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# New Topology on Omega Algebra

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**Abstract.** The purpose of this paper is to define a new topology called omega topology over a new structure called omega algebra and discuss some of its topological properties. Four different examples of omega topology are introduced. Furthermore, we define a new topology over a semiring in conventional algebra and we study the relationship between omega topology and weaker kinds of normality.

**2020 Mathematics Subject Classifications**: 16Y60, 54F15

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## 1. Introduction

Tropical geometry is the most recent but fast growing branch of mathematical science, which is analytically based on idempotent analysis and algebraically on idempotent semirings also known as tropical semirings. These are basically extended sets of real numbers $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ and $\mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\}$ which are given monoidal structures by using min and max operations for addition, respectively. In order to adhere the semiring structure, the additive operation of $\mathbb{R}$ is used as the multiplication operation. By these choices, both $\mathbb{R}_\infty$ and $\mathbb{R}_{-\infty}$ become idempotent semirings. In the literature, they are also termed as min and max plus algebras, respectively. In both cases, 0 of $\mathbb{R}$ becomes a multiplicative identity and $\infty$ and $-\infty$ become additive identities of these semirings, respectively. Interestingly, some authors associated $\mathbb{R}_{-\infty}$ to tropical geometry, while other authors associated $\mathbb{R}_\infty$ to tropical geometry, see for instance [8], [10], [12] and [14]. Omega algebra, or "w-algebra" for short, unifies the different terms and introduces an original structure, which, in fact,is an "abstract tropical algebra". The $\mathbb{R}_{-\infty}$ and $\mathbb{R}_\infty$ and their nearby structures, like $\min - \max$ and $\max -$ times algebras, etc., are all subsumed under omega algebra.

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All these are idempotent semirings, which are also called dioids. In the previous studies, for the construction of all such semirings, an ordered infinite abelian group is mandatory. In $\omega-$ algebra, the definition is extended to cyclically ordered abelian groups and also to finite sets under some suitable ordering. Note that cyclically ordered abelian groups are more general than that of ordered abelian groups [16]. The aim of this paper is to define a new topology, which is called omega topology over omega algebra, and discuss some of its topological properties. Four different examples of omega topology are introduced. Furthermore, we defined a new topology over a semiring in conventional algebra and study the relationship between omega topology and weaker kinds of normality. This paper is divided as follows. In Section 2, we review an abstract definition and some basic facts about abstract omega algebras. We support these by presenting three concrete examples. In Section 3, we define a new topology on omega algebra and discuss some of its topological properties. In Section 4, we provide four different examples of omega topology: the first and fourth examples are from an ordered infinite sets, the second example is from a cyclically ordered infinite set, and the third example is from a finite set. Furthermore, we define a new topology over a semiring in conventional algebra. Finally, we study the relationship between omega topology and weaker kinds of normality in section 5. Throughout this paper, we do not assume $T_2$ in the definition of compactness. We also do not assume regularity in the definition of Lindel$\ddot{o}$fness. This paper is produced from the PhD thesis of Mr. Mesfer Hayyan Alqahtani in King Abdulaziz University.

## 2. Preliminaries

In this section, we provide an abstract definition and review some basic facts about abstract omega algebras. Furthermore, we support these by presenting concrete examples: one from an ordered infinite set, another from a cyclically ordered infinite set, and a third one from a finite set. For more details, see [11].

Let $(G, \circ, e)$ be an abelian group. Let $A$ be a closed subset of $G$ and $e \in A$. Then $(A, \circ, e)$ is a submonoid of $G$. Assume that $\omega$ is an indeterminate (may belong to $A$ or $G$, as we will see in Examples 1 and 2). Obviously, in this case $\omega$ is no longer an indeterminate. Because the terms are generated from tropical geometry, this indeterminate can be called a tropical indeterminate.

**Definition 1.** [11]

We say that $A_\omega = A \cup \{\omega\}$ is an *omega algebra* (in short *$\omega-$ algebra*) over the group $G$ in case $A_\omega$ is closed under two binary operations,

$$\oplus, \otimes : A_\omega \times A_\omega \longrightarrow A_\omega,$$

Then for all $a_1, a_2, a_3 \in A$, the following axioms are satisfied:

(i) $a_1 \oplus a_2 = a_1 \text{ or } a_2$;

(ii) $a_1 \oplus \omega = a_1 = \omega \oplus a_1$;

(iii) $\omega \oplus \omega = \omega$;

(iv) $a_1 \otimes a_2 = a_2 \otimes a_1 \in A$;

(v) $(a_1 \otimes a_2) \otimes a_3 = a_1 \otimes (a_2 \otimes a_3)$;

(vi) $a_1 \otimes e = a_1$;

(vii) $a_1 \otimes \omega = \omega \otimes a_1 = \begin{cases} \omega & if \quad \omega \neq e \\ a_1 & if \quad \omega = e \end{cases}$;

(viii) $\omega \otimes \omega = \omega$;

(ix) $a_1 \otimes (a_2 \oplus a_3) = (a_1 \otimes a_2) \oplus (a_1 \otimes a_3)$.

**Remark 1.** [11]

(1) $\oplus$ is a pairwise comparison operation such as max, min, inf, sup, up, down, lexicographic ordering, or anything else that compairs two elements of $A_\omega$. Obviously, it is associative and commutative and the tropical indeterminate $\omega$ play the role of the identity. Hence $(A_\omega, \oplus, \omega)$ is a commutative monoid.

(2) $\otimes$ is also associative and commutative on $A_\omega$, and $e$ plays the role of the multiplicative identity of $A_\omega$. Hence, $(A_\omega, \otimes, e)$ is also a commutative monoid.

(3) The left distributive law (ix) also gives the right distributive law.

(4) Every element of $A_\omega$ is an idempotent under $\oplus$.

(5) Altogether, we write both structures as: $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$. This is an idempotent semiring, which is also called "dioid" in the literature.

**Remark 2.** [11] $\omega-$ algebra can similarly be defined over a commutative monoid, ring, or even a semiring. More generally, one may construct analogously such algebras on other more weaker structures.

In this note, we confined ourselves to only $\omega-$ algebras over abelian groups and rings.

