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On the symmetric block design with parameters (210,77,28) admitting a Frobenius group of order 57

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Abstract. In this paper we have proved that for a putative symmetric block design $\mathcal{D}$ with parameters (210,77,28), admitting a Frobenius group $G = \langle \rho, \mu | \rho^{19} = \mu^3 = 1, \rho^\mu = \rho^7 \rangle$ of order 57, there are exactly six possible orbit structures up to isomorphism with the orbit distribution $[1; 19; 19; 19; 19; 19; 19; 19; 19; 19; 19; 19]$.

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Key Words and Phrases: Symmetric block design, Orbit structure, Automorphism group

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1. Introduction and preliminaries

A $2-(v,k,\lambda)$ design $(\mathcal{P}, \mathcal{B}, I)$ is said to be *symmetric* if the relation $|\mathcal{P}| = |\mathcal{B}| = v$ holds and in that case we often speak of a symmetric design with parameters (v,k,λ). The collection of the parameter sets (v,k,λ) for which a symmetric $2-(v,k,\lambda)$ design exists is often called the "spectrum". The determination of the spectrum for symmetric designs is a widely open problem. For example, a finite projective plane of order n is a symmetric design with parameters $(n^2+n+1, n+1, 1)$ and it is still unknown whether finite projective planes of non–prime–power order may exist at all.

The existence/non-existence of a symmetric design has often required "ad hoc" treatments even for a single parameter set (v,k,λ). The most famous instance of this circumstance is perhaps the non-existence of the projective plane of order 10, see [9].

It is of interest to study symmetric designs with additional properties, which often involve the assumption that a non–trivial automorphism group acts on the design under consideration, see for instance [3].

Among symmetric block designs of square order, a study of symmetric block designs of order 49 is of a particular interest. There are 15 possible parameters (v,k,λ) for symmetric designs of order 49, but until now only a few results are known (see [2], [4]). Due to the fact that symmetric designs of order 49 have a big number of points (blocks), the study of

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sporadic cases is very difficult, except, possibly, when the existence of a collineation group is assumed.

A few methods for the construction of symmetric designs are known and all of them have shown to be effective in certain situations. Here, we shall use the method of tactical decompositions, assuming that a certain automorphism group acts on the design we want to construct, used by Z. Janko in [7]; see also [3] and [5]. The present paper is concerned with a symmetric design $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ with parameters $(280, 63, 14)$: the existence/non–existence of such a design is still in doubt as far as we know. We shall further assume that the given design admits a certain automorphism group of order 93. We assume the reader is familiar with the basic facts of design theory, see for instance [8], [1] and [10]. If g is an automorphism of a symmetric design $\mathcal{D}$ with parameters (v, k, λ), then g fixes an equal number of points and blocks, see [10, Theorem 3.1, p.78]. We denote the sets of these fixed elements by $F_{\mathcal{P}}(g)$ and $F_{\mathcal{B}}(g)$ respectively, and their cardinality simply by $|F(g)|$. We shall make use of the following upper bound for the number of fixed points, see [10, Corollary 3.7, p. 82]:

$$|F(g)| \leq k + \sqrt{k - \lambda}. \tag{1}$$

It is also known that an automorphism group G of a symmetric design has the same number of orbits on the set of points $\mathcal{P}$ as on the set of blocks $\mathcal{B}$: [10, Theorem 3.3, p.79]. Denote that number by t.

