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Some results on blow-up phenomenon for nonlinear porous medium equations with weighted source

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Abstract. This paper deals with the blow-up phenomena for a type of nonlinear porous medium equations with weighted source $u_t - \triangle u^m = a(x)f(u)$ subject to Dirichlet (or Neumann) boundary conditions. Based on the auxiliary functions and differential-integral inequalities, the blow-up criterions which ensure that u cannot exist all time are given under two different assumptions, and the corresponding estimates on the upper bounds for blow-up time and blow-up rate are derived respectively. Moreover, we use three different methods to determine the lower bounds for blow-up time and blow-up rate estimates if blow-up does occurs.

2020 Mathematics Subject Classifications: 35A01, 35B44, 35K20, 35K61

Key Words and Phrases: Porous medium equation, Upper and lower bounds, Blow-up rate, Weighted source

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1. Introduction

In this paper, we deal with the blow-up time and blow-up rate estimates of the solutions to the following problem:

$$u_t - \triangle u^m = a(x)f(u), \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

$$u(x,t) = 0 \ \text{ or } \ \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

$$u(x,0) = g(x) \geq 0, \quad x \in \Omega, \tag{1.3}$$

where $m > 1$ and $\Omega \subset R^n$ $(n \geq 3)$ is a smooth bounded domain, ν is the outward normal vector, $g(x)$ is a continuous nonnegative function and satisfies the compatible condition. Here, the nonlinear function f satisfies

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(f_1): $f(s) \geq 0$ for all $s \geq 0$;
And the weighted function $a(x) \in C^1(\Omega) \cap C^0(\bar{\Omega})$ satisfies
(a_1): $a(x) \geq C > 0$ on $\bar{\Omega}$ for some constant C or
(a_2): $a(x) > 0$ in Ω and $a(x) = 0$ on $\partial\Omega$.

It is well known that the porous medium equations have extensive physical background and rich theoretical connotation. They have been used to model the processes involving the chemical reaction, heat transfer or diffusion, population dynamics and so on. We refer readers to see [1, 26] and references therein, where a series of physical application of Eq.(1.1) are also summarized. For instance, the nonlinear term $f(u)$ of Eq.(1.1) describes the nonlinear source in the diffusion phenomena, and it is called to be "heat source". If the "heat source" occurs, the solutions of Eq.(1.1) might be unbounded at finite time, namely, the solutions might be blowing up in finite time.

The Eq.(1.1) includes many important physical models. If the exponent $m = 1$ and weighted function $a(x) \equiv 1$, the model (1.1) reduces to the semilinear heat equations

$$u_t - \triangle u = f(u), \quad x \in \Omega, \quad t > 0. \tag{1.4}$$

About this model, many results about the blow-up phenomenon of the solutions have been obtained, we refer to see [13, 16–19, 23, 25] and references therein. In [18, 19], Payne and Schaefer obtained a lower bound on blow-up time of the solutions to the Eq.(1.4) under null Dirichlet boundary condition and homogeneous Neumann boundary condition, respectively. Later, Payne et al. [16, 17] studied the blow-up phenomenon of the solutions for Eq.(1.4) with nonlinear boundary conditions. When the nonlinear source term $f(u) = \int_\Omega u^q dx - ku^s$, Song [23] obtained the lower bounds for blow-up time of the solutions with either homogeneous Dirichlet or homogeneous Neumann boundary conditions in three dimensional space. Afterwards, Liu [13] studied the lower bounds for blow-up time under nonlinear boundary conditions in three dimensional space. In [25], Tang et al. extended the results of literature [13] in higher dimensional space.

When the exponent $m \neq 1$ and weighted function $a(x) \equiv 1$, the model (1.1) becomes the following porous medium equations

$$u_t - \triangle u^m = f(u), \quad x \in \Omega, \quad t > 0. \tag{1.5}$$

This type of equations appears in several branches of applied mathematics [10, 12]. There is large body of literature on the study of the Eq.(1.5), such as the existence and uniqueness in [4, 5, 9, 11, 26], blow-up in [5, 7–10, 26], asymptotic behavior in [3, 20, 21, 26] and other interesting results in [2, 4, 12, 26] and references therein. For instance, in the case of nonlinear source $f(u) = u^p$, Galaktionov et al. [5] obtained the finite time blow-up of the solutions for $1 < p < m + \frac{2}{n}$, and proved the global existence of the solutions for $p > m + \frac{2}{n}$. For the critical case, Galaktionov and Levine [6], Kawanago [9] revealed that all nonnegative nontrivial mild solutions blow up in finite time. Jiang, Zheng and Song [8] gave some sufficient or necessary blow-up conditions, and the blow-up rate estimates, where $f(u) = u^p$, $m > 1, p > 1$.

If the exponent $m = 1$ and weighted function $a(x) \not\equiv 1$, the model (1.1) reduces to the semilinear parabolic equations with weighted source

$$u_t - \triangle u = a(x)f(u), \quad x \in \Omega, \quad t > 0. \tag{1.6}$$

Recently, the studying on the blow-up phenomenon had some new development, where more attention was paid on the parabolic equations with weighted source. These models can be used to illustrate the processes of heat transfer arising in physical and engineering applications, such as a model of phase separation in binary alloys [22]. The existence and nonexistence of global solutions, bounds for blow-up time, blow-up rate, blow-up sets and asymptotic behavior for this type of equations were investigated by many authors. We refer the reader to see [14, 15, 24] and papers cited therein. For example, Song and Lv [14, 24] studied the initial boundary value problem for the above equations with nonlinear Neumann boundary condition, and they derived the upper and lower bounds for blow-up time in three dimensional space [14]. In [24], they further investigated the estimates of blow-up rate and the bounds for blow-up time in higher dimensional space. Ma and Fang [15] changed the diffusion term $\triangle u$ into $\sum_{i,j=1}^{N} \left(a^{i,j}(x)u_{x_i}\right)_{x_j}$ in Eq.(1.6), where the upper and lower bounds for the blow-up time were derived in higher dimensional space.