**Example 1.** [11] *Max-plus algebra, min-plus algebra* and all such "so called" *algebras* are particular cases of the $\omega-$ *algebra* over the ring $\mathbb{R}$ or its associated subrings. A simpler example is the following. In the abelian group $(\mathbb{Z}, +)$, for any integer $m$, we have $W(m) = \{0, m, 2m, \cdots\}$. This is an additive submonoid of $(\mathbb{Z},+)$. Let $\omega = -\infty$, $a_1 \oplus a_2 = \max(a_1, a_2)$ and $a_1 \otimes a_2 = a_1 + a_2, \forall a_1, a_2 \in W(m)$. Then.

$$W(m)_{-\infty} = \{W(m)_{-\infty}, \oplus, \otimes, -\infty, 0\}$$

is $-\infty -$ algebra over the abelian group of integers $\mathbb{Z}$. Hence, we have a sequence of $\omega-$ subalgebras

$$W(m) \geq W(2m) \geq \cdots .$$

**Example 2.** [11] *A cyclically ordered abelian group.* This example is constructed exclusively over an abelian group. A cyclically ordered abelian group is more general than that of a linearly ordered abelian group. Every linearly ordered abelian group is cyclically ordered, but the converse,in general, is not true. The following example is that of a cyclically ordered abelian group, which is not an ordered abelian group [16]. For more details about a cyclically ordered abelian groups. Consider the cyclically ordered abelian group in the form of the unit circle

$$C = \{z \in \mathbb{C} \mid \ |z| = 1\}.$$

Let

$$W = \{0, 1, 2, \cdots\}.$$

For some $\theta \in [0, 1)$, define $\rho_x = e^{2\pi i\theta x}$, where $x \in W$, in particular, $\rho_0 = 1$. Set

$$A := \{\rho_x \mid x \in W\} \subset C.$$

Because $\rho_{x_1}\rho_{x_2} = \rho_{x_1+x_2}$, $\forall x_1, x_2 \in W$, $A$ is multiplicatively closed.

**Theorem 1.** [11] $A_{\rho_0} = A \cup \{\rho_0\}$ is an omega algebra with the identical additive and multiplicative identities. This omega algebra contains infinite omega subalgebras.

*Proof.* Define $\oplus$ on $A$ by

$$\rho_{x_1} \oplus \rho_{x_2} = \rho_{x_3} \text{ where } x_1, x_2, x_3 \in W, \text{ with } x_3 = \max(x_1, x_2)$$

and define $\otimes$ on $A$ by

$$\rho_{x_1} \otimes \rho_{x_2} = \rho_{x_1+x_2}, \text{ where } x_1, x_2 \in W.$$

Clearly, both operations are associative, and as $\rho_0 \oplus \rho_{x_1} = \rho_{x_1}$ and $\rho_0 \otimes \rho_{x_1} = \rho_{x_1}$, so $(A, \oplus, \rho_0)$ and $(A, \otimes, \rho_0)$ are monoids. Finally, $\forall x_1, x_2, x_3 \in W$,

$$\begin{aligned} \rho_{x_1} \otimes (\rho_{x_2} \oplus \rho_{x_3}) &= \rho_{x_1} \otimes \rho_{\max(x_2,x_3)} \\ &= \rho_{x_1+\max(x_2,x_3)} \\ &= \rho_{\max(x_1+x_2,x_1+x_3)} \\ &= (\rho_{x_1+x_2} \oplus \rho_{x_1+x_3}) \\ &= (\rho_{x_1} \otimes \rho_{x_2}) \oplus (\rho_{x_1} \otimes \rho_{x_3}). \end{aligned}$$

As, $\forall x_1 \in W$,

$$\rho_{x_1} \oplus \rho_{x_1} = \rho_{x_1}$$

and $\rho_0 = 1$ we conclude that $A = (A, \oplus, \otimes, 1, 1)$ is an omega algebra. Finally, consider $W(m) = \{0, m, 2m, \cdots\}$, where $m = 1, 2, \cdots$ . For each $m$, one can construct an omega subalgebra.

**Example 3.** [11] A *Laxicographic Ordering.* Consider the binary linear code of length 2;

$$\mathbb{Z}_2^{(2)} = \{00,\ 01,\ 10,\ 11\}.$$

Under componentwise addition $+$ and componentwise multiplication $\circ$, $(\mathbb{Z}_2^{(2)}, +, \circ)$ is a ring with code-words $0 = 00$ and $1 = 11$ as additive and multiplicative identities. We define the laxicographic odering on the elements of $\mathbb{Z}_2^{(2)}$ and arrange them as:

$$00 < 01 < 10 < 11$$

Let $A = \{00, 01\}$. Consider $\omega = 11$. Note that, in this example, $\omega \notin A$ but $\omega \in G$. We define addition on $A_\omega = \{00, 01, 11\}$ by:

$$a \oplus b = \min(a, b).$$

Hence we get the table:

| $\oplus$ | 00 | 01 | 11 |
|---|---|---|---|
| 00 | 00 | 00 | 00 |
| 01 | 00 | 01 | 01 |
| 11 | 00 | 01 | 11 |

Define multiplication as the boolean sum, namely,

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 1 = 1.$$

Hence we get the table:

| $\otimes$ | 00 | 01 | 11 |
|---|---|---|---|
| 00 | 00 | 01 | 11 |
| 01 | 01 | 01 | 11 |
| 11 | 11 | 11 | 11 |

We conclude that $(A_\omega, \oplus, 11)$ and $(A_\omega, \otimes, 00)$ are the additive and multiplicative monoides. Clearly, this is a simple $\omega-$ algebra.

## 3. Omega topology

In this section, we define a new topology on omega algebra and discuss some of its topological properties.

**Proposition 1.** *Let $(G, \circ, e)$ be an abelian group and $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$ an $\omega-$algebra over the group $G$. We define a new topology on $A_\omega$ is called an omega topology, denoted by $\tau_\omega$, as follow:*

$\tau_\omega = \{\emptyset, A_\omega\} \cup \{U \subseteq A_\omega : \omega \in U$ *and for any* $a \in U \setminus \{\omega\},$ *the multiplicative inverse of exists in* $U$ $\}$