2. Point- and block-orbits

We adopt the notation and terminology of Section 1 in [3]. In the following, for the sake of completeness, some fundamental relations are explicitly provided. Let $\mathcal{D}$ be a symmetric design with parameters (v, k, λ) and let G be a subgroup of the automorphism group $Aut(\mathcal{D})$ of $\mathcal{D}$. Denote the point orbits of G on $\mathcal{P}$ by $\mathcal{P}_1, \mathcal{P}_2, \ldots \mathcal{P}_t$ and the line orbits of G on $\mathcal{B}$ by $\mathcal{B}_1, \mathcal{B}_2, \ldots \mathcal{B}_t$. Put $|\mathcal{P}_r| = \omega_r$ and $|\mathcal{B}_i| = \Omega_i$. Obviously,

$$\sum_{r=1}^{t} \omega_r = \sum_{i=1}^{t} \Omega_i = v. \tag{2}$$

Let γ_{ir} be the number of points from $\mathcal{P}_r$, which lie on a line from $\mathcal{B}_i$; clearly this number does not depend on the chosen line. Similarly, let Γ_{js} be the number of lines from $\mathcal{B}_j$ which pass through a point from $\mathcal{P}_s$. Then, obviously,

$$\sum_{r=1}^{t} \gamma_{ir} = k \text{ and } \sum_{j=1}^{t} \Gamma_{js} = k. \tag{3}$$

By [1, Lemma 5.3.1. p.221], the partition of the point set $\mathcal{P}$ and of the block set $\mathcal{B}$ forms a tactical decomposition of the design $\mathcal{D}$ in the sense of [1, p.210]. Thus, the following equations hold:

$$\Omega_i \cdot \gamma_{ir} = \omega_r \cdot \Gamma_{ir} \tag{4}$$

$$\sum_{r=1}^{t} \gamma_{ir}\Gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda) \tag{5}$$

$$\sum_{i=1}^{t} \Gamma_{ir}\gamma_{is} = \lambda\omega_s + \delta_{rs}(k-\lambda) \tag{6}$$

where δ_{ij}, δ_{rs} are the Kronecker symbols.

For a proof of these equations, the reader is referred to [1] and [3]. Equation (5), together with (4) yields

$$\sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr} = \lambda\Omega_j + \delta_{ij}(k-\lambda). \tag{7}$$

Definition 1. *We denote*

$$[L_i, L_j] = \sum_{r=1}^{t} \frac{\Omega_j}{\omega_r}\gamma_{ir}\gamma_{jr}, 1 \leq i, j \leq t$$

and call these expressions the orbit products. The $(t \times t)$-matrix (γ_{ir}) is called the orbit structure of the design $\mathcal{D}$.

An automorphism of an orbit structure is a permutation of rows followed by a permutation of columns leaving that matrix unchanged. It is clear that the set of all such automorphisms is a group, which we call the automorphism group of that orbit structure.

The first step in the construction of a design is to find all possible orbit structures. The second step of the construction is usually called indexing. In fact for each coefficient γ_{ir} of the orbit matrix one has to specify which γ_{ir} points of the point orbit $\mathcal{P}_r$ lie on the lines of the block orbit $\mathcal{B}_i$. Of course, it is enough to do this for a representative of each block orbit, as the other lines of that orbit can be obtained by producing all G-images of the given representative.

3. Action of the Frobenius group of order 57

In our construction of symmetric $2-(210, 77, 28)$ designs we assume the existence of an automorphism group $G = \langle \rho, \mu | \rho^{19} = \mu^3 = 1, \rho^\mu = \rho^7 \rangle$, which is a so called Frobenius group of order 57 with Frobenius kernel of order 19 (see [6]).

Theorem 1. *Up to isomorphism there are exactly six orbit structures for symmetric (210,77,28) designs and the automorphism group $G = F_{19\cdot 3}$ acting with the orbit distribution $\mathcal{O} = [1; 19; 19; 19; 19; 19; 19; 19; 19; 19; 19; 19]$.*

