**EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS**
Vol. 13, No. 4, 2020, 814-829
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

# New refinement of Niezgoda's inequality with applications to Ky Fan inequality

Sadia Chanan[1,*], Asif R. Khan[1]

[1] *Department of Mathematics, University of Karachi, University Road, Karachi-75270, Pakistan*

**Abstract.** The aim of this article is to give the refinement of Niezgoda's inequality with its applications to Ky Fan inequality and cyclic mixed symmetric means.

**2020 Mathematics Subject Classifications**: 26A51, 39B62, 26D15, 26D20, 26D99

**Key Words and Phrases**: Convex functions, Niezgoda's inequality, Ky Fan Inequality, cyclic mixed symmetric means

## 1. Introduction and Preliminaries

Jensen's inequality for convex functions is one of the most celebrated inequality in Mathematics and Statistics. Due to its high importance there are given numerous variants, generalizations and refinements of Jensen's inequalities (for reference see [8, 9, 12, 13, 29]). We also adduce to [25] and [28] for detailed discussion on Jensen's inequality and for some remarks on literature and history of the topic.

A variant of Jensen's inequality named as Jensen-Mercer inequality was established by Mercer [24] given as follows:

**Theorem 1.** *Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and let $w_1, w_2, \ldots, w_n$ be nonnegative real numbers such that $\sum_{i=1}^{n} w_i = 1$. If $\phi$ is a convex function defined on an interval containing all $x_i$'s for $1 \leq i \leq n$. Then*

$$\phi\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq \phi(x_1) + \phi(x_n) - \sum_{i=1}^{n} w_i \phi(x_i). \tag{1}$$

Now, we recall a prerequisite concept of majorization from [23].

Let $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ denote two $m$-tuples and $x_{[1]} \geq \cdots \geq x_{[m]}$, $y_{[1]} \geq \cdots \geq y_{[m]}$ be their ordered components.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i4.3776

*Email addresses:* sadiachanankhan@yahoo.com (S. Chanan), asifrk@uok.edu.pk (A. R. Khan)

 

**Definition 1.** *For* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$,

$$\mathbf{x} \prec \mathbf{y} \quad \textit{if} \quad \begin{cases} \sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad , \quad k \in \{1, \ldots, m-1\}, \\ \sum_{i=1}^{m} x_{[i]} = \sum_{i=1}^{m} y_{[i]} \end{cases}$$

*when* $\mathbf{x} \prec \mathbf{y}$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ *or* $\mathbf{y}$ majorizes $\mathbf{x}$.

In the book [10] we find a very power result namely majorization theorem (see also [23]).

**Theorem 2.** *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ *then following inequality is true for all continuous convex functions* $\phi : \mathbb{R} \to \mathbb{R}$,

$$\sum_{i=1}^{n} \phi(x_i) \leq \sum_{i=1}^{n} \phi(y_i)$$

*if and only if* $\mathbf{x} \prec \mathbf{y}$.

We now define an extension of Jensen-Mercer inequality which is referred as Niezgoda's inequality by Niezgoda [26]. For recent work on Niezgoda inequality we refer the reader [1, 15–17, 27].

**Theorem 3.** *Suppose that* $\boldsymbol{a}$ *be an* $m$*-tuple such that* $a_i \in J$ *and a* $n \times m$ *matrix* $\boldsymbol{X} = (\mathbf{x}_j) = (x_{ij})$ *with* $x_{ij} \in J$ *for all* $i \in \{1, \ldots, n\}$ *and* $j \in \{1, \ldots, m\}$.

*If* $\boldsymbol{a}$ *majorizes each row of* $\boldsymbol{X}$, *that is,*

$$\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im}) \prec (a_1, \ldots, a_m) = \boldsymbol{a} \textit{ for each } i \in \{1, \ldots, n\},$$

*then for a continuous convex function* $\phi$ *on* $J$ *following inequality holds.*

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{m-1} \sum_{i=1}^{n} w_i \phi(x_{ij}), \tag{2}$$

*with* $w_i \geq 0$ *such that* $\sum_{i=1}^{n} w_i = 1$.