*Proof.* Condition $\emptyset, A_\omega \in \tau_\omega$ is satisfied from the definition of $\tau_\omega$. Now let $V_1, V_2 \in \tau_\omega$ be arbitrary. If either $V_1$ or $V_2$ is empty, then $V_1 \cap V_2 = \emptyset \in \tau_\omega$. Assume now, $V_1 \neq \emptyset \neq V_2$. If either $V_1$ or $V_2$ is a whole set $A_\omega$, then $V_1 \cap V_2 = V_1$ *or* $V_2 \in \tau_\omega$. So, assume that $V_1 \neq A_\omega \neq V_2$, then $V_1 \cap V_2 \in \tau_\omega$, because $\omega \in V_1$ and $\omega \in V_2$. Hence $\omega \in V_1 \cap V_2$. Also for any element $(a \neq \omega) \in V_1 \cap V_2$, we have $a \in V_1$ and $a \in V_2$, then $a$ and the multiplicative inverse of $a$ must belong to $V_1$ and $V_2$. Hence $a$ and the multiplicative inverse of $a$ belong to $V_1 \cap V_2$, then $V_1 \cap V_2 \in \tau_\omega$. For the third condition, let $S_\gamma \in \tau_\omega$ for any $\gamma \in I$. If $S_\gamma = \emptyset$ for all $\gamma \in I$, then $\bigcup_{\gamma \in I} S_\gamma = \emptyset \in \tau_\omega$. So, assume that some member is non-empty, but since the empty set does not affect any union, assume that, without loss of generality $S_\gamma \neq \emptyset$ for all $\gamma \in I$. If there exist a $\gamma_1 \in I$ where $S_{\gamma_1} = A_\omega$, then $\bigcup_{\gamma \in I} S_\gamma = A_\omega \in \tau_\omega$. So, assume now that $S_\gamma \neq A_\omega$ for all $\gamma \in I$, then $\bigcup_{\gamma \in I} S_\gamma \in \tau_\omega$, because $\omega \in S_\gamma$ for all $\gamma \in I$. Hence $\omega \in \bigcup_{\gamma \in I} S_\gamma$. Also for any $(a \neq \omega) \in \bigcup_{\gamma \in I} S_\gamma$ , there exists $\gamma_a \in I$ such that $a \in S_{\gamma_a}$, hence $a$ and the multiplicative inverse of $a$ belong to $S_{\gamma_a}$, then $a$ and the multiplicative inverse of $a$ belong to $\bigcup_{\gamma \in I} S_\gamma$. Hence $\bigcup_{\gamma \in I} S_\gamma \in \tau_\omega$. Therefore, $(A_\omega, \tau_\omega)$ is topological space.

**Corollary 1.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then $A_\omega$ is the only open set in $(A_\omega, \tau_\omega)$ containing $a$.*

Let us denoted for the multiplicative inverse of $a \in A_\omega$ by $a^{-1}$. If $(A_\omega \setminus \{\omega\}, \otimes)$ is a group, where $\omega$ and $e$ are the zero and multiplicative identity elements, respectively, then for any $a \in A_\omega \setminus \{\omega\}$, we have $a^{-1} \in A$ .

**Proposition 2.** *The omega topological space $(A_\omega, \tau_\omega)$ has a base*

$$\mathfrak{B} = \{A_\omega, \{\omega\}, \{\omega, a, a^{-1}\} : a \in A_\omega \setminus \{\omega\},\ \textit{which has a multiplicative inverse}\}.$$

*Proof.* For the first condition, let $B \in \mathfrak{B}$ be arbitrary, if $B = \{\omega\}$ or $A_\omega$, then $B \in \tau_\omega$ is satisfied from the definition of $\tau_\omega$. Assuming that $B = \{\omega, a, a^{-1}\}$ for any $a \in A_\omega \setminus \{\omega\}$ which has a multiplicative inverse, then $B \in \tau_\omega$, because $\omega \in B$, and the element $a$ in $B$ its multiplicative inverse exists in $B$. Hence, $\mathfrak{B} \subseteq \tau_\omega$. For the second condition, let $a \in A_\omega$ be arbitrary and let $U$ be any open neighborhood of $a$ in $A_\omega$. Then we have two cases:
Case 1: If $a = \omega$, then we have $B = \{\omega\} \in \mathfrak{B}$, where $\omega \in B \subseteq U$, because the smallest open neighborhood in $A_\omega$ containing $\omega$ is $\{\omega\}$.
Case 2: Let $a \neq \omega$
Subcase 2.1: If $a$ has no multiplicative inverse, then there exists $B = A_\omega \in \mathfrak{B}$, such that $a \in B \subseteq U$, because the smallest open neighborhood in $A_\omega$ containing $a$ is $A_\omega$.
Subcase 2.2: If $a$ has a multiplicative inverse, then there exists $B = \{\omega, a, a^{-1}\} \in \mathfrak{B}$, such that $a \in B \subseteq U$, because the smallest open neighborhood in $A_\omega$ containing $a$ is $\{\omega, a, a^{-1}\}$.
Therefore, $\mathfrak{B}$ is a base for the omega topological space $(A_\omega, \tau_\omega)$.

**Corollary 2.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group, then the omega topological space $(A_\omega, \tau_\omega)$ has a base*

$$\mathfrak{B} = \{\{\omega\}, \{\omega, a, a^{-1}\} : a \in A_\omega \setminus \{\omega\}\}.$$

**Proposition 3.** *If $A_\omega$ has a finite number of elements, which have a multiplicative inverses, then the omega topological space $(A_\omega, \tau_\omega)$ is second countable.*

*Proof.* Suppose that $a_1, a_2, \cdots, a_m$, where $m \in \mathbb{Z}^+$ are the finite number of elements in $A_\omega$, which have a multiplicative inverses. Then
$\mathfrak{B} = \{A_\omega, \{\omega\}, \{\omega, a_1, a_1^{-1}\}, \cdots, \{\omega, a_m, a_m^{-1}\}\}$ is a countable base for $(A_\omega, \tau_\omega)$.

**Proposition 4.** *The omega topological space $(A_\omega, \tau_\omega)$ is first countable.*

*Proof.* Let $a \in A_\omega$ be arbitrary. If $a = \omega$, then $\mathfrak{B}(\omega) = \{\{\omega\}\}$ is a countable local base at $\omega$. Assume that, $a \neq \omega$.
Case 1: If $a$ has a multiplicative inverse, then $\mathfrak{B}(a) = \{\{\omega, a, a^{-1}\}\}$ is a countable local base at $a$.
Case 2: If $a$ has no multiplicative inverse, then $\mathfrak{B}(a) = \{A_\omega\}$ is a countable local base at $a$. Hence for any $a \in A_\omega$, there exists a countable local base at $a$. Then $(A_\omega, \tau_\omega)$ is first countable.