Proof. We denote by ∞ the fixed point of ρ and put $\mathcal{P}_I = \{I_0, I_1, \cdots, I_{18}\}, I = 1,2,3,4,5,6,7,8,9,10,11$, for the non–trivial orbits of the group G. Thus, G acts on these point orbits as a permutation group in a unique way. Hence, for the two generators of G we may put

$$\rho = (\infty)(I_0, I_1, \cdots, I_{18}), I = 1,2,\cdots,11,$$

where ∞ is the fixed point of collineation, whereas non–trivial $\langle\rho\rangle-$orbits are numbers $1,2,3,4,5,6,7,8,9,10,11$ and $\infty, 1_0, 1_1, \cdots, 11_{18}$ all points of the symmetric block design $\mathcal{D}$, and the collineation μ of order 3 acts in the symmetric block design as permutation $(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)$ on orbit numbers, whereas on indices acts $\mu : x \to 7x \pmod{19}$ or

$$\mu = (\infty)(K_0)(K_1, K_7, K_{11})(K_2, K_{14}, K_3)(K_4, K_9, K_6)(K_5, K_{16}, K_{17})$$
$$(K_8, K_{18}, K_{12})(K_{10}, K_{13}, K_{15}), \quad K = 1,2,3,4,5,6,7,8,9,10,11.$$

We immediately obtain the following.

Corollary 1. *The element μ of G of order 3 fixes precisely 12 points and 12 blocks of $\mathcal{D}$. Each block orbit contains a unique block stabilized by μ.*

In what follows, we are going to construct a representative block for each block orbit. A representative block for the block orbit of length 19 will be the block fixed by μ. Hence the multiplicities of orbit numbers in orbit blocks, will be $\equiv 0, 1 \pmod 3$.

The $\langle\rho\rangle-$fixed block can be writen in the form:

$$L_1 = \infty(1_0 1_1 \cdots 1_{18})(2_0 2_1 \cdots 2_{18})(3_0 3_1 \cdots 3_{18})(4_0 4_1 \cdots 4_{18})$$

or

$$L_1 = \infty 1_{19} 2_{19} 3_{19} 4_{19}.$$

Let $L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}$ be the representative blocks for the eleven non–trivial block orbits. There are exactly four non–fixed orbit blocks passing through the fixed point ∞. Let them be L_2, L_3, L_4, L_5. We write

$$L_2 = \infty 1_{a_1} 2_{a_2} 3_{a_3} 4_{a_4} 5_{a_5} 6_{a_6} 7_{a_7} 8_{a_8} 9_{a_9} 10_{a_{10}} 11_{a_{11}}$$

$$L_3 = \infty 1_{b_1} 2_{b_2} 3_{b_3} 4_{b_4} 5_{b_5} 6_{b_6} 7_{b_7} 8_{b_8} 9_{b_9} 10_{b_{10}} 11_{b_{11}}$$

$$L_4 = \infty 1_{c_1} 2_{c_2} 3_{c_3} 4_{c_4} 5_{c_5} 6_{c_6} 7_{c_7} 8_{c_8} 9_{c_9} 10_{c_{10}} 11_{c_{11}}$$

$$L_5 = \infty 1_{d_1} 2_{d_2} 3_{d_3} 4_{d_4} 5_{d_5} 6_{d_6} 7_{d_7} 8_{d_8} 9_{d_9} 10_{d_{10}} 11_{d_{11}}$$

where a_i, b_i, c_i, d_i denote the multiplicities of the appearance of orbit numbers in the orbit blocks L_2, L_3, L_4, L_5, respectively.

The multiplicities of the appearances of orbit numbers satisfy the following conditions:

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} = 76.$$

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} = 76.$$

$$c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} = 76.$$

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 + d_{10} + d_{11} = 76.$$

Because $|L_i \cap L_1| = 28, i = 2, 3, 4, 5$ and $\infty \in L_i, i = 1, 2, 3, 4, 5$ we have $a_1 + a_2 + a_3 + a_4 = 27$, $b_1 + b_2 + b_3 + b_4 = 27$, $c_1 + c_2 + c_3 + c_4 = 27$, $d_1 + d_2 + d_3 + d_4 = 27$ and consequently $a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} = 49$, $b_5 + b_6 + b_7 + b_8 + b_9 + b_{10} + b_{11} = 49$, $c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} = 49$, $d_5 + d_6 + d_7 + d_8 + d_9 + d_{10} + d_{11} = 49$. From (7) we have

