p)}(G)$. Since $g \in W^1_{p)}(G)$ and $F(g_0) = r_0$, we have

$$0 \leq I\left(\frac{g-g_0}{2}\right) = \frac{1}{2}F(g) + \frac{1}{2}F(g_0) - F\left(\frac{g+g_0}{2}\right) \leq \frac{r_0}{2} + \frac{r_0}{2} - r_0 = 0,$$

$$I(g - g_0) = 0.$$

By $\|g_m - g_0\|_{W^1_{p)}(G)} \to 0,\ m \to \infty$, it follows that the function g coincides with g_0 as an element of the space $W^1_{p)}(G)$. Again from the theorem on trace in space $W^1_{p)}(G)$, we have

$$\|(g_m - g_0)|_{\partial G}\|_{L_{t-\varepsilon}(\partial G)} \leq C\,\|g_m - g_0\|_{W^1_{p)}(G)} \to 0,\ \ m \to \infty.$$

Since

$$\|\,g_m|_{\partial G} - \ \varphi|_{\partial G}\,\|_{L_{t-\varepsilon}(\partial G)} \to 0,\ \ m \to \infty,$$

therefore

$$\|\,g_0|_{\partial G} - \ \varphi|_{\partial G}\,\|_{L_{t-\varepsilon}(\partial G)} \to 0\ \ m \to \infty.$$

Taking into account the condition $\frac{d}{d\mu}\left(F(g_0 + \mu\omega)\right)_{\mu=0} = 0$, show that the function $g_0 \in W^1_{p)}(G)$, minimizing the integral $F(g)$ satisfies the following equation

$$I(g_0, \omega) - (f, \omega) = 0. \tag{8}$$

Now prove that the function $g_0 \in W^1_{p)}(G)$ minimizing the integral $F(g)$ is the weak solution of the problem (3)-(4). By $\theta(t)$ we denote some monotonically decreasing function on the segment $\frac{1}{2} \leq t \leq 1$ and having the following properties

$$\theta\left(\frac{1}{2}+0\right) = 1,\ \theta(1-0) = -1,\ \theta^{(s)}\left(\frac{1}{2}+0\right) = \theta^{(s)}(1-0) = 0,\ s = 1, 2, \ldots\ .$$

The function

$$\gamma(t) = \begin{cases} \theta'(t), & \frac{1}{2} \le t \le 1, \\ 0, & -\infty < t < \frac{1}{2},\ 1 < t < \infty \end{cases}$$

is infinitely differentiable and finite on the real line. Note that the function γ satisfy condition

$$\gamma^{(s)}\left(\frac{1}{2}+0\right) = \gamma(1-0),\ (s = 1, 2, \ldots).$$

Let $\delta > 0$ and let $G_\delta = \{y : \rho(y, R^n \backslash G) > \delta\}$ be arbitrary point of the domain G, and $r = \rho(x, x_0)$. There $\rho(x, x_0)$ is the Euclidean distance between x and x_0, where $x \in G$ and x_0 be a fixed point in G. Following Sobolev [24], we introduce the function

$$\omega(x) = \gamma\left(\frac{r}{l_1}\right) - \gamma\left(\frac{r}{l_2}\right),$$

for $0 < l_1 < l_2 < \delta$. It is obvious that $\omega(x)$ is a infinitely differentiable finite function with a support lying on a annular domain $\frac{l_1}{2} < r < l_2$. Therefore $\omega \in C_0^\infty(G)$ and $D^{(s)}\omega|_{\partial G} = 0$ for all $s = 1, 2, \ldots$. Then from (8) by definition of the weak derivative it follows that

$$\int_G K\left(\frac{r}{l_1}\right) g(x)\, dx = \int_G K\left(\frac{r}{l_2}\right) g(x)\, dx, \tag{9}$$

where

$$K\left(\frac{r}{l_i}\right) = div\left(\left|\nabla\gamma\left(\frac{r}{l_i}\right)\right|^{p-q} \nabla\gamma\left(\frac{r}{l_i}\right)\right) - div\ f,\quad i = 1, 2.$$

Note that the function $K\left(\frac{r}{l_i}\right)$ having all properties of kernel. Namely, the following properties hold:
1) K is infinitely differentiable function with support in the ball $r \le l_i$;
2) The function K and all its derivatives on sphere $R = h$ are zero;
3)

$$\frac{1}{\tau_n\, l_i^n} \int_G K\left(\frac{r}{l_i}\right) dx = 1,$$

where

$$\tau_n = \frac{2\,\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 \xi^{n-1} K(\xi)\, d\xi.$$

Then for the function $g_0(x)$ we can constructed Sobolev's averaging $g_{0,l_i}(x)$, $i = 1, 2$ on the ball l_i $(i = 1, 2)$ with centered at the point x as

$$g_{0,l_i}(x) = \frac{1}{\tau_n\ l_i^n} \int_{R^n} K\left(\frac{|z-x|}{l_i}\right) g_0(z)\, dz,\quad i = 1, 2.$$