In the present work, we main study the blow-up phenomena for the porous medium equations with weighted nonlinear source. As far as we known, there is little information on the blow-up results of the solutions for problem (1.1)-(1.3). Obviously, the existence and uniqueness of local solutions for this problem can be obtained by applying the classical Faedo-Galerkin method or Contraction Mapping Principle. Naturally, we would like to study the estimates of blow-up rate and the bounds for blow-up time of the solutions in any smooth bounded domain $\Omega \subset R^n$ $(n \geq 3)$. Here, the appearance of the diffusion term $\triangle u^m$ and weighted nonlinear source $a(x)f(u)$ cause some difficulties in dealing with the qualitative properties of problem (1.1)-(1.3). Hence, we shall use some modified auxiliary functions and differential-integral inequality skills to over these difficulties.

In detail, this paper is organized as follows: the blow-up criterions are given under two different assumptions, and the corresponding estimates on the upper bounds for blow-up time and blow-up rate are derived in Subsection 2.1 and 2.2. In section 3, we will use three methods to give the lower bounds for blow-up time and blow-up rate of the solutions if the blow-up does occurs.

2. Upper estimates for blow-up time and blow-up rate

The purpose of this section is to establish some estimates about the upper bounds for blow-up time and blow-up rate of the solutions to problem (1.1)-(1.3) under two different assumptions, respectively.

2.1. The first method

To obtain the results of this subsection, we first assume that

(f_2): there exists a positive constant $C_1 > 2$ such that

$$\int_\Omega a(x)s^m f(s)dx \geq C_1 \int_\Omega a(x)F(s)dx,$$

for any function $s(x) \geq 0$, where $F(s) = m\int_0^s \theta^{m-1}f(\theta)d\theta$;

(g_1): the initial data $g(x)$ satisfies

$$\int_\Omega |\nabla g^m|^2 dx < 2\int_\Omega a(x)F(g)dx.$$

Then, inspired by Payne et al. [16, 17], we further define the following auxiliary function

$$\varphi(t) = \int_\Omega u^{m+1}dx. \tag{2.1}$$

Theorem 1. *Assume that the conditions* (f_1), (f_2), (g_1), (a_1), (a_2) *hold, and* u *is a nonnegative solution of problem* (1.1)-(1.3). *Then, we conclude that the solution* u *becomes unbounded in* $L^{m+1}-$*norm at* $t = t^*$. *Moreover, an upper bound for blow-up time* t^* *is given by*

$$t^* \leq \frac{(m+1)\varphi(0)}{(m-1)\phi(0)}, \tag{2.2}$$

and the upper estimate of blow-up rate can be given by

$$\|u\|_{m+1} \leq \left(\frac{(m+1)\varphi(0)^{\frac{2m}{m+1}}}{(m-1)\phi(0)}\right)^{\frac{1}{m-1}} (t^*-t)^{-\frac{1}{m-1}}, \tag{2.3}$$

where $\varphi(0) = \|g\|_{m+1}^{m+1}$ *and* $\phi(0) = -(m+1)\int_\Omega |\nabla g^m|^2 dx + 2(m+1)\int_\Omega a(x)F(g)dx > 0$.

Proof. Firstly, differentiating (2.1) with respect to t and using Eq.(1.1), then we have

$$\begin{aligned}\varphi'(t) &= (m+1)\int_\Omega u^m u_t dx\\ &= (m+1)\int_\Omega u^m\big(\triangle u^m + a(x)f(u)\big)dx\\ &= -(m+1)\int_\Omega |\nabla u^m|^2 dx + (m+1)\int_\Omega a(x)u^m f(u)dx. \end{aligned}\tag{2.4}$$

By the combination of (2.4) and condition (f_2), we obtain

$$\varphi'(t) \geq -(m+1)\int_\Omega |\nabla u^m|^2 dx + C_1(m+1)\int_\Omega a(x)F(u)dx > \phi(t), \tag{2.5}$$

where

$$\phi(t) = -(m+1)\int_\Omega |\nabla u^m|^2 dx + 2(m+1)\int_\Omega a(x)F(u)dx. \tag{2.6}$$

On the other hand, a simple computation yields

$$\begin{aligned}\phi'(t) &= -2(m+1)\int_\Omega \nabla u^m\cdot(\nabla u^m)_t dx\\ &\quad + 2m(m+1)\int_\Omega a(x)u^{m-1}u_t f(u)dx\\ &= 2m(m+1)\int_\Omega u^{m-1}u_t\big(\triangle u^m + a(x)f(u)\big)dx\\ &= 2m(m+1)\int_\Omega u^{m-1}u_t^2 dx \geq 0. \end{aligned}\tag{2.7}$$