$$[L_2, L_2] = 19/1 \cdot 1 \cdot 1 + 19/19 \cdot a_1^2 + 19/19 \cdot a_2^2 + 19/19 \cdot a_3^2 + 19/19 \cdot a_4^2 + 19/19 \cdot a_5^2 + 19/19 \cdot a_6^2$$

$$+19/19 \cdot a_7^2 + 19/19 \cdot a_8^2 + 19/19 \cdot a_9^2 + 19/19 \cdot a_{10}^2 + 19/19 \cdot a_{11}^2 = 28 \cdot 19 + 77 - 28 = 581$$

$$[L_3, L_3] = 19/1 \cdot 1 \cdot 1 + 19/19 \cdot b_1^2 + 19/19 \cdot b_2^2 + 19/19 \cdot b_3^2 + 19/19 \cdot b_4^2 + 19/19 \cdot b_5^2 + 19/19 \cdot b_6^2$$

$$+19/19 \cdot b_7^2 + 19/19 \cdot b_8^2 + 19/19 \cdot b_9^2 + 19/19 \cdot b_{10}^2 + 19/19 \cdot b_{11}^2 = 28 \cdot 19 + 77 - 28 = 581$$

$$[L_4, L_4] = 19/1 \cdot 1 \cdot 1 + 19/19 \cdot c_1^2 + 19/19 \cdot c_2^2 + 19/19 \cdot c_3^2 + 19/19 \cdot c_4^2 + 19/19 \cdot c_5^2 + 19/19 \cdot c_6^2$$

$$+19/19 \cdot c_7^2 + 19/19 \cdot c_8^2 + 19/19 \cdot c_9^2 + 19/19 \cdot c_{10}^2 + 19/19 \cdot c_{11}^2 = 28 \cdot 19 + 77 - 28 = 581$$

$$[L_5, L_5] = 19/1 \cdot 1 \cdot 1 + 19/19 \cdot d_1^2 + 19/19 \cdot d_2^2 + 19/19 \cdot d_3^2 + 19/19 \cdot d_4^2 + 19/19 \cdot d_5^2 + 19/19 \cdot d_6^2$$

$$+19/19 \cdot d_7^2 + 19/19 \cdot d_8^2 + 19/19 \cdot d_9^2 + 19/19 \cdot d_{10}^2 + 19/19 \cdot d_{11}^2 = 28 \cdot 19 + 77 - 28 = 581$$

$$[L_2, L_3] = 19 \cdot 1 \cdot 1 + 19/19 \cdot a_1b_1 + 19/19 \cdot a_2b_2 + 19/19 \cdot a_3b_3 + 19/19 \cdot a_4b_4 + 19/19 \cdot a_5b_5$$

$$+19/19 \cdot a_6b_6 + 19/19 \cdot a_7b_7 + 19/19 \cdot a_8b_8 + 19/19 \cdot a_9b_9 + 19/19 \cdot a_{10}b_{10} + 19/19 \cdot a_{11}b_{11} = 28 \cdot 19 = 532$$

$$[L_2, L_4] = 19 \cdot 1 \cdot 1 + 19/19 \cdot a_1c_1 + 19/19 \cdot a_2c_2 + 19/19 \cdot a_3c_3 + 19/19 \cdot a_4c_4 + 19/19 \cdot a_5c_5$$

$$+19/19 \cdot a_6c_6 + 19/19 \cdot a_7c_7 + 19/19 \cdot a_8c_8 + 19/19 \cdot a_9c_9 + 19/19 \cdot a_{10}c_{10} + 19/19 \cdot a_{11}c_{11} = 28 \cdot 19 = 532$$

$$[L_2, L_5] = 19 \cdot 1 \cdot 1 + 19/19 \cdot a_1d_1 + 19/19 \cdot a_2d_2 + 19/19 \cdot a_3d_3 + 19/19 \cdot a_4d_4 + 19/19 \cdot a_5d_5$$

$$+19/19 \cdot a_6d_6 + 19/19 \cdot a_7d_7 + 19/19 \cdot a_8d_8 + 19/19 \cdot a_9d_9 + 19/19 \cdot a_{10}d_{10} + 19/19 \cdot a_{11}d_{11} = 28 \cdot 19 = 532$$

where $0 \leq a_i \leq 23, i = 1, 2, \cdots, 11$.