**Proposition 5.** *The omega topological space $(A_\omega, \tau_\omega)$ is separable.*

*Proof.* There exist $\{\omega\} \subseteq A_\omega$, such that for any nonempty $U \in \tau_\omega$, we have $U \cap \{\omega\} \neq \emptyset$, because any nonempty open set in $(A_\omega, \tau_\omega)$ must be contaning $\omega$. Hence, $\overline{\{\omega\}} = A_\omega$ (which means $\{\omega\}$ is a dense subset of $A_\omega$). Then $A_\omega$ has a countable dense subset. Therefore, $(A_\omega, \tau_\omega)$ is separable.

Let us recall this definition.

**Definition 2.** A topological space $X$ is called *hyperconnected* if every non-empty open subset is dense in $X$.

**Proposition 6.** *The omega topological space $(A_\omega, \tau_\omega)$ is hyperconnected.*

*Proof.* If $A_\omega$ is singleton, then $(A_\omega, \tau_\omega)$ is hyperconnected. Suppose that $A_\omega$, which has more than one element. Let $U$ be arbitrary non-empty open subset of $A_\omega$, then $U$ intersects every non-empty open subset of $A_\omega$, because any non-empty open subset of $A_\omega$ contained $\omega$. Hence $U$ is dense in $A_\omega$. Since $U$ was chosen arbitrary, then every non-empty open subset of $A_\omega$ is dense. Therefore $(A_\omega, \tau_\omega)$ is hyperconnected.

Since any hyperconnected space is connected and locally connected, then we conclude the following corollaries.

**Corollary 3.** *The omega topological space $(A_\omega, \tau_\omega)$ is connected.*

**Corollary 4.** *The omega topological space $(A_\omega, \tau_\omega)$ is locally connected.*

**Proposition 7.** *The omega topological space $(A_\omega, \tau_\omega)$ is not $T_1$.*

*Proof.* If $A_\omega$ is singleton, then it is $T_1$. Assume that $A_\omega$, which has more than one element. There exists $a, \omega \in A_\omega$ such that $a \neq \omega$. Since any nonempty open set must be containing $\omega$, then we can not find two open sets $U$ and $V$ such that $a \in U, \omega \notin U, a \notin V$ and $\omega \in V$. Therefore, $(A_\omega, \tau_\omega)$ is not $T_1$.

**Proposition 8.** *Let $(A_\omega \setminus \{\omega\}, \otimes)$ be a group, which has more than two elements. Then the omega topological space $(A_\omega, \tau_\omega)$ is not $T_0$.*

*Proof.* If $A_\omega$ has only two elements, then it is $T_0$. Assume that $A_\omega$, which has more than two elements. If $a \in A_\omega$ is arbitrary, such that $a \neq \omega$ and $a \neq e$, then $a \neq a^{-1}$ in $A_\omega$ and any open set containing $a$ must be containing $a^{-1}$. Hence, $(A_\omega, \tau_\omega)$ is not $T_0$.

**Proposition 9.** *If $A_\omega$, has more than one element, then the omega topological space $(A_\omega, \tau_\omega)$ is not regular.*

*Proof.* If $A_\omega$ is singleton then it is regular. There exists $K = A_\omega \setminus \{e, \omega\}$ be a closed subset of $A_\omega$ and $\omega \notin K$. We can not separated $\omega$ and $K$ by any open sets (because any open set in $A_\omega$ is containing $\omega$). Hence, $(A_\omega, \tau_\omega)$ is not regular.

**Proposition 10.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group, which has more than two elements, then the omega topological space $(A_\omega, \tau_\omega)$ is not normal.*

*Proof.* If $A_\omega = \{e = \omega\}$ *or* $\{e, \omega\}$, then it is normal. Assume that $A_\omega$ has more than two elements, then there exists $a \in A_\omega$ such that $a \neq e$, $a \neq \omega$ and its multiplicative inverse exists in $A_\omega$. Then, there exists $K = \{a, a^{-1}\}$ and $H = \{e\}$ are two disjoint nonempty closed subsets of $A_\omega$, such that we can not separat them by any open sets (because any open set in $A_\omega$ is containing $\omega$). Hence, $(A_\omega, \tau_\omega)$ is not normal.

**Proposition 11.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group and $A$ is uncountable infinite set, then the omega topological space $(A_\omega, \tau_\omega)$ is not compact (Lindelöf).*

*Proof.* There exists $\{\{\omega\}, \{\omega, a, a^{-1}\} : a \in A_\omega \setminus \{\omega\}\}$, which s an open cover of $A_\omega$, and has no finite (countable) subcover of $A_\omega$.

**Proposition 12.** *Let $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse. Then the omega topological space $(A_\omega, \tau_\omega)$ is compact.*

*Proof.* Let $\{C_\alpha : \alpha \in \Lambda\}$ be any open cover of $A_\omega$. Since $a \in A_\omega$, then for some $\beta \in \Lambda$, there exists $C_\beta$ containing $a$. But $C_\beta = A_\omega$, because $A_\omega$ is the only open set containing $a$. Hence, $\{C_\beta\}$ is a finite subcover of $\{C_\alpha : \alpha \in \Lambda\}$, which cover $A_\omega$. Therefore, $(A_\omega, \tau_\omega)$ is a compact space.

Since any compact space is Lindelöf and countably compact, then we conclude the following corollaries.

**Corollary 5.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then the omega topological space $(A_\omega, \tau_\omega)$ is Lindelöf.*

**Corollary 6.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then the omega topological space $(A_\omega, \tau_\omega)$ is countably compact.*

**Remark 3.** Since every nonempty open set in $(A_\omega, \tau_\omega)$ containing $\omega$, then the closure of any nonempty open set is equal $A_\omega$.

## 4. Some of the fundamental properties for different examples on omega topology

In this section, we give four different examples of omega topologies. The first and fourth examples are from an ordered infinite set, the second is from a cyclically ordered infinite set, and the third is from a finite set. Furthermore, we define a new topology over semiring in conventional algebra.

**Example 4.** By Example 1, we have $(W_{-\infty}, \tau_{-\infty})$ which is a topological space, where $W = \{0, 1, 2, 3, \cdots\}$. If $a \in W \setminus \{0\}$ be arbitrary, then $a^{-1}$ does not exists in $(W_{-\infty} \setminus \{-\infty\}, \otimes)$, where $a^{-1}$ is the multiplicative inverse of $a$. Hence, we have

$$\tau_{-\infty} = \{W_{-\infty}, \emptyset, \{-\infty\}, \{-\infty, 0\}\}.$$

A direct check shows that $(W_{-\infty}, \tau_{-\infty})$ is a topological space.