Especially, the inequality stated below is also valid for $w_i = \frac{1}{n}, i \in \{1, \ldots, n\}$

$$\phi\left(\sum_{j=1}^{m} a_j - \frac{1}{n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} x_{ij}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \frac{1}{n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} \phi(x_{ij}). \tag{3}$$

The cyclic refinement of the Jensen's inequality in paper [3] is given as follows:

**Theorem 4.** *Let $\phi : I \to \mathbb{R}$ be a convex function and $I$ be an interval in $\mathbb{R}, \mathbf{x} = (x_1, \ldots, x_n) \in I^n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a nonnegative $n$-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for some $k$, $2 \leq k \leq n$. Then*

$$\phi\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \leq \frac{1}{n}\sum_{i=1}^{n} \phi\left(\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \leq \frac{1}{n}\sum_{i=1}^{n} \phi(x_i), \tag{4}$$

*where $i+j$ means $i+j-n$ in case of $i+j>n$.*

The cyclic refinement of Jensen-Mercer inequality in paper [5] is defined as follows:

**Theorem 5.** *Let $I \subset \mathbb{R}$ be an interval, $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ such that $\left(c+d-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \in I$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a positive $n$-tuple such that $\sum_{i=1}^{k} \lambda_i = 1$ for some $k$, $2 \leq k \leq n$, then for convex function $\phi : I \to \mathbb{R}$, $[c,d] \subset I$, following inequalities hold:*

$$\begin{aligned}\phi\left(c+d-\sum_{i=1}^{n} w_i x_i\right) &\leq \sum_{i=1}^{n} w_i \phi\left(c+d-\sum_{j=0}^{k-1} \lambda_{j+1} x_{i+j}\right) \\ &\leq \phi(c)+\phi(d)-\sum_{i=1}^{n} w_i \phi(x_i),\end{aligned} \tag{5}$$

*where $i+j$ means $i+j-n$ in case of $i+j>n$.*

In this article we are going to use some of the following assumptions:

- $(C_1)$: Let $\phi : J \to \mathbb{R}$ be a convex function.
- $(C_2)$: Let $\mathbf{a}$ be a $m$-tuple such that $a_j \in J^n$ and a $n \times m$ matrix $X = (x_{ij}) \in J^n, \forall i \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, m\}$ such that $(x_{i1+k}, \ldots, x_{im+k}) = (x_{i1}, \ldots, x_{im}), \forall i \in \{1, \ldots, n\})$ and $\lambda : (\lambda_1, \ldots, \lambda_n)$ ba a $n$-tuple such that $\sum_{k=1}^{l} \lambda_k = 1, l \in \{2, \ldots, n\}$. Moreover, $w_i$'s are non-negative real weights for $1 \leq i \leq n$ such that $\sum_{i=1}^{n} w_i = 1$
- $(C_3)$: Let $\phi, \psi : J \to \mathbb{R}$ be continuous and strictly monotone functions.

Under the assumptions stated above it should be noted that

$$\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right) \in J.$$

The aim of this paper is to present new refinement of Theorem 3. In main result section, we will give refinement for weighted version of Niezgoda's inequality, then we will define its special case for equal weights. In application section, with the help of main results we will give refinements of Ky Fan and arithmetic-geometric means inequalities and their related results. We also define cyclic mixed symmetric means, power mean and generalized quasi-arithmetic means and study their properties. We follow the techniques given in [3]. The final section gives suggestions for further work and future ideas.

## 2. Main Result

**Theorem 6.** *Let the assumptions stated in Theorem* 3 *be true. In addition we suppose that the assumptions given in* $(C1)$ *and* $(C2)$ *are also valid. Then we have*

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leq \sum_{i=1}^{n} w_i \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right) \leq \sum_{j=1}^{m} \phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \phi\left(x_{ij}\right). \quad (6)$$

*Proof.* To prove first inequality of (6), since $\phi$ is a convex function and

$$\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right) \in J,$$

therefore by Jensen's Inequality,

$$\begin{aligned}
&\sum_{i=1}^{n} w_i \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right) \\
&\geq \phi\left(\sum_{i=1}^{n} w_i \sum_{j=1}^{m} a_j - \sum_{i=1}^{n}\sum_{j=1}^{m-1}\sum_{k=0}^{l-1} w_i \lambda_{k+1} x_{ij+k}\right) \\
&= \phi\left(\sum_{j=1}^{m} a_j - \left(\sum_{k=1}^{l} \lambda_k\right)\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \\
&= \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right).
\end{aligned}$$

On the other hand, to prove second inequality of (6), we consider following expression

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right)$$

for fixed $i \in \{1, 2, \ldots, n\}$ and proceed as follows:

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}\right)$$

$$= \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij}\right)$$
$$= \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=1}^{l}\lambda_{k}x_{ij}\right)$$
$$= \phi\left(\sum_{j=1}^{m} a_j - \left(\sum_{k=1}^{l}\lambda_{k}\right)\sum_{j=1}^{m-1} x_{ij}\right)$$
$$= \phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right)$$

Using majorization property we have

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1} x_{ij}\right) = \phi\left(x_{im}\right) \leq \sum_{j=1}^{m}\phi\left(a_j\right) - \sum_{j=1}^{m-1}\phi\left(x_{ij}\right)$$

Or we can write,

$$\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}\right) \leq \sum_{j=1}^{m}\phi\left(a_j\right) - \sum_{j=1}^{m-1}\phi\left(x_{ij}\right). \tag{7}$$

Now multiplying inequality (7) with $w_i$ and summing over $i$ from 1 to $n$ we get our required result.

**Corollary 1.** *Under the assumptions of Theorem* 6 *and for* $w_i = \frac{1}{n}, i \in \{1, \ldots, n\}$, *we have*

$$\phi\left(\sum_{j=1}^{m} a_j - \frac{1}{n}\sum_{j=1}^{m-1}\sum_{i=1}^{n} x_{ij}\right)$$
$$\leq \frac{1}{n}\sum_{i=1}^{n}\phi\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}\right)$$
$$\leq \sum_{j=1}^{m}\phi(a_j) - \frac{1}{n}\sum_{j=1}^{m-1}\sum_{i=1}^{n}\phi\left(x_{ij}\right). \tag{8}$$

**Remark 1.** *If we set* $m = 2$, $a_1 = c$, $a_2 = d$ *and* $x_{i1} = x_i$ *for* $i \in \{1, \ldots, n\}$, *then Theorem* 5 *will become special case of Theorem* 6.

## 3. Refinement of the Ky Fan inequality

Throughout this section, let the assumptions stated in Theorem 6 be valid with $0 < c < d$. We define generalized (or modified) arithmetic, geometric and harmonic mean respectively as follow (for general discussion on mean and related inequalities we refer [4]):

$$\hat{A}_n = \sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij},$$

$$\hat{G}_n = \frac{\prod_{j=1}^{m} a_j}{\prod_{j=1}^{m-1}\prod_{i=1}^{n}(x_{ij})^{w_i}},$$

$$\hat{H}_n = \left(\sum_{j=1}^{m}(a_j)^{-1} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i (x_{ij})^{-1}\right)^{-1}.$$

Also for $x_{ij} \in (0, \frac{1}{2}]$, we define arithmetic, geometric and harmonic means as follows:

$$\hat{A}'_n = \sum_{j=1}^{m}(1-a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i (1-x_{ij}),$$

$$\hat{G}'_n = \frac{\prod_{j=1}^{m}(1-a_j)}{\prod_{j=1}^{m-1}\prod_{i=1}^{n}((1-x_{ij})^{w_i}},$$

$$\hat{H}'_n = \left(\sum_{j=1}^{m}(1-a_j)^{-1} - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i (1-x_{ij})^{-1}\right)^{-1}.$$

We also define new notations $\hat{A}(\lambda; \mathbf{x})$ and $\hat{G}(\lambda; \mathbf{x})$ as under:

$$\hat{A}(\lambda; \mathbf{x}) = \sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k},$$

$$\hat{G}(\lambda; \mathbf{x}) = \frac{\prod_{j=1}^{m} a_j}{\prod_{j=1}^{m-1}\prod_{k=0}^{l-1}(x_{ij+k})^{\lambda_{k+1}}}.$$

Also for $x_{ij} \in (0, \frac{1}{2}]$, we define

$$\hat{A}'(\lambda; \mathbf{x}) = \sum_{j=1}^{m}(1 - a_j) - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}(1 - x_{ij+k}),$$

$$\hat{G}'(\lambda; \mathbf{x}) = \frac{\prod_{j=1}^{m}(1 - a_j)}{\prod_{j=1}^{m-1}\prod_{k=0}^{l-1}(1 - x_{ij+k})^{\lambda_{k+1}}}.$$

Now, we present the refinement of the Ky Fan type inequality. For Ky Fan inequality and related results, see [2], [6] and [5] and references given therein.