In order to reduce isomorphic cases that may appear in the orbit structures at the last stage, without loss of generality, for the block L_2, we can use the reduction $a_1 \geq a_2 \geq a_3 \geq a_4,\ a_5 \geq a_6 \geq a_7 \geq a_8 \geq a_9 \geq a_{10} \geq a_{11}$.

Using the computer we have proved that there exist exactly twenty one different orbit types for the block L_2 satisfying the above mentioned conditions:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{19}	a_{11}
1.	10	9	4	4	9	7	7	7	7	6	6
2.	10	7	7	3	10	7	7	7	6	6	6
3.	10	7	7	3	9	9	7	6	6	6	6
4.	10	7	6	4	10	9	6	6	6	6	6
5.	10	7	6	4	10	7	7	7	7	7	4
6.	10	7	6	4	9	9	7	7	7	6	4
7.	9	9	6	3	10	7	7	7	6	6	6
8.	9	9	6	3	9	9	7	6	6	6	6
9.	9	7	7	4	10	9	7	7	6	6	4
10.	9	7	7	4	9	9	9	6	6	6	4
11.	9	7	7	4	9	9	7	7	7	7	3
12.	9	6	6	6	12	7	6	6	6	6	6
13.	9	6	6	6	10	10	7	6	6	6	4
14.	9	6	6	6	10	9	7	7	7	6	3
15.	9	6	6	6	9	9	9	7	7	4	4
16.	9	6	6	6	9	9	9	7	6	6	3
17.	7	7	7	6	12	7	7	7	6	6	4
18.	7	7	7	6	10	10	7	7	7	4	4
19.	7	7	7	6	10	10	7	7	6	6	3
20.	7	7	7	6	10	9	9	7	6	4	4
21.	7	7	7	6	10	9	9	6	6	6	3

Further on, we find the possible candidates for L_3 considering each orbit type for L_2.

Among the candidates for block L_3 are blocks L_4, L_5. Therefore, for each case for L_2, from the candidates for L_3, must be found triples of blocks $\{L_3, L_4, L_5\}$, which are pairwise compatibile.

Using the computer we obtain the results which we present in the table below:

Table 1:

Block L_2	Number of orbit types for L_3	Number of quadruples $\{L_2, L_3, L_4, L_5\}$
Type 1.	969	8
Type 2.	1201	15
Type 3.	2073	41
Type 4.	1847	13
Type 5.	2355	6
Type 6.	1172	57
Type 7.	1401	8
Type 8.	1792	21
Type 9.	1001	67
Type 10.	2001	43
Type 11.	1301	23
Type 12.	1170	0
Type 13.	1371	11
Type 14.	1295	35
Type 15.	1584	8
Type 16.	1902	55
Type 17.	1001	11
Type 18.	2340	13
Type 19.	1026	28
Type 20.	1131	46
Type 21.	2098	69

Therefore, there are 578 quadruples L_2, L_3, L_4, L_5 (i.e. 578 quintuples L_1, L_2, L_3, L_4, L_5).

The sixth orbit block L_6 has the form:

$$L_6 = 1_{e_1} 2_{e_2} 3_{e_3} 4_{e_4} 5_{e_5} 6_{e_6} 7_{e_7} 8_{e_8} 9_{e_9} 10_{e_{10}} 11 e_{11}$$

where $e_i, i = 1, 2, \cdots, 11$ are multiplicities of the appearance of orbit numbers 1,2,3, 4,5,6,7,8,9,10 and 11 in orbit block L_6.