Here, we have used the fact that $u(x,t)=0$ (or $\frac{\partial u}{\partial \nu}=0$) on $\partial\Omega$. Using Schwarz's inequality, we get

$$\left(\int_\Omega u^m u_t dx\right)^2 \leq \int_\Omega u^{m+1}dx\int_\Omega u^{m-1}u_t^2 dx. \tag{2.8}$$

Hence, multiplying $\varphi(t)$ by $\phi'(t)$, it follows from (2.5) that

$$\begin{aligned}\varphi(t)\phi'(t) &= 2m(m+1)\int_\Omega u^{m+1}dx\int_\Omega u^{m-1}u_t^2 dx\\ &\geq 2m(m+1)\left(\int_\Omega u^m u_t dx\right)^2 = \frac{2m}{m+1}[\varphi'(t)]^2\\ &\geq \frac{2m}{m+1}\varphi'(t)\phi(t). \end{aligned}\tag{2.9}$$

Thus, the above inequality implies that

$$\left(\phi(t)\,[\varphi(t)]^{-\frac{2m}{m+1}}\right)' = [\varphi(t)]^{-\frac{3m+1}{m+1}}\left\{\varphi(t)\phi'(t) - \frac{2m}{m+1}\varphi'(t)\phi(t)\right\} \geq 0. \tag{2.10}$$

Utilizing the assumption (g_1) and (2.1), (2.7), we know that

$$\varphi(0) = \|g\|_{m+1}^{m+1} > 0, \tag{2.11}$$

and

$$\phi(t) \geq \phi(0) = -(m+1)\int_\Omega |\nabla g^m|^2 dx + 2(m+1)\int_\Omega a(x)F(g)dx > 0. \tag{2.12}$$

Integrating (2.10) from 0 to t, we obtain

$$\phi(t)\,[\varphi(t)]^{-\frac{2m}{m+1}} \geq \phi(0)\,[\varphi(0)]^{-\frac{2m}{m+1}} = M > 0. \tag{2.13}$$

By (2.5) and (2.13), we get

$$\frac{m+1}{1-m}\left([\varphi(t)]^{\frac{1-m}{m+1}}\right)' = \varphi'(t)\,[\varphi(t)]^{-\frac{2m}{m+1}} \geq \phi(t)\,[\varphi(t)]^{-\frac{2m}{m+1}} > 0. \tag{2.14}$$

Integrating (2.14) from 0 to t, we have

$$[\varphi(t)]^{\frac{1-m}{m+1}} \leq [\varphi(0)]^{\frac{1-m}{m+1}} - \frac{m-1}{m+1}Mt. \tag{2.15}$$

Clearly, the above inequality cannot hold for all $t > 0$. Consequently, u blows up at some finite time t^* and

$$t^* \leq \frac{m+1\varphi(0)}{(m-1)\phi(0)}.$$

Furthermore, from (2.5) and (2.13) again, we have

$$\varphi'(t) \geq \phi(t) \geq \phi(0)\,[\varphi(0)]^{-\frac{2m}{m+1}}\,[\varphi(t)]^{\frac{2m}{m+1}}. \tag{2.16}$$

Integrating (2.16) from t to t^*, we obtain

$$\varphi(t) \leq \left(\frac{(m+1)\varphi(0)^{\frac{2m}{m+1}}}{(m-1)\phi(0)}\right)^{\frac{m+1}{m-1}} (t^*-t)^{-\frac{m+1}{m-1}}, \tag{2.17}$$

which implies that the upper estimate of blow-up rate is given by (2.3).

2.2. The second method

We first assume that

(f_3): there exists a positive function $G(\theta)$ such that

$$\int_\Omega a(x)f(s)dx \geq C_2 G\left[\int_\Omega s dx\right] \quad \text{with} \quad \int_0^{+\infty} \frac{d\theta}{G(\theta)} < +\infty,$$

for any function $s(x) \geq 0$. Then, we define the following auxiliary function

$$\varphi_1(t) = \int_\Omega u dx. \tag{2.18}$$

Theorem 2. *Assume that the conditions (f_1), (f_3), (a_1), (a_2) hold, and u is a nonnegative solution of problem* (1.1)-(1.3)*. Then, we conclude that the solution u becomes unbounded in L^1-norm at $t = t^*$. Moreover, an upper bound for blow-up time t^* is given by*

$$t^* \leq \int_{\varphi_1(0)}^{+\infty} \frac{d\theta}{G(\theta)} < +\infty, \tag{2.19}$$

and the upper estimate of blow-up rate can be given by

$$\|u\|_{L^1} \leq Y^{-1}(t^*-t), \tag{2.20}$$

where the function $Y(s) := \int_s^{+\infty} \frac{d\theta}{G(\theta)}$ for any function $s(x) \geq 0$, and $\varphi_1(0) = \int_\Omega g dx$.