**Theorem 7.** *Let assumptions stated in Theorem 6 be true. Then following inequality holds:*

$$\frac{\hat{A}_n}{\hat{A'}_n} \leq \prod_{i=1}^{n}\left(\frac{\hat{A}(\lambda, \mathbf{x})}{\hat{A}'(\lambda, \mathbf{x})}\right)^{w_i} \leq \frac{\hat{G}_n}{\hat{G'}_n}.$$

*Proof.* By applying the convex function $\phi(x) = ln\left(\frac{x}{1-x}\right)$ for all $x \in (0, \frac{1}{2}]$, to the inequality (6), we get,

$$\ln\left(\frac{\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}}{1 - \sum_{j=1}^{m} a_j + \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}}\right)$$
$$\leq \sum_{i=1}^{n} w_i \ln\left(\frac{\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}}{1 - \sum_{j=1}^{m} a_j + \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}}\right)$$
$$\leq \sum_{j=1}^{m}\ln\left(\frac{a_i}{1 - a_j}\right) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i \ln\left(\frac{x_{ij}}{1 - x_{ij}}\right)$$

consequently,

$$\ln\left(\frac{\hat{A}_n}{\hat{A}'_n}\right) \leq \ln\prod_{i=1}^{n}\left(\frac{\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}}{1 - \sum_{j=1}^{m} a_j + \sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}x_{ij+k}}\right)^{w_i} \leq \ln\left(\frac{\hat{G}_n}{\hat{G}'_n}\right)$$

Finally, we obtain

$$\left(\frac{\hat{A}_n}{\hat{A}'_n}\right) \leq \prod_{i=1}^{n}\left(\frac{\hat{A}(\lambda, \mathbf{x})}{\hat{A}'(\lambda, \mathbf{x})}\right)^{w_i} \leq \left(\frac{\hat{G}_n}{\hat{G}'_n}\right),$$

which completes the proof.

**Remark 2.** *For $w_i = \frac{1}{n}$, we obtain the special case of Theorem 7 as follows:*

$$\frac{A_n}{A'_n} \leq \prod_{i=1}^{n} \left( \frac{A(\lambda, \mathbf{x})}{A'(\lambda, \mathbf{x})} \right)^{\frac{1}{n}} \leq \frac{G_n}{G'_n},$$

*where*

$$A_n = \sum_{j=1}^{m} a_j - \frac{1}{n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} x_{ij},$$
$$G_n = \frac{\prod_{j=1}^{m} a_j}{\left( \prod_{j=1}^{m-1} \prod_{i=1}^{n} x_{ij} \right)^{\frac{1}{n}}},$$

*and*

$$A'_n = \sum_{j=1}^{m} (1 - a_j) - \frac{1}{n} \sum_{j=1}^{m-1} \sum_{i=1}^{n} (1 - x_{ij}),$$
$$G'_n = \frac{\prod_{j=1}^{m} (1 - a_j)}{\left( \prod_{j=1}^{m-1} \prod_{i=1}^{n} (1 - x_{ij}) \right)^{\frac{1}{n}}}.$$

Now, we present refinement of arithmetic-geometric mean type inequality as follows:

**Corollary 2.** *Let the assumptions stated in Theorem 6 be true. Then following inequality holds:*

$$\hat{A}_n \geq \prod_{i=1}^{n} \left( \hat{A}(\lambda, \mathbf{x}) \right)^{w_i} \geq \hat{G}'_n.$$

*Proof.* By applying the convex function $\phi(x) = -\ln(x), x \in (0, \frac{1}{2}]$ to Theorem 6 we obtain required result.

Now, we present refinement of harmonic and geometric means inequality as follows:

**Corollary 3.** *Let the assumptions stated in Theorem 6 be true. Then following inequalities hold:*

$$\left( \hat{G}'_n \right)^{-1} \leq \sum_{i=1}^{n} w_i \left( \hat{G}'(\lambda, \mathbf{x}) \right)^{-1} \leq \left( \hat{H}'_n \right)^{-1}.$$

*Proof.* By applying the convex function $\phi(x) = exp(x), x \in (0, \frac{1}{2}]$ to Theorem 6 and by replacing $a_j$ and $x_{ij}$ by $\ln\left(\frac{1}{1-a_j}\right)$ and $\ln\left(\frac{1}{1-x_{ij}}\right)$ respectively, we get

$$\exp\left(\sum_{j=1}^{m}\ln\left(\frac{1}{1-a_j}\right)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\ln\left(\frac{1}{1-x_{ij}}\right)\right)$$
$$\leq\sum_{i=1}^{n}w_i\exp\left(\sum_{j=1}^{m}\ln\left(\frac{1}{1-a_j}\right)-\sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}\ln\left(\frac{1}{1-x_{ij+k}}\right)\right)$$
$$\leq\sum_{j=1}^{m}\exp\left(\ln\left(\frac{1}{1-a_j}\right)\right)-\sum_{j=1}^{m-1}\sum_{i=0}^{n}w_i\exp\left(\ln\left(\frac{1}{1-x_{ij}}\right)\right),$$