We have: $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 + e_9 + e_{10} + e_{11} = 77$,

$$[L_6, L_6] = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2 + e_6^2 + e_7^2 + e_8^2 + e_9^2 + e_{10}^2 + e_{11}^2 = 28 \cdot 19 + 77 - 28 = 581,$$

$$[L_6, L_i] = 28 \cdot 19 = 532, \ (i = 2, 3, 4, 5).$$

$[L_6 \cap L_1] = 28$ implies $e_1 + e_2 + e_3 + e_4 = 28$, therefore $c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} + c_{11} = 77 - 28 = 49$.

$[L_6, L_6] = 581$ implies $0 \leq c_i \leq 24, \ i = 1, 2, \cdots, 11$.

Candidates for L_6 are also blocks $L_7, L_8, L_9, L_{10}, L_{11}$, and L_{12}, so we look for septuples $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}\}$ which are pairwise compatibile. Using the computer for the number of quadruples $\{L_2, L_3, L_4, L_5\}$ given in Table 1, we obtain the results which we present in the table below:

Type for L_2	Subcase for quadruples $\{L_2, L_3, L_4, L_5\}$	Number of orbit types for L_6	Number of septuples $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}\}$
1	1	48	0
1	2	48	0
2	3	40	0
2	4	40	0
2	6	60	0
2	8	60	0
3	3	32	0
3	4	22	0
3	12	78	0
3	13	33	0
3	14	20	0
3	15	23	0
3	16	23	0
3	18	33	0
3	19	79	0
5	1	120	12
5	6	48	0
6	42	48	0
7	8	25	0
9	22	22	0
9	59	7	1
9	60	22	1
9	63	36	0
10	41	45	4
11	1	48	0
11	2	48	0
11	10	32	0
11	17	39	0
13	7	45	4
13	10	32	0
13	11	39	0
14	4	25	0
15	3	22	1
15	5	25	0
15	6	25	0
15	7	36	0
15	8	36	0
16	23	36	0
16	24	7	1
16	29	39	0
16	54	74	0
16	55	74	0

Type for L_2	Subcase for quadruples $\{L_2, L_3, L_4, L_5\}$	Number of orbit types for L_6	Number of septuples $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}\}$
17	3	60	0
17	5	23	0
17	11	36	0
18	1	40	0
18	2	78	0
18	3	60	0
19	5	33	0
19	6	20	0
19	15	22	1
19	20	32	0
19	21	36	0
19	22	22	1
19	23	32	0
19	26	74	0
19	27	36	0
20	7	40	0
20	9	33	0
20	10	23	0
20	34	32	0
21	11	78	0
21	43	36	0
21	45	7	1
21	46	39	0
21	47	45	4
21	48	7	1
21	56	39	0
21	57	7	1
21	59	39	0
21	61	7	1
21	63	74	0
21	64	39	0
21	65	45	4
21	66	36	0
	Other subcases	< 7	0

From the table above it can be seen that there are 75 subcases, for which the number of orbit types for L_6 is greater than 6, and for those subcases we have searched for septuples $\{L_6, L_7, L_8, L_9, L_{10}, L_{11}, L_{12}\}$, respectively, we have searched for orbit structures. For these 75 subcases we have found 38 orbit structures. After the removal of the isomorphic cases, there remained exactly 6 orbit structures, which are given below:

Table 2:

OS1.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	10	7	6	4	10	7	7	7	7	7	4
	1	7	10	4	6	4	7	7	7	7	7	10
	1	6	4	7	10	7	10	7	7	7	4	7
	1	4	6	10	7	7	4	7	7	7	10	7
	0	10	4	7	7	7	4	9	7	6	6	10
	0	7	7	10	4	4	10	9	7	6	7	6
	0	7	7	7	7	9	9	3	6	4	9	9
	0	7	7	7	7	7	7	6	3	12	7	7
	0	7	7	7	7	6	6	4	12	9	6	6
	0	7	7	4	10	6	7	9	7	6	10	4
	0	4	10	7	7	10	6	9	7	6	4	7