Proof. Integrating the Eq.(1.1) by parts, from the condition (f_3) and (2.18) we have

$$\int_\Omega u_t dx = \int_\Omega a(x) f(u) dx \geq C_2 G\left[\int_\Omega u dx\right], \tag{2.21}$$

which means that

$$\varphi_1'(t) \geq C_2 G\left[\varphi_1(t)\right] > 0. \tag{2.22}$$

Here, we have used the fact that $u(x,t) = 0$ (or $\frac{\partial u}{\partial \nu} = 0$) on $\partial\Omega$. It then follows from (2.22) that $\varphi_1(t)$ is a increasing function, so we have

$$\varphi_1(t) > \varphi_1(0) = \int_\Omega g(x) dx \geq 0. \tag{2.23}$$

Integrating (2.22) from 0 to t and using (2.23), (f_3), we discover

$$t \leq \int_{\varphi_1(0)}^{\varphi_1(t)} \frac{d\theta}{G(\theta)} \leq \int_{\varphi_1(0)}^{+\infty} \frac{d\theta}{G(\theta)} < +\infty. \tag{2.24}$$

Obviously, (2.24) cannot hold for all time t. Consequently, we can derive an upper bound t^* such that

$$t^* \leq \int_{\varphi_1(0)}^{+\infty} \frac{d\theta}{G(\theta)} < +\infty,$$

and

$$\lim_{t \to t^*} \varphi_1(t) = +\infty, \tag{2.25}$$

where $(0, t^*)$ is the interval of existence of the solutions u in L^1-norm. In fact, if the equality (2.25) doesn't hold, then there exists a time $t_1 > t^*$ such that $\varphi_1(t^*) < \varphi_1(t_1) < +\infty$ and t_1 satisfies the inequality (2.23),(2.24), which contradict the maximum existence of t^*.

Furthermore, integrating (2.22) from t to t^*, it follows that

$$t^* - t \leq \int_{\varphi_1(t)}^{+\infty} \frac{d\theta}{G(\theta)} := Y(\varphi_1(t)). \tag{2.26}$$

We note that Y is a decreasing function, which means its inverse function Y^{-1} exists and is also a decreasing function. Therefore, we have

$$\varphi_1(t) \leq Y^{-1}(t^* - t), \tag{2.27}$$

which implies that the estimate (2.20) of blow-up rate holds.

Remark 1. This result can be generalized to the case of problem (1.1)-(1.3) subject to $\frac{\partial u}{\partial \nu} = b(x,t) \geq 0$. In this case, it follows that

$$\varphi_1'(t) \geq m \int_{\partial\Omega} u^{m-1} b(x,t) ds + C_2 G\left[\varphi_1(t)\right] > 0. \tag{2.28}$$

We also can obtain the inequalities (2.19) and (2.20).

3. Lower estimates for blow-up time and blow-up rate

In this section, we will give three methods to establish the lower bounds for blow-up time and blow-up rate of the solution to problem (1.1)-(1.3).

3.1. The first method

Firstly, let us assume that

(f_4): there exists positive constants C_3, C_4 such that

$$a(x)f(s) \leq C_3 + C_4 s^{l+1},$$

for any function $s(x) \geq 0$, where $0 < l \leq \frac{2nm-(n-2)(m+1)}{n}$. And then we introduce the auxiliary function $\varphi(t) = \int_\Omega u^{m+1}dx$ as (2.1).

Next, we shall state and prove the main results of this subsection as follows:

Theorem 3. *Assume that the conditions (f_1), (f_4), (a_1), (a_2) hold, and u is a nonnegative solution of problem (1.1)-(1.3) which becomes unbounded in $L^{m+1}-$norm at $t = t^*$. Then, we conclude that a lower bound for blow-up time t^* is given by*

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_1\eta + k_2\eta^{1+\frac{2}{n\epsilon_2}}}. \tag{3.1}$$

and the lower estimate of blow-up rate is

$$\|u\|_{m+1} \geq \left(\frac{4k_2}{n\epsilon_2}\right)^{-\frac{n\epsilon_2}{2(m+1)}}(t^*-t)^{-\frac{n\epsilon_2}{2(m+1)}}, \tag{3.2}$$

where k_1, k_2 and ϵ_2 are positive constants which will be given later.

Proof. Differentiating (2.1) with respect to t and using the condition (f_4) and (1.1), we have

$$\begin{aligned}
\varphi'(t) &= (m+1)\int_\Omega u^m u_t dx \\
&= (m+1)\int_\Omega u^m\big(\triangle u^m + a(x)f(u)\big)dx \\
&= -(m+1)\int_\Omega |\nabla u^m|^2 dx + (m+1)\int_\Omega a(x)u^m f(u)dx \\
&\leq -(m+1)\int_\Omega |\nabla u^m|^2 dx + (m+1)C_3\int_\Omega u^m dx \\
&\quad + (m+1)C_4\int_\Omega u^{m+l+1}dx.
\end{aligned} \tag{3.3}$$

From Hölder's inequality, Young's inequality and condition (f_4), we have

$$\int_\Omega u^m dx \leq \frac{m}{m+l+1}\int_\Omega u^{m+l+1}dx + \frac{l+1}{m+l+1}|\Omega|, \tag{3.4}$$

and

$$
\begin{aligned}
\int_\Omega u^{m+l+1}dx &= \int_\Omega u^{m+1-\epsilon_1}u^{l-\epsilon_1}dx \\
&\leq \Big(\int_\Omega u^{m+1}dx\Big)^{\frac{m+1-\epsilon_1}{m+1}}\Big(\int_\Omega u^{\frac{(l+\epsilon_1)(m+1)}{\epsilon_1}}dx\Big)^{\frac{\epsilon_1}{m+1}} \\
&= \Big(\int_\Omega u^{m+1}dx\Big)^{\frac{m+1-\epsilon_1}{m+1}}\Big(\int_\Omega u^{\frac{2nm}{n-2}}dx\Big)^{\frac{\epsilon_1}{m+1}} \\
&\leq \frac{\epsilon_2 C(\epsilon)}{1+\epsilon_2}\Big(\int_\Omega u^{m+1}dx\Big)^{\frac{(m+1-\epsilon_1)(\epsilon_2+1)}{(m+1)\epsilon_2}} \\
&\quad + \frac{\epsilon}{1+\epsilon_2}\Big(\int_\Omega u^{\frac{2nm}{n-2}}dx\Big)^{\frac{\epsilon_1(1+\epsilon_2)}{m+1}}, \qquad (3.5)
\end{aligned}
$$

where $\epsilon_1 = \frac{(n-2)(m+1)l}{2nm-(n-2)(m+1)} > 0$, $\epsilon_2 = \frac{n(2m-l)-(n-2)(m+1)}{nl} > 0$, and ϵ will be determined later.