The we can rewrite equality (9) in the form $g_{0,l_1}(x) = g_{0,l_2}(x)$. Consequently, for $l < \delta$

$$g_{0,l}(x) = g_0(x).$$

Since the average functions $g_{0,l_i}(x)$, $i=1,2$ are continuous and has continuous derivatives for any order, then $g_0\left(x\right)$ also is a kernel. Integrating by parts in the equality $I\left(g_0,\omega\right)-\left(f,\omega\right)=0$, whence is the limit case

$$\sum_{i=1}^{n}\int_G \omega(x)\frac{\partial}{\partial x_i}\left(\left|\nabla g_0\right|^{p-q}\frac{\partial}{\partial x_i}g_0\left(x\right)\right)dx=\sum_{i=1}^{n}\int_G \omega(x)\frac{\partial}{\partial x_i}f(x)\ \ dx\ .$$

Hence by the arbitrariness of the functions $\omega(x)$ it follows that

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\left|\nabla g_0\right|^{p-q}\frac{\partial}{\partial x_i}g_0\left(x\right)\right)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}f(x)$$

i.e

$$div\left(\left|\nabla g_0\right|^{p-q}\nabla g_0\right)=div f.$$

Thus, solution of the variational problem (5) from the class $W^1_{p)}\left(G\right)$ is also solution of Dirichlet problem (3)-(4) and this solution is unique.

3. Conclusion

In conclusion, we note that for a p-harmonic type equation in the grand Sobolev space, a result is obtained on the existence and uniqueness of a weak solution.

References

[1] G Afrouzi and A Hadjian. Non Trivial Solutions for p- Harmonic type Equations via a local minimum theorem for functionals. *Taiwaness Journal of Math.*, 19(6):1731–1742, 2015.

[2] G Aronsson and P Lingvist. On p- Harmonic Functions in the Plane and their Stream Functions. *Jour. of Diff. Equations*, 74:157–188, 1988.

[3] O V Besov, V P Ilyin, and S M Nikolskii. *Integral Representations Functions and Embeddind Theorems.* M. Nauka, Moscow, 1996.

[4] G Boccardo. Non linear elliptic and parabolic equations involving measure data. *Jour. Func. Anal.*, 87:149–169, 1989.

[5] Y Deng and H Pi. Multiple solutions for p- harmonic type equations. *Nonlinear Anal.*, 71:4952–4959, 2009.

[6] A Fiorenza, M R Formica, and A Gogatishvili. On grand and small Lebesgue and Sobolev spaces and some applcations to PDEs. *Diff. Equat. and Applic.*, 10(1):21–46, 2018.

[7] A Fiorenza and C Sbordone. Existence and uniqueness results for solutions of nonlinear equations with right hand side in L^1. *Stud. Math.*, 127(3):4959–4969, 1998.

[8] S Gala, Q Liu, and M A Ragusa. A new regularity criterion for the nematic liquid crystal flows. *Applicable Analysis*, 91(9):1741–1747, 2012.

[9] S Gala and M A Ragusa. Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices. *Applicable Analysis*, 95(6):1271–1279, 2016.

[10] L Greco, T Iwaniec, and C Sbordone. Inverting the p- harmonic operator. *Manuscripta Math.*, 92(2):249–258, 1997.

[11] L Greco and A Verde. A reguliraty property of p- harmonic functions. *Annal. Academ. Scien. Fennicae Math.*, 25:317–323, 2000.

[12] H Luiro and M Parviainen. Gradient walks and p- harmonic functions. *Proc. Amer. of the Math. Soc.*, 145:4313–4324, 2017.

[13] J Manfredi. P- harmonic functions in the plane. *Proc. of the Amer. Math. Soc.*, 103(2):473–479, 1988.

[14] A M Najafov. Problem on smoothness of solution of one class of hypoelliptic equations. *Proc. A. Razm. Math. Inst.*, 140:131–139, 2006.

[15] A M Najafov. The Differential Properties of Functions from Sobolev-Morrey type Spaces of Fractional Order. *Jour. Math. Res.*, 7(3):1–10, 2015.

[16] A M Najafov and A T Orujova. On the Solution of a Class of Partial Differential Equations. *Electron. Jour. Qual. Theory Diff. Equ.*, 2017(44):1–9, 2017.

[17] A M Najafov and N R Rustamova. Some Differential Properties of Anisotropic Grand Sobolev-Morrey spaces. *Trans. A. Razm. Math. Inst.*, 172:82–89, 2018.

[18] A M Najafov, N R Rustamova, and S T Alekberli. On Solvability of a Quasi-Elliptic Partial Differential Equations. *Jour. of Ellip. and Parab. Equ.*, 2019(5):175187, 2019.

[19] N S Papageorgiou and A Scapellato. Nonlinear Robin problems with general potential and crossing reaction. *Rend. Lincei-Mat. Appl.*, 30:1–29, 2019.

[20] N S Papageorgiou and A Scapellato. Concave-Convex Problems for the Robin p-Laplacian Plus an Indefinite Potential. *Mathematics*, 8(3,421):1–27, 2020.

[21] S Polidoro and M A Ragusa. Harnack Inequality for Hypoelliptic Ultraparabolic Equations with a Singular Lower Order Term. *Revista Matematica Iberoamericana*, 24(3):1011–1046, 2008.

[22] M A Ragusa and A Tachikawa. Regularity for Minimizers for Functionals of Double Phase with Variable Exponents. *Advances in Nonlinear Analysis*, 9(1):710–728, 2020.

[23] C Sbordone. Grand Sobolev Spaces and their Applications to Variational Problems. *Le Matematiche (Catania)*, 51(2):335–347, 1996.

[24] S L Sobolev. *Some Applications of Functional Analysis in Mathematical Physies.* Novosibirsk, Russian, 1950.