Consequently,

$$\exp\left(-\ln\left(\frac{\prod_{j=1}^{m}(1-a_j)}{\prod_{i=1}^{n}\prod_{j=1}^{m-1}(1-x_{ij})^{w_i}}\right)\right)$$
$$\leq\sum_{i=1}^{n}w_i\exp\left[-\ln\left(\frac{\prod_{j=1}^{m}(1-a_j)}{\prod_{j=1}^{m-1}\prod_{k=0}^{l-1}\left(\frac{1}{1-x_{ij+k}}\right)^{\lambda_{j+1}}}\right)\right]$$
$$\leq\left(\sum_{j=1}^{m}\left(\frac{1}{(1-a_j)}\right)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\left(\frac{1}{1-x_{ij}}\right)\right),$$

which is equivalent to

$$\left(\hat{G}_n'\right)^{-1}\leq\sum_{i=1}^{n}w_i\left(\hat{G}'(\lambda,\mathbf{x})\right)^{-1}\leq\left(\sum_{j=1}^{m}\left(\frac{1}{(1-a_j)}\right)-\sum_{j=1}^{m-1}\sum_{i=1}^{n}w_i\left(\frac{1}{1-x_{ij}}\right)\right),$$

finally, we obtain

$$\left(\hat{G}_n'\right)^{-1}\leq\sum_{i=1}^{n}w_i\left(\hat{G}'(\lambda,\mathbf{x})\right)^{-1}\leq\left(\hat{H}_n'\right)^{-1},$$

which completes the proof.

We would also establish refinements related to arithmetic-harmonic means inequalities as follows:

**Corollary 4.** *Let the assumptions stated in Theorem* 6 *be true. Then following inequalities hold:*

$$\frac{1}{\hat{A}_n} \leq \sum_{i=1}^{n} w_i \left( \frac{1}{\hat{A}(\lambda, \mathbf{x})} \right) \leq \frac{1}{\hat{H}_n}, \tag{9}$$

$$\frac{1}{\hat{A}'_n} \leq \sum_{i=1}^{n} w_i \left( \frac{1}{\hat{A}'(\lambda, \mathbf{x})} \right) \leq \frac{1}{\hat{H}'_n}. \tag{10}$$

*Proof.* By applying convex function $f(x) = \frac{1}{x}, x \in (0, \frac{1}{2}]$ to Theorem 6 we get inequality (9). Similarly, by using convex function $f(x) = \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 6 we get inequality (10).

We establish a refinement of the difference of the arithmetic and harmonic means.

**Corollary 5.** *Let the assumptions stated in Theorem* 6 *be true. Then following inequalities hold:*

$$\frac{1}{\hat{A}_n} - \frac{1}{\hat{A}'_n} \leq \sum_{i=1}^{n} w_i \left( \frac{1}{\hat{A}(\lambda, \mathbf{x})} - \frac{1}{\hat{A}'(\lambda, \mathbf{x})} \right) \leq \frac{1}{\hat{H}_n} - \frac{1}{\hat{H}'_n}.$$

*Proof.* By applying convex function $f(x) = \frac{1}{x} - \frac{1}{1-x}, x \in (0, \frac{1}{2}]$ to Theorem 6 we obtain required result.

## 4. Cyclic mixed symmetric means

The Jensen's inequality and Jensen-Mercer inequality are much fertile to study about mixed means (see [11] and [21]). Let the assumptions stated in Theorem 6 be true. Then we define power mean of the order $r \in \mathbb{R}$, for positive $n$-tuple $\mathbf{x}$ as follows:

$$\hat{M}_r \left( x_{ij}, \ldots, x_{ij+l-1}; \lambda_1, \ldots, \lambda_l \right) = \begin{cases} \left( \sum_{j=1}^{m} a_j^r - \sum_{j=1}^{m-1} \sum_{k=0}^{l-1} \lambda_{k+1} x_{ij+k}^r \right)^{\frac{1}{r}}, & r \neq 0, \\ \dfrac{\prod_{j=1}^{m} a_j}{\prod_{j=1}^{m-1} \prod_{k=0}^{l-1} \left( x_{ij+k} \right)^{\lambda_{k+1}}}, & r = 0, \end{cases}$$