Noting that

$$
\int_\Omega u^{\frac{2nm}{n-2}}dx \leq C_5^{\frac{2n}{n-2}}\Big(\int_\Omega |\nabla u^m|^2\Big)^{\frac{n}{n-2}}, \qquad (3.6)
$$

where C_5 is the optimal constant of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$.

Furthermore, from the choice of ϵ_1 and ϵ_2, it is easy to see that $\frac{\epsilon_1(1+\epsilon_2)n}{(m+1)(n-2)} = 1$. Inserting (3.4)-(3.6) into (3.3), it follows that

$$
\begin{aligned}
\varphi'(t) &\leq -(m+1)\int_\Omega |\nabla u^m|^2dx + \frac{(m+1)C_3(l+1)}{m+l+1}|\Omega| \\
&\quad + \Big[\frac{m(m+1)C_3}{m+l+1} + (m+1)C_4\Big]\int_\Omega u^{m+l+1}dx \\
&\leq -\left[m+1 - \frac{(m+1)(mC_3+mC_4+lC_4+C_4)C_5^2\epsilon}{(1+\epsilon_2)(m+l+1)}\right]\int_\Omega |\nabla u^m|^2dx \\
&\quad + \frac{(m+1)(mC_3+mC_4+lC_4+C_4)\epsilon_2 C(\epsilon)}{(1+\epsilon_2)(m+l+1)}\Big(\int_\Omega u^{m+1}dx\Big)^{\frac{(m+1-\epsilon_1)(\epsilon_2+1)}{(m+1)\epsilon_2}} \\
&\quad + \frac{(m+1)C_3(l+1)}{m+l+1}|\Omega|. \qquad (3.7)
\end{aligned}
$$

Taking ϵ small enough such that

$$
m+1 - \frac{(m+1)(mC_3+mC_4+lC_4+C_4)C_5^2\epsilon}{(1+\epsilon_2)(m+l+1)} > 0. \qquad (3.8)
$$

Hence, we have

$$
\varphi'(t) \leq k_1 + k_2[\varphi(t)]^{\frac{(m+1-\epsilon_1)(1+\epsilon_2)}{(m+1)\epsilon_2}} = k_1 + k_2[\varphi(t)]^{1+\frac{2}{n\epsilon_2}}, \qquad (3.9)
$$

where $k_1 = \frac{(m+1)C_3(l+1)}{m+l+1}|\Omega|$ and $k_2 = \frac{(m+1)(mC_3+mC_4+lC_4+C_4)\epsilon_2 C(\epsilon)}{(1+\epsilon_2)(m+l+1)}$.

Integrating (3.9) from 0 to t, we get

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{k_1\eta + k_2\eta^{1+\frac{2}{n\epsilon_2}}} \leq t. \tag{3.10}$$

If u blows up in the measure $\varphi(t)$ as $t \to t^*$, then we can obtain the lower bound

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_1\eta + k_2\eta^{1+\frac{2}{n\epsilon_2}}}.$$

Moreover, integrating the inequality (3.9) from t to t^*, we obtain

$$t^* - t \geq \int_{\varphi(t)}^{+\infty} \frac{d\eta}{k_1\eta + k_2\eta^{1+\frac{2}{n\epsilon_2}}} := Y_1(\varphi(t)). \tag{3.11}$$

We note that Y_1 is a decreasing function, which means its inverse function Y_1^{-1} exists and is also a decreasing function. Therefore, we have

$$\varphi(t) \geq Y_1^{-1}(t^* - t), \tag{3.12}$$

which gives the lower estimate of blow-up rate. In fact, if t closes t^* enough such that $\varphi(t) \gg 1$ and $k_2\eta^{1+\frac{2}{n\epsilon_2}} > k_1\eta$ in the inequality (3.11), then we have

$$t^* - t \geq \frac{n\epsilon_2}{4k_2}\big[\varphi(t)\big]^{-\frac{2}{n\epsilon_2}}, \tag{3.13}$$

which means that

$$\varphi(t) \geq \big(\frac{4k_2}{n\epsilon_2}\big)^{-\frac{n\epsilon_2}{2}}(t^* - t)^{-\frac{n\epsilon_2}{2}}. \tag{3.14}$$

Thus, the estimate (3.2) of blow-up rate also holds.

3.2. The second method

Firstly, we need the following assumption:

(f_5): there exists positive constants C_6, C_7, q and Q such that

$$a(x)f(s) \leq C_6 + C_7 s^p \left(\int_\Omega s^{q+1}dx\right)^Q,$$

for any function $s(x) \geq 0$.