**Proposition 13.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is second countable.*

*Proof.* There exists an element $2 \in W_{-\infty} \setminus \{-\infty\}$, which has no multiplicative inverse, then by Proposition 3, $(W_{-\infty}, \tau_{-\infty})$ is second countable.

**Corollary 7.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is first countable.*

**Corollary 8.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is separable.*

**Proposition 14.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is not $T_0$.*

*Proof.* There exists $(2 \neq 3)$ in $W_{-\infty}$. Let $U$ be any open set, which either contains 2 or 3. However, there exists only one open set $U = W_{-\infty}$ containing 2 and 3. Hence, $(W_{-\infty}, \tau_{-\infty})$ is not $T_0$.

**Proposition 15.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is not regular.*

*Proof.* We have a closed set $C = W_{-\infty} \setminus \{-\infty, 0\}$ and $0 \notin C$, such that for any open sets $V_1$ and $V_2$ containing 0 and $C$ ,respectively, we have $V_1 \cap V_2 \neq \emptyset$. Hence, $(W_{-\infty}, \tau_{-\infty})$ is not regular.

**Proposition 16.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is normal.*

*Proof.* Let $K_1$ and $K_2$ be any two closed sets, where $K_1 \cap K_2 = \emptyset$ and $K_1, K_2 \subseteq W_{-\infty}$. Since all closed subsets of $W_{-\infty}$ are $W_{-\infty}, \emptyset, W_{-\infty} \setminus \{-\infty\}$ and $W_{-\infty} \setminus \{-\infty, 0\}$, then $K_1$ or $K_2$ is equal $\emptyset$. If $K_1 = \emptyset$, then there exists $U_1 = \emptyset$ and $U_2 = W_{-\infty}$ are open sets and $U_1 \cap U_2 = \emptyset$ in $W_{-\infty}$, where $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$. If $K_2 = \emptyset$, then there exists $U_1 = \emptyset$ and $U_2 = W_{-\infty}$ are open sets and $U_1 \cap U_2 = \emptyset$ in $W_{-\infty}$, where $K_2 \subseteq U_1$ and $K_1 \subseteq U_2$. Hence, $(W_{-\infty}, \tau_{-\infty})$ is normal.

**Proposition 17.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is hyperconnected.*

*Proof.* Using the same proof of Proposition 6.

**Corollary 9.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is connected.*

**Corollary 10.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is locally connected.*

**Proposition 18.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is compact.*

*Proof.* There exists an element $2 \in W_{-\infty} \setminus \{-\infty\}$, which has no multiplicative inverse. Hence, by Proposition 12, $(W_{-\infty}, \tau_{-\infty})$ is compact.

**Corollary 11.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is countably compact.*

**Corollary 12.** *The omega topological space $(W_{-\infty}, \tau_{-\infty})$ is Lindelöf.*

**Example 5.** By Example 2, $(A_{\rho_0}, \tau_{\rho_0})$ is a topological space. If $\rho_x \in A_{\rho_0} \setminus \{\rho_0\}$ is arbitrary, then $\rho_x^{-1}$ does not exists in $(A_{\rho_0} \setminus \{\rho_0\}, \otimes)$, where $\rho_x^{-1}$ is the multiplicative inverse of $\rho_x$ (because $\rho_x \otimes \rho_x^{-1} = \rho_x \otimes \rho_{x^{-1}} = \rho_{x+x^{-1}} = \rho_{x+(-x)} = \rho_0$ and $-x \notin W$). Then we have

$$\tau_{\rho_0} = \{A_{\rho_0}, \emptyset, \{\rho_0\}\}.$$

A direct check shows that $(A_{\rho_0}, \tau_{\rho_0})$ is a topological space.

**Remark 4.** The omega topological space $(A_{\rho_0}, \tau_{\rho_0})$ is second countable, first countable, separable, normal, hyperconnected, connected, locally connected, compact, countably compact, Lindelöf, does not satisfy $T_0$ and regular.

**Proposition 19.** *The omega topological space $(A_{\rho_0}, \tau_{\rho_0})$ is not homeomorphic to $(W_{-\infty}, \tau_{-\infty})$.*

*Proof.* There exists an open set in $W_{-\infty}$, which consists two elements, and such that an open set does not exists in $A_{\rho_0}$.

**Example 6.** By Example 3, $(A_{11}, \tau_{11})$ is a topological space. Let $a \in A_{11} \setminus \{11\}$ be arbitrary. If $a = 00$, then the multiplicative inverse of $a$ in $A_{11}$ is 00. If $a = 01$, then the multiplicative inverse of $a$ in $A_{11}$ does not exists. Then we have

$$\tau_{11} = \{A_{11}, \emptyset, \{11\}, \{11, 00\}\}.$$

A direct check shows that $(A_{11}, \tau_{11})$ is a topological space.

**Proposition 20.** *The omega topological space* $(A_{11}, \tau_{11})$ *is* $T_0$ *and does not satisfy* $T_1$*.*

*Proof.* If the space $A_{11}$ consists of three elements $00, 01$ and $11$ then we have three cases:
Case 1: If $00 \neq 01$ in $A_{11}$, then we have $\{00, 11\}$ an open set, where $00 \in \{00, 11\}$ and $01 \notin \{00, 11\}$.
Case 2: If $00 \neq 11$ in $A_{11}$, then we have $\{11\}$ an open set, where $11 \in \{11\}$ and $00 \notin \{11\}$.
Case 3: If $01 \neq 11$ in $A_{11}$, then we have $\{11\}$ an open set, where $11 \in \{11\}$ and $01 \notin \{11\}$.
Hence, $(A_{11}, \tau_{11})$ is $T_0$.
Suppose that $(A_{11}, \tau_{11})$ is $T_1$, then $\{11\}$ is closed. However, $A_{11} \setminus \{11\} = \{00, 01\}$ is not open, thus a contradiction. Then $(A_{11}, \tau_{11})$ is not $T_1$.

**Remark 5.** The omega topological space $(A_{11}, \tau_{11})$ is second countable, first countable, separable, not regular, normal, hyperconnected, connected, locally connected, compact, countably compact and Lindelöf.