and cyclic mixed symmetric means corresponding to (6) is given as:

$$\hat{M}_{r,s}(\mathbf{x},\lambda)=\begin{cases}\left(\sum_{i=1}^{n} w_i \hat{M}_r^s\left(x_{ij},\cdots,x_{ij+l-1},\lambda_1,\cdots,\lambda_l\right)\right)^{\frac{1}{s}}, & s\neq 0,\\ \left(\prod_{i=1}^{n} \hat{M}_r\left(x_{ij},\cdots,x_{ij+l-1},\lambda_1,\cdots,\lambda_l\right)\right)^{w_i}, & s=0.\end{cases}$$

The standard power mean of order $r \in \mathbb{R}$ for the positive $n$-tuple $\mathbf{x}$ are defined as follows:

$$\hat{M}_r\left(\mathbf{x}\right)=\begin{cases}\left(\sum_{j=1}^{m} a_j^r-\sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}^r\right)^{\frac{1}{r}}, & r\neq 0,\\ \dfrac{\prod_{j=1}^{m} a_j}{\prod_{j=1}^{m-1}\prod_{i=1}^{n}\left(x_{ij}\right)^{w_i}}, & r=0.\end{cases}$$

**Corollary 6.** *For $r \leq 1$ and by considering the assumptions stated in $(C_1)$ for positive $m$-tuple $\mathbf{a}$ and $\mathbf{x}$, the following inequality hold.*

$$\hat{M}_r\left(\mathbf{x}\right) \leq \hat{M}_r\left(\mathbf{x},\lambda\right) \leq \hat{A}_n. \tag{11}$$

*For $r \geq 0$, the inequality (11) is reversed.*

*Proof.* For $r \leq 1, r \neq 0$, by applying the convex function $\phi(x)=x^{\frac{1}{r}}$ to Theorem 6 and replacing $a_j$ and $x_{ij}$ with $a_j^r$ and $x_{ij}^r$ respectively and for $r=0$ applying convex function $\phi(x)=\exp(x)$ to the Theorem 6, replacing $a_j$and $x_{ij}$ with $\ln a_j$ and $\ln xij$ respectively, we obtain 11.

If $r \geq 1$, then the function $\phi(x)=x^{\frac{1}{r}}$ is concave, so the inequalities in (11) is reversed.

Now, we define the bounds for power mean and cyclic mixed symmetric means as follows:

**Corollary 7.** *Let $r, s \in \mathbb{R}$ such that $r \leq s$ and considering the assumption stated in $(C_1)$ for positive $n$-tuple $\mathbf{x}$, following inequalities hold.*

$$\hat{M}_r\left(\mathbf{x}\right) \leq \hat{M}_{r,s}\left(\mathbf{x},\lambda\right) \leq \hat{M}_s\left(\mathbf{x}\right). \tag{12}$$

*Proof.* Let $r, s \neq 0$. By applying Theorem 6 for convex function $\phi\left(x\right)=x^{\frac{s}{r}}$, $x>0$ and by replacing $a_j$ and positive $n$-tuple $\mathbf{x}$ by $a_j^r$ and $(\mathbf{x}^r)$ respectively, and then raising the power $\frac{1}{s}$ we get,

$$\hat{M}_r\left(\mathbf{x}\right) \leq \hat{M}_{r,s}\left(\mathbf{x},\lambda\right) \leq \hat{M}_s\left(\mathbf{x}\right).$$

For $s = 0$ or $r = 0$, we obtain the required result by applying appropriate limits.

Let $\phi : [a, b] \to \mathbb{R}$ be a continuous and strictly monotone function then cyclic quasi-arithmetic means are defined as

$$\hat{M}_{\phi}(\mathbf{x}) := \phi^{-1}\left(\sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij})\right) \tag{13}$$

.