(e_1): we also assume that

$$0 \leq p \leq 1, \quad 0 \leq q \leq m, \quad \text{and} \quad (q+1)Q + p > 1.$$

To obtain the main results, we define the auxiliary function $\varphi(t) = \int_\Omega u^{m+1}dx$ again.

Next, we will state our results below:

Theorem 4. *Assume that the conditions* (f_1), (f_5), (e_1), (a_1), (a_2) *hold, and* u *is a nonnegative solution of problem* (1.1)-(1.3) *which becomes unbounded in* $L^{m+1}-$*norm at* $t = t^*$. *Then, we conclude that a lower bound for blow-up time* t^* *is given by*

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_3\eta^{\frac{m}{m+1}} + k_4\eta^{\frac{m+p+(q+1)Q}{m+1}}}, \tag{3.15}$$

and the lower estimate of blow-up rate is

$$\|u\|_{m+1} \geq \big[\frac{2k_4((q+1)Q+p-1)}{m+1}\big]^{-\frac{1}{(q+1)Q+p-1}}(t^*-t)^{-\frac{1}{(q+1)Q+p-1}}, \tag{3.16}$$

where $k_3 = C_6(m+1)|\Omega|^{\frac{1}{m+1}}$, $k_4 = C_7(m+1)|\Omega|^{\frac{1-p+(m-q)Q}{m+1}}$.

Proof. Under the assumption condition (f_5), we have from (1.1) and (2.1) that

$$\begin{aligned}
\varphi'(t) &= (m+1)\int_\Omega u^m u_t dx \\
&= (m+1)\int_\Omega u^m\big(\triangle u^m + a(x)f(u)\big)dx \\
&= -(m+1)\int_\Omega |\nabla u^m|^2 dx + (m+1)\int_\Omega a(x)u^m f(u)dx \\
&\leq C_6(m+1)\int_\Omega u^m dx + C_7(m+1)\int_\Omega u^{m+p}dx\left(\int_\Omega u^{q+1}dx\right)^Q.
\end{aligned} \tag{3.17}$$

Applying Hölder's inequality and condition (e_1), we know that

$$\int_\Omega u^m dx \leq \left(\int_\Omega u^{m+1}dx\right)^{\frac{m}{m+1}}|\Omega|^{\frac{1}{m+1}}, \tag{3.18}$$

$$\int_\Omega u^{m+p} dx \leq \left(\int_\Omega u^{m+1}dx\right)^{\frac{m+p}{m+1}}|\Omega|^{\frac{1-p}{m+1}}, \tag{3.19}$$

and

$$\int_\Omega u^{q+1} dx \leq \left(\int_\Omega u^{m+1}dx\right)^{\frac{q+1}{m+1}}|\Omega|^{\frac{m-q}{m+1}}. \tag{3.20}$$

Inserting (3.18)-(3.20) into (3.17), it follows that

$$\begin{aligned}
\varphi'(t) \leq\ & C_6(m+1)|\Omega|^{\frac{1}{m+1}}\left(\int_\Omega u^{m+1}dx\right)^{\frac{m}{m+1}} \\
&+ C_7(m+1)|\Omega|^{\frac{1-p+(m-q)Q}{m+1}}\left(\int_\Omega u^{m+1}dx\right)^{\frac{m+p+(q+1)Q}{m+1}},
\end{aligned} \tag{3.21}$$

where from the condition (e_1), it is easy to see that $\frac{m+p+(q+1)Q}{m+1} > 1$.

Then, integrating the above inequality from 0 to t yields that

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\eta}{k_3\eta^{\frac{m}{m+1}} + k_4\eta^{\frac{m+p+(q+1)Q}{m+1}}} \leq t,$$

where $k_3 = C_6(m+1)|\Omega|^{\frac{1}{m+1}}$, $k_4 = C_7(m+1)|\Omega|^{\frac{1-p+(m-q)Q}{m+1}}$.

If u blows up in the measure $\varphi(t)$ as $t \to t^*$, then we can obtain the lower bound

$$t^* \geq \int_{\varphi(0)}^{+\infty} \frac{d\eta}{k_3\eta^{\frac{m}{m+1}} + k_4\eta^{\frac{m+p+(q+1)Q}{m+1}}}.$$

Furthermore, integrating the inequality (3.21) from t to t^*, we obtain

$$t^* - t \geq \int_{\varphi(t)}^{+\infty} \frac{d\eta}{k_3\eta^{\frac{m}{m+1}} + k_4\eta^{\frac{m+p+(q+1)Q}{m+1}}} := Y_2(\varphi(t)). \tag{3.22}$$

We note that Y_2 is a decreasing function, which means its inverse function Y_2^{-1} exists and it is also a decreasing function. Therefore, we have

$$\varphi(t) \geq Y_2^{-1}(t^* - t), \tag{3.23}$$

which gives the lower estimate of blow-up rate. In fact, the auxiliary function $\varphi(t)$ becomes unbounded at time $t = t^*$, so we know that $\varphi(t) \gg 1$ and the inequality $k_4\eta^{\frac{m+p+(q+1)Q}{m+1}} > k_3\eta^{\frac{m}{m+1}}$ as $t \to t^{*-}$. Hence, when t is close to t^*, inserting the above inequality into (3.22) and then a direct calculation yields that

$$t^* - t \geq \frac{m+1}{2k_4[(q+1)Q+p-1]}\big[\varphi(t)\big]^{-\frac{(q+1)Q+p-1}{m+1}}, \tag{3.24}$$

which means that

$$\varphi(t) \geq \Big[\frac{2k_4((q+1)Q+p-1)}{m+1}\Big]^{-\frac{m+1}{(q+1)Q+p-1}}(t^* - t)^{-\frac{m+1}{(q+1)Q+p-1}}. \tag{3.25}$$

Hence, the estimate (3.16) of blow-up rate holds.