**Example 7.** In the ring $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}, +)$ is an additive submonoid of an abelian group $(\mathbb{R}, +)$. Let $\omega = -\infty$, $a_1 \oplus a_2 = \max(a_1, a_2)$ and $a_1 \otimes a_2 = a_1 + a_2, \forall a_1, a_2 \in \mathbb{R}$. Then, $\mathbb{R}_{-\infty} = (\mathbb{R}_{-\infty}, \oplus, \otimes, -\infty, 0)$ is $-\infty -$ algebra over the ring $(\mathbb{R}, +, \cdot)$. Then, using the same proof as that of Proposition 1 $(\mathbb{R}_{-\infty}, \tau_{-\infty})$ is a topological space.

**Remark 6.** The omega topological space $(\mathbb{R}_{-\infty}, \tau_{-\infty})$ is first countable, separable, hyperconnected, connected and locally connected and does not satisfy any of these $T_0$; regular and normal.

**Example 8.** In the ring $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}, +)$ is an additive submonoid of an abelian group $(\mathbb{R}, +)$. Let $\omega = +\infty$, $a_1 \oplus a_2 = \min(a_1, a_2)$ and $a_1 \otimes a_2 = a_1 + a_2, \forall a_1, a_2 \in \mathbb{R}$. Then, $\mathbb{R}_{+\infty} = (\mathbb{R}_{+\infty}, \oplus, \otimes, +\infty, 0)$ is $+\infty -$ algebra over the ring $(\mathbb{R}, +, \cdot)$. Then, using the same proof as that of Proposition 1 $(\mathbb{R}_{+\infty}, \tau_{+\infty})$ is a topological space.

**Proposition 21.** *The omega topological spaces* $(\mathbb{R}_{-\infty}, \tau_{-\infty})$ *and* $(\mathbb{R}_{+\infty}, \tau_{+\infty})$ *are homeomorphic, where* $\mathbb{R}_{-\infty}$ *is a max−plus algebra and* $\mathbb{R}_{+\infty}$ *is a min−plus algebra. These are special cases of omega algebra.*

*Proof.* We have a map $h : (\mathbb{R}_{-\infty}, \tau_{-\infty}) \to (\mathbb{R}_{+\infty}, \tau_{+\infty})$ is defined by:

$$h(x_1) = \begin{cases} x_1 & if \quad x_1 \in \mathbb{R} \\ +\infty & if \quad x_1 = -\infty \end{cases} ;$$

Let $x_1, x_2 \in \mathbb{R}_{-\infty}$ be arbitrary. Let $h(x_1) = h(x_2)$, then $x_1 = x_2$. Hence, $h$ is an injective.
If $x_1 \in \mathbb{R}_{+\infty}$ is arbitrary, then we have two cases:
Case 1: If $x_1 \neq +\infty$, then there exists $x_1 \in \mathbb{R}_{-\infty} \setminus \{-\infty\}$, such that $h(x_1) = x_1$.
Case 2: If $x_1 = +\infty$, then there exists $x_1 = -\infty \in \mathbb{R}_{-\infty}$, such that $h(-\infty) = +\infty$. Hence, $h$ is surjective.
Let $B \in \tau_{+\infty}$ be any basic open set. Since $(\mathbb{R}_{-\infty} \setminus \{-\infty\}, \otimes)$ and $(\mathbb{R}_{+\infty} \setminus \{+\infty\}, \otimes)$ are groups, then by Problem 2, we have $\mathfrak{B} = \{\{-\infty\}, \{-\infty, c, c^{-1}\} : c \in \mathbb{R}\}$ and

$\mathfrak{B} = \{\{+\infty\}, \{+\infty, c, c^{-1}\} : c \in \mathbb{R}\}$ are a base for $\mathbb{R}_{-\infty}$ and $\mathbb{R}_{+\infty}$, respectively.
To prove that $h$ is continuous, we have two cases:
Case 1: If $B = \{+\infty\}$, then $h^{-1}(B) = h^{-1}(\{+\infty\}) = \{-\infty\} \in \tau_{-\infty}$.
Case 2: If $B = \{+\infty, c, c^{-1}\}$, then $h^{-1}(B) = h^{-1}(\{+\infty, c, c^{-1}\}) = \{-\infty, c, c^{-1}\} \in \tau_{-\infty}$.
Hence, $h$ is continuous.
To prove that $h^{-1}$ is continuous, we have two cases: (since $h$ is one to one and onto, then $(h^{-1})^{-1}(B) = h(B)$).
Case 1: If $B = \{-\infty\}$, then $(h^{-1})^{-1}(B) = h(B) = h(\{-\infty\}) = \{+\infty\} \in \tau_{+\infty}$.
Case 2: If $B = \{-\infty, c, c^{-1}\}$, then $(h^{-1})^{-1}(B) = h(B) = h(\{-\infty, c, c^{-1}\}) = \{+\infty, c, c^{-1}\} \in \tau_{+\infty}$. Hence, $h^{-1}$ is continuous (which means $h$ is open).
In conclusion, if $h$ is homeomorphism, then $(\mathbb{R}_{-\infty}, \tau_{-\infty})$ and $(\mathbb{R}_{+\infty}, \tau_{+\infty})$ are homeomorphic.

**Theorem 2.** *Let $X$ be any semiring in conventional algebra, such that $e$ is the zero element. We define a topology on $X$ is called zero element topology, as follows:*

$\tau_e = \{\emptyset, X\} \cup \{U \subseteq X : e \in U$ *and for any* $a \in U \setminus \{e\}$, *the multiplicative inverse of* $a$ *exists in* $U\}$.