Let $\phi, \psi : [a, b] \to \mathbb{R}$ be strictly monotonic and continuous functions and under the assumptions stated in $(A_1)$ and $(A_2)$, we define generalized means with respect to (6) as follows:

$$M_{\phi,\psi}(\mathbf{x}, \lambda) = \phi^{-1}\left[\phi o\psi^{-1}\left(\sum_{j=1}^{m} a_j\right) - \sum_{i=1}^{n} w_i\left(\phi o\psi^{-1}\right)\left(\sum_{j=1}^{m-1}\sum_{k=0}^{l-1}\lambda_{k+1}\psi(x_{ij+k})\right)\right]. \tag{14}$$

Now, we establish the relation among generalized means and quasi-arithmetic means as follows:

**Corollary 8.** *Let assumptions $(C_1)$ and $(C_3)$ be true. Then*

$$\hat{M}_{\psi}(\mathbf{x}) \le \hat{M}_{\phi,\psi}(\mathbf{x}, \lambda) \le \hat{M}_{\phi}(\mathbf{x}),$$

*if either $\phi o\psi^{-1}$ is convex and $\psi$ is strictly increasing or $\phi o\psi^{-1}$ is concave and $\psi$ is strictly decreasing.*

*Proof.* By applying Theorem 6 to the convex function $\phi o\psi^{-1}$ and replacing $a_j$ by $\psi(a_j)$ and $n$-tuples $\mathbf{x}$ by $\psi(\mathbf{x})$, we get

$$\begin{aligned}\phi o\psi^{-1}&\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\psi(x_{ij})\right)\\ &\le \sum_{i=1}^{n} w_i\phi o\psi^{-1}\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{k=0}^{l-1}\sum_{j=1}^{m-1}\lambda_{k+1}\psi(x_{ij+k})\right)\\ &\le \sum_{j=1}^{m}\phi o\psi^{-1}(\psi(a_j)) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi o\psi^{-1}(\psi(x_{ij})),\end{aligned}$$

consequently,

$$\phi o\psi^{-1}\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\psi(x_{ij})\right)$$

$$\leq \sum_{i=1}^{n} w_i \phi o \psi^{-1}\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{k=0}^{l-1}\sum_{j=1}^{m-1}\lambda_{k+1}\psi(x_{ij+k})\right)$$
$$\leq \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij}),$$

by applying $\phi^{-1}$, we get

$$\phi^{-1}\phi o\psi^{-1}\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\psi(x_{ij})\right)$$
$$\leq \phi^{-1}\left(\sum_{i=1}^{n} w_i\phi o\psi^{-1}\left(\sum_{j=1}^{m}\psi(a_j) - \sum_{k=0}^{l-1}\sum_{j=1}^{m-1}\lambda_{k+1}\psi(x_{ij+k})\right)\right)$$
$$\leq \phi^{-1}\left(\sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij})\right),$$

and after some simplification we obtained required result.

## 5. Further Results and Future Work

Under the assumptions of Theorem 6, we define two positive linear functionals as

$$\begin{aligned}
\varphi_1(\mathbf{x},\lambda,\phi) &= \sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij}) \\
&- \sum_{i=1}^{n} w_i\phi\left(\sum_{j=1}^{m} a_j - \sum_{k=0}^{l-1}\sum_{j=1}^{m-1}\lambda_{k+1}x_{ij+k}\right) \\
\varphi_2(\mathbf{x},\lambda,\phi) &= \sum_{i=1}^{n} w_i\phi\left(\sum_{j=1}^{m} a_j - \sum_{k=0}^{l-1}\sum_{j=1}^{m-1}\lambda_{k+1}x_{ij+k}\right) \\
&- \left(\sum_{j=1}^{m}\phi(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i\phi(x_{ij})\right)
\end{aligned}$$

We can state different results for these two functionals defined above which may be listed as follows:

(i) We can state Lagrange type and Cauchy type mean value theorems and results related to $n-$exponential and logarithmic convexity by using similar techniques as stated in [3] and [14].

(ii) We can also state number of applications by using method of article [22].

(iii) We can state further results using technique of index set function with series of refinements and plenty of applications including Rado and Popovicu series of inequality by using method of [18] and [19].

(iv) We can also prove all inequalities in reverse direction by considering concave function instead of convex function by using simple relation: $f$ is concave iff and $-f$ is convex.

Here we state some future ideas for interested readers:

(i) One can also work on similar results as stated in this article for generalized convex functions including functions with nondecreasing increments and functions with nondecreasing increments of convex type see for example [1], [7] and [20].

(ii) One can also state similar results as stated in this article for arbitrary real numbers (not only non-negative real numbers) for example by working with assumptions of Jensen-Steffensen inequality.

(iii) One can also try its integral version as well.

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