3.3. The third method

This subsection is devoted to the estimates of the lower bounds for blow-up time and blow-up rate of the solutions to problem (1.1)-(1.3) by utilizing the method appearing in [18, 19].

For this purpose, we need to assume that

(f_6): $s^{m-1} \geq \alpha\left(\int_s^{+\infty} \frac{d\eta}{f(\eta)}\right)^{-\gamma}$, for any function $s(x) \geq 0$, where α, γ are positive constants and $0 < \gamma < 1$;

(f_7): there exist positive constants k and β such that $k > \frac{4(n-2)-n\gamma}{2n}$,

$$f(s)\left(\int_s^{+\infty}\frac{d\eta}{f(\eta)}\right)^{nk+1} \to +\infty, \quad \text{as } s \to 0^+,$$

and

$$f'(s)\int_s^{+\infty}\frac{d\eta}{f(\eta)} \le nk+1-\beta,$$

for any function $s(x) \ge 0$. Then, we define the following auxiliary function

$$\varphi_2(t) = \int_\Omega V^{nk}(u)dx, \qquad V(u) = \left(\int_u^{+\infty}\frac{d\eta}{f(\eta)}\right)^{-1}. \tag{3.26}$$

Next, we will state our results below:

Theorem 5. *Assume that the conditions* (f_1), (f_6), (f_7), (a_1), (a_2) *hold, and* u *is a nonnegative solution of problem* (1.1)-(1.3) *which becomes unbounded in* $\varphi_2(t)$*-form at* $t = t^*$*. Then, we conclude that a lower bound for blow-up time* t^* *is given by*

$$t^* \ge \int_{\varphi_2(0)}^{+\infty}\frac{d\eta}{k_5 + k_6\eta^{\frac{3n-6}{3n-8}}}, \tag{3.27}$$

and the lower estimate of blow-up rate can be given by

$$\varphi_2(t) \ge \left(\frac{4k_6}{3n-8}\right)^{-\frac{3n-8}{2}}(t^*-t)^{-\frac{3n-8}{2}}, \tag{3.28}$$

where k_5*,* k_6 *will be given later, and* $\varphi_2(0) = \int_\Omega\left[\int_g^{+\infty}\frac{d\eta}{f(\eta)}\right]^{-nk}dx$*.*

Proof. Under the assumptions (f_7), we have from (1.1) and (3.26) that

$$\begin{aligned}
\varphi_2'(t) &= nk\int_\Omega V^{nk+1}[f(u)]^{-1}u_t dx\\
&= nk\int_\Omega V^{nk+1}[f(u)]^{-1}[\triangle u^m + a(x)f(u)]dx\\
&= -mnk(nk+1)\int_\Omega V^{nk+2}[f(u)]^{-2}|\nabla u|^2u^{m-1}dx + nk\int_\Omega V^{nk+1}a(x)dx\\
&\quad + mnk\int_\Omega V^{nk+1}[f(u)]^{-2}f'(u)u^{m-1}|\nabla u|^2dx.\\
&\le -mnk(nk+1)\int_\Omega V^{nk+2}[f(u)]^{-2}|\nabla u|^2u^{m-1}dx + nk\int_\Omega V^{nk+1}a(x)dx\\
&\quad + mnk(nk+1-\beta)\int_\Omega V^{nk+2}[f(u)]^{-2}u^{m-1}|\nabla u|^2dx
\end{aligned}$$

$$= nk\int_\Omega V^{nk+1}a(x)dx - mnk\beta\int_\Omega V^{nk+2}[f(u)]^{-2}u^{m-1}|\nabla u|^2dx. \tag{3.29}$$

Since

$$|\nabla V^{\frac{nk+\gamma}{2}}|^2 = \left(\frac{nk+\gamma}{2}\right)^2 V^{nk+\gamma+2}[f(u)]^{-2}|\nabla u|^2. \tag{3.30}$$

In view of assumptions (f_6), (3.29) and (3.30), we discover that

$$\varphi_2'(t) \le nk\int_\Omega V^{nk+1}a(x)dx - \frac{4nk\alpha\beta}{(nk+\gamma)^2}\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2dx. \tag{3.31}$$

For convenience, we denote

$$\epsilon_3 = \frac{3nk(n-2)+n(nk+\gamma)-4(n-2)(nk+1)}{3nk(n-2)+n(nk+\gamma)},$$

$$\epsilon_4 = \frac{4(n-2)(nk+1)}{3nk(n-2)+n(nk+\gamma)},$$

where the assumptions (f_7) implies that $\epsilon_3 > 0$, $\epsilon_4 > 0$ and $\epsilon_3+\epsilon_4 = 1$. So by Hölder's inequality, we have

$$\int_\Omega V^{nk+1}a(x)dx \le \left(\int_\Omega V^{\frac{3nk(n-2)+n(nk+\gamma)}{4(n-2)}}dx\right)^{\epsilon_4}\left(\int_\Omega a(x)^{\frac{1}{\epsilon_3}}dx\right)^{\epsilon_3}. \tag{3.32}$$