OS2.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	10	7	6	4	10	7	7	7	7	7	4
	1	7	10	4	6	4	7	7	7	7	7	10
	1	6	4	7	10	7	10	7	7	7	4	7
	1	4	6	10	7	7	4	7	7	7	10	7
	0	10	4	7	7	6	4	10	7	7	6	9
	0	7	7	10	4	4	9	6	9	9	6	6
	0	7	7	7	7	9	7	4	10	3	7	9
	0	7	7	7	7	9	7	4	3	10	7	9
	0	7	7	7	7	6	10	9	4	4	10	6
	0	7	7	4	10	6	6	6	9	9	9	4
	0	4	10	7	7	9	6	10	7	7	4	6

OS3.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	7	7	7	6	10	9	9	6	6	6	3
	1	7	7	7	6	3	9	9	6	6	6	10
	1	7	7	4	9	9	6	4	9	6	6	9
	1	6	6	9	6	6	4	6	7	10	10	6
	0	10	10	4	4	7	6	7	6	9	7	7
	0	10	3	6	9	6	9	6	6	7	9	6
	0	7	7	10	4	7	9	4	10	6	6	7
	0	7	7	7	7	7	4	10	9	3	9	7
	0	6	6	9	7	10	6	7	3	7	6	10
	0	6	6	7	9	6	6	9	9	10	3	6
	0	3	10	6	9	6	9	6	6	7	9	6

OS4.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	7	7	7	6	10	9	9	6	6	6	3
	1	7	7	7	6	3	9	9	6	6	6	10
	1	7	7	4	9	9	6	4	9	6	6	9
	1	6	6	9	6	6	4	6	7	10	10	6
	0	10	9	6	3	6	7	6	10	7	7	6
	0	9	6	4	9	6	9	6	4	9	9	6
	0	9	3	10	6	9	7	6	6	6	6	9
	0	6	10	6	6	9	4	9	4	7	7	9
	0	6	6	7	9	6	6	9	9	10	3	6
	0	6	6	7	9	6	6	9	9	3	10	6
	0	3	9	9	7	7	10	4	7	7	7	7

OS5.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	9	7	7	4	10	9	7	7	6	6	4
	1	6	7	7	7	9	3	6	6	10	6	9
	1	6	7	7	7	3	9	6	6	9	10	6
	1	6	6	6	9	6	7	9	9	3	6	9
	0	10	6	6	6	6	7	10	3	7	7	9
	0	10	6	6	6	6	7	3	10	7	7	9
	0	7	6	6	9	6	7	9	9	10	4	4
	0	6	10	3	9	9	7	6	6	6	9	6
	0	6	9	9	4	6	4	9	9	6	9	6
	0	6	3	10	9	9	7	6	6	6	9	6
	0	4	9	9	6	7	10	6	6	7	4	9

OS6.	1	19	19	19	19	19	19	19	19	19	19	19
	1	19	19	19	19	0	0	0	0	0	0	0
	1	9	7	7	4	10	9	7	7	6	6	4
	1	6	7	7	7	7	6	7	6	10	3	10
	1	6	7	7	7	7	6	7	6	3	10	10
	1	6	6	6	9	4	7	7	9	9	9	4
	0	10	6	6	6	6	7	3	10	7	7	9
	0	10	6	6	6	6	6	9	3	9	9	7
	0	7	6	6	9	10	3	9	9	6	6	6
	0	6	10	3	9	7	10	7	6	6	6	7
	0	6	9	9	4	4	7	10	9	6	6	7
	0	6	3	10	9	7	10	7	6	6	6	7
	0	4	9	9	6	9	6	4	6	9	9	6

□

Remark 1. *The actual indexing of these six orbit structures in order to produce an example is still an open problem.*

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