*Then $(X, \tau_e)$ is a topological space.*

*Proof.* Condition $\emptyset, X \in \tau_e$ is satisfied from the definition of $\tau_e$. Now let $U_1, U_2 \in \tau_e$ be arbitrary. If either $U_1 = \emptyset$ or $U_2 = \emptyset$, then $U_1 \cap U_2 = \emptyset \in \tau_e$. Assume, $U_1 \neq \emptyset$ and $U_2 \neq \emptyset$. If either $U_1 = X$ or $U_2 = X$, then $U_1 \cap U_2 = U_1 \text{ or } U_2 \in \tau_e$. So assume, $U_1 \neq X$ and $U_2 \neq X$, then $U_1 \cap U_2 \in \tau_e$, because $e \in U_1$ and $e \in U_2$. Hence $e \in U_1 \cap U_2$, also for any element $(a \neq e) \in U_1 \cap U_2$, if $a \in U_1$ and $a \in U_2$, then $a$ and the multiplicative inverse of $a$ must belong to $U_1$ and $U_2$. Hence $a$ and the multiplicative inverse of $a$ belong to $U_1 \cap U_2$, then $U_1 \cap U_2 \in \tau_e$. For the third condition, let $S_\gamma \in \tau_e$ for any $\gamma \in I$. If $S_\gamma = \emptyset$ for all $\gamma \in I$, then $\bigcup_{\gamma \in I} S_\gamma = \emptyset \in \tau_e$. So, assume that some member is non-empty. However, since the empty set does not affect any union, assume that, without loss of generality $S_\gamma \neq \emptyset$ for all $\gamma \in I$. If there exists a $\gamma_1 \in I$ such that $S_{\gamma_1} = X$, then $\bigcup_{\gamma \in I} S_{\gamma_1} = X \in \tau_e$. So, assume now that $S_\gamma \neq X$ for all $\gamma \in I$, then $\bigcup_{\gamma \in I} S_\gamma \in \tau_e$, because $e \in S_\gamma$ for all $\gamma \in I$. Hence $e \in \bigcup_{\gamma \in I} S_\gamma$. Also, for any $(a \neq e) \in \bigcup_{\gamma \in I} S_\gamma$ , there exists $\gamma_a \in I$ such that $a \in S_{\gamma_a}$. Hence $a$ and the multiplicative inverse of $a$ belong to $S_{\gamma_a}$. Furthermore, $a$ and the multiplicative inverse of $a$ belong to $\bigcup_{\gamma \in I} S_\gamma$. Hence $\bigcup_{\gamma \in I} S_\gamma \in \tau_e$. Therefore, $(X, \tau_e)$ is topological space.

**Example 9.** The space $(\mathbb{R}, +, \cdot)$ is a ring, where 0 and 1 are the zero and unit elements, respectively. Then $(\mathbb{R}, \tau_0)$ is a topological space. The proof is similar to Theorem 2; just replacing $X$, $\tau_e$ and $e$ by $\mathbb{R}$, $\tau_0$ and 0, respectively.

**Remark 7.** The topological space $(\mathbb{R}, \tau_0)$ is first countable, separable, hyperconnected, connected and locally connected, and does not satisfy any of these $T_0$, regular, normal, compact and Lindelöf.

## 5. Omega topology and other properties

Recall that, a subset $A$ of a space $X$ is said to be *regularly-open* or an *open domain* if it is the interior of its own closure, see [7]. A set $A$ is said to be a *regularly-closed* or a *closed domain* if its complement is an open domain. A subset $A$ of a space $X$ is called a $\pi$*-closed* if it is a finite intersection of closed domain sets, see [17]. A subset $A$ is called a $\pi$*-open* if its complement is a $\pi$-closed. If $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a set $X$ such that $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T}'$ is called the coarser topology than $\mathcal{T}$ and $\mathcal{T}$ is called the finer. A space $X$ is said to be a $\pi$*-normal*, [3], if any pair of disjoint closed subsets $A$ and $B$ of $X$, one of which is $\pi$-closed, can be separated by two disjoint open subsets. A space $X$ is said to be a *almost-normal*, [3], if any pair of disjoint closed subsets $A$ and $B$ of $X$, one of which is closed domain, can be separated by two disjoint open subsets. A space $X$ is said to be a *mildly normal*, [15], if any pair of disjoint closed domain subsets $A$ and $B$ of $X$ can be separated by two disjoint open subsets. A space $(X, \mathcal{T})$ is said to be a *epi-normal*, [5], if there exists a coarser topology $\mathcal{T}'$ on $X$ such that $(X, \mathcal{T}')$ is $T_4$- space (normal and $T_1$-space). A space $(X, \mathcal{T})$ is said to be a *epi-mildly normal*, [9], if there exists a coarser topology $\mathcal{T}'$ on $X$ such that $(X, \mathcal{T}')$ is $T_2$ and almost normal space.

**Theorem 3.** *If $(A_\omega \backslash \{\omega\}, \otimes)$ be a group has more than one element, then omega topological space $(A_\omega, \tau_\omega)$ is $\pi$-normal.*

*Proof.* Since the only $\pi$-closed sets are the ground set $A_\omega$ and the empty set, then $(A_\omega, \tau_\omega)$ is a $\pi$-normal.

It is clear from the definitions that

$$normal \Rightarrow \pi - normal \Rightarrow almost\ normal \Rightarrow mildly\ normal. \tag{1}$$

By (1) and Theorem 3, we conclude the following Corollaries.

**Corollary 13.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group has more than one element, then omega topological space $(A_\omega, \tau_\omega)$ is almost normal.*

**Corollary 14.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group has more than one element, then omega topological space $(A_\omega, \tau_\omega)$ is mildly normal.*

**Proposition 22.** *Any omega topological space $(A_\omega, \tau_\omega)$ is not Epi-mildly Normal.*

*Proof.* Suppose that, $(A_\omega, \tau_\omega)$ is Epi-mildly Normal. Then there exists a coarser topology $\mathcal{T}'$ on $A_\omega$ such that $(A_\omega, \mathcal{T}')$ is $T_2$ and mildly normal space. Hence $(A_\omega, \tau_\omega)$ is $T_2$, thus a contradiction, because $(A_\omega, \tau_\omega)$ is not $T_1$ (see Proposition 7). Then $(A_\omega, \tau_\omega)$ is not Epi-mildly Normal.

**Proposition 23.** *Any omega topological space $(A_\omega, \tau_\omega)$ is not Epi-almost Normal.*

*Proof.* Using the same proof of Proposition 22.

**Definition 3.** Let $X$ be a space. Then:

1) A space $X$ is called a *$C$-normal* if there exist a normal space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$, [2].

2) A space $X$ is called a *$CC$-normal* if there exist a normal space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each countably compact subspace $A \subseteq X$.[4].

3) A space $X$ is called a *$L$-normal* if there exist a normal space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each lindelöf subspace $A \subseteq X$.[6].

4) A space $X$ is called a *$S$- normal* if there exist a normal space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each seprable subspace $A \subseteq X$.[1].