Using Hölder's inequality again, it follows that

$$\int_\Omega V^{\frac{3nk(n-2)+n(nk+\gamma)}{4(n-2)}}dx \le \left(\int_\Omega V^{nk}dx\right)^{\frac{3}{4}}\left(\int_\Omega V^{\frac{n(nk+\gamma)}{n-2}}dx\right)^{\frac{1}{4}}. \tag{3.33}$$

Furthermore, applying the Sobolev's inequality, we obtain that

$$\left(\int_\Omega V^{\frac{n(nk+\gamma)}{n-2}}dx\right)^{\frac{n-2}{2n}} = \left(\int_\Omega (V^{\frac{nk+\gamma}{2}})^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{2n}} \le C_5\left(\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2dx\right)^{\frac{1}{2}}, \tag{3.34}$$

where C_5 is the optimal Sobolev's embedding constant defined as (3.6). Substituting (3.34) into (3.33) yields that

$$\int_\Omega V^{\frac{3nk(n-2)+n(nk+\gamma)}{4(n-2)}}dx \le C_5^{\frac{n}{2(n-2)}}\left(\int_\Omega V^{nk}dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2dx\right)^{\frac{n}{4(n-2)}}. \tag{3.35}$$

By Young's inequality, (3.32) and (3.35), we obtain

$$\int_\Omega V^{nk+1}a(x)dx$$

$$\leq \left(C_5^{\frac{n}{2(n-2)}} \left(\int_\Omega V^{nk} dx \right)^{\frac{3}{4}} \left(\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx \right)^{\frac{n}{4(n-2)}} \right)^{\epsilon_4} \left(\int_\Omega a(x)^{\frac{1}{\epsilon_3}} dx \right)^{\epsilon_3}$$

$$\leq \epsilon_4 C_5^{\frac{n}{2(n-2)}} \left(\int_\Omega V^{nk} dx \right)^{\frac{3}{4}} \left(\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx \right)^{\frac{n}{4(n-2)}} + \epsilon_3 \int_\Omega a(x)^{\frac{1}{\epsilon_3}} dx. \qquad (3.36)$$

Applying Young's inequality again, we have

$$\left(\int_\Omega V^{nk} dx \right)^{\frac{3}{4}} \left(\int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx \right)^{\frac{n}{4(n-2)}}$$
$$\leq \frac{n\delta}{4(n-2)} \int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx + \frac{(3n-8)C(\delta)}{4(n-2)} \left(\int_\Omega V^{nk} dx \right)^{\frac{3n-6}{3n-8}}, \qquad (3.37)$$

where $\frac{n}{4(n-2)} > 1$, $\frac{3n-8}{4(n-2)} > 1$ with $\frac{n}{4(n-2)} + \frac{3n-8}{4(n-2)} = 1$.

Consequently, it follows that

$$\int_\Omega V^{nk+1} a(x) dx \leq \epsilon_3 \int_\Omega a(x)^{\frac{1}{\epsilon_3}} dx + \frac{\epsilon_4 n\delta}{4(n-2)} \int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx$$
$$+ \frac{\epsilon_4 (3n-8)C(\delta)}{4(n-2)} \left(\int_\Omega V^{nk} dx \right)^{\frac{3n-6}{3n-8}}. \qquad (3.38)$$

Substituting (3.38) into (3.29), we get

$$\varphi_2'(t) \leq \epsilon_3 \int_\Omega a(x)^{\frac{1}{\epsilon_3}} dx + \frac{\epsilon_4 nk(3n-8)C(\delta)}{4(n-2)} \left(\int_\Omega V^{nk} dx \right)^{\frac{3n-6}{3n-8}}$$
$$- \left[\frac{4nk\alpha\beta}{(nk+\gamma)^2} - \frac{\epsilon_4 n^2 k\delta}{4(n-2)} \right] \int_\Omega |\nabla V^{\frac{nk+\gamma}{2}}|^2 dx. \qquad (3.39)$$

Now, we can choose δ small enough to make the coefficient $\frac{4nk\alpha\beta}{(nk+\gamma)^2} - \frac{\epsilon_4 n^2 k\delta}{4(n-2)} > 0$. Hence, we have

$$\varphi_2'(t) \leq k_5 + k_6^{\frac{3n-6}{3n-8}}, \qquad (3.40)$$

where $k_5 = \epsilon_3 \int_\Omega a(x)^{\frac{1}{\epsilon_3}} dx$ and $k_6 = \frac{\epsilon_4 nk(3n-8)C(\delta)}{4(n-2)}$.

Then, integrating (3.40) from 0 to t yields that

$$\int_{\varphi_2(0)}^{\varphi_2(t)} \frac{d\eta}{k_5 + k_6 \eta^{\frac{3n-6}{3n-8}}} \leq t. \qquad (3.41)$$

If u blows up in the measure $\varphi_2(t)$ as $t \to t^*$, then we can obtain the lower bound

$$t^* \geq \int_{\varphi_2(0)}^{+\infty} \frac{d\eta}{k_5 + k_6 \eta^{\frac{3n-6}{3n-8}}}.$$

Similar to the above derivation of (3.14) and (3.25), it is easy to get

$$\varphi_2(t) \geq \left(\frac{4k_6}{3n-8}\right)^{-\frac{3n-8}{2}} (t^*-t)^{-\frac{3n-8}{2}}.$$

This completes the proof of Theorem 5.

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