5) A space $X$ is called a *$C$-paracompact ($C_2$-paracompact) if there exist a paracompact (Hausdorff paracompact) space $Y$ and a bijective function $f : X \to Y$ such that the restriction function $f|_A : A \to f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.[13].*

**Theorem 4.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then omega topological space $(A_\omega, \tau_\omega)$ is $C$-normal.*

*Proof.* Let $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse. Let $V$ be any non-empty closed subset of $A_\omega$. Then $a \in V$. Suppose not, $a \notin V$, then $a \in A_\omega \setminus V$. By the definition of $\tau_\omega$, $A_\omega \setminus V$ is not open, thus a contradiction. Hence, $a$ belong to any non-empty closed subsets of $A_\omega$. Let $K$ and $H$ be any two disjoint closed subsets of $A_\omega$. Then $K$ or $H$ is equal $\emptyset$. If $K = \emptyset$, then there exists $U = \emptyset$ and $V = A_\omega$ are two disjoint open sets in $A_\omega$ containing $K$ and $H$, respectively. If $H = \emptyset$, then there exists $U = \emptyset$ and $V = A_\omega$ are two disjoint open sets in $A_\omega$ containing $H$ and $K$, respectively. Therefore, $(A_\omega, \tau_\omega)$ is normal. Then there exist $Y = A_\omega$ is a normal space and the identity function $id : A_\omega \to A_\omega$ is bijective. Let $C$ be any compact subset of $(A_\omega, \tau_\omega)$. Then the restriction function $id\restriction_C : C \to f(C)$ is a homeomorphism. Therefore, $(A_\omega, \tau_\omega)$ is a $C-$normal.

Since any normal space is $CC$-normal, $L$-normal and $S$-normal, just by taking $X = Y$ and $f$ to be the identity function. Hence, we conclude the following Theorems.

**Theorem 5.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then omega topological space $(A_\omega, \tau_\omega)$ is $CC$-normal.*

*Proof.* Using the same proof of Theorem 4.

**Theorem 6.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then omega topological space $(A_\omega, \tau_\omega)$ is $L$-normal.*

*Proof.* Using the same proof of Theorem 4.

**Theorem 7.** *If $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse, then omega topological space $(A_\omega, \tau_\omega)$ is $S$-normal.*

*Proof.* Using the same proof of Theorem 4.

**Example 10.** By Example 4, $(A_{-\infty}, \tau_{-\infty})$ is $C$-normal, $CC$-normal, $L$-normal and $S$-normal.

**Theorem 8.** *If $(A_\omega \setminus \{\omega\}, \otimes)$ be a group has more than one element, then omega topological space $(A_\omega, \tau_\omega)$ is not $S$-normal.*

*Proof.* From the proposition any separable $S$-normal must be normal (see [1]) and since $(A_\omega, \tau_\omega)$ is separable and not normal (see Proposition 5 and Proposition 10, repectively), then $(A_\omega, \tau_\omega)$ is not $S$-normal.

**Example 11.** By Example 7, $(\mathbb{R}_{-\infty}, \tau_{-\infty})$ is not a $S$-normal.

**Theorem 9.** Every omega topological space $(A_\omega, \tau_\omega)$ is not $C_2$-paracompact.

*Proof.* Since any $C_2$-paracompact Fréchet space is Housdorff, see [13], and $(A_\omega, \tau_\omega)$ is First countable not Housdorff space, then $(A_\omega, \tau_\omega)$ can not be $C_2$-paracompact.

**Theorem 10.** Let $a \in A_\omega \setminus \{\omega\}$ has no multiplicative inverse. Then omega topological space $(A_\omega, \tau_\omega)$ is not $C$-paracompact.

*Proof.* Assume that $(A_\omega, \tau_\omega)$ is $C$-paracompact. Let $Y$ be a paracompact space and $f : A_\omega \to Y$ be bijective such that the restriction $f \restriction_C : C \to f(C)$ is a homeomorphism for all compact subspace $C$ of $(A_\omega, \tau_\omega)$. Hence, $A_\omega \equiv Y$, since $A_\omega$ is compact (see Proposition 12). However, $A_\omega$ is paracompact, thus a contradiction. Because any paracompact space is Hausdorff space and $A_\omega$ is not a Hausdorff space. Therefore, $(A_\omega, \tau_\omega)$ is not a $C$-paracompact.

## References


[1] M. Alhomieyed and L. Kalantan. $S$-Normality. *Journal of Mathematical Analysis*, 9(3):48–54, 2018.

[2] S. Alzahrani and L. Kalantan. $c$-normal topological property. *Filomat*, 31:2:407–411, 2017.

[3] L. Kalantan. $\pi$-normal topological spaces. *Filomat*, 22-1:173–181, 2008.

[4] L. Kalantan and M. Alhomieyed. $CC$-normal topological spaces. *Turk. J. Math.*, 41:749–755, 2017.

[5] L. Kalantan and S. Alzahrani. Epinormality. *J. nonlinear Sci. Appl.*, 9:5398–5402, 2016.

[6] L. Kalantan and M. Saeed. $l$-normality. *Topology Proceedings*, 50:141–149, 2017.

[7] C. Kuratowski. *Topology I.* Hafner, New yor, 1958.

[8] G. L. Litvinov. The maslov dequantization idempotent and tropical mathematics. *Journal of Mathematical Sciences*, 3:426–444, 2007.

[9] L.Kalantan and I. Alshammari. Epi-mild normality. *open Mat. J.*, 16:1170–1175, 2018.

[10] D. Maclagan and B. Sturmfels. *Introduction to Tropical Geometry, Graduate Studies in Mathematics.*, volume 161. American Mathematical Society, 2015.

[11] S. Khalid Nauman, C. Ozel, and H. Zekraoui. Abstract omega algebra that subsumes min and max plus algebras. *Turkish Journal of Mathematics and Computer*, 11(Special Issue):1–10, 2019.

[12] J.-E. Pin. *Tropical semirings, Idempotency.*, volume 11. Cambridge Univ, Cambridge, 1998.

[13] Maha Mohammed Saeed, Lutfi Kalantan, and Hala Alzumi. $C$-Paracompactness and $C_2$ -paracompactness. *Turk. J. Math.*, 43:9–20, 2019.

[14] I. Simon. *Recognizable sets with multiplicities in the tropical semiring.*, volume 324. Springer, Berlin, 1988.

[15] M.K. Singal and A.R. Singal. Mildly normal spaces. *Kyungpook Math J.*, 13:27–31, 1973.

[16] S. Świerczkowski. On cyclically ordered groups. *Fundamenta Mathematicae*, 47:161–166, 1959.

[17] V. Zaitsev. On certain classes of topological spaces and their bicompactifications. *Dokl. Akad. Nauk SSSR*, 178:778–779, 1968.