



Efficient Zero Ring Labeling of Graphs

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Abstract. A zero ring is a ring in which the product of any two elements is zero, which is the additive identity. A zero ring labeling of a graph is an assignment of distinct elements of a zero ring to the vertices of the graph such that the sum of the labels of any two adjacent vertices is not the zero element in the ring. Given a zero ring labeling of a graph, if the cardinality of the set of distinct sums obtained from all adjacent vertices is equal to the maximum degree of the graph, then the zero ring labeling is *efficient*. In this paper, we showed the existence of an efficient zero ring labeling for some classes of trees and their disjoint union. In particular, we showed that an efficient zero ring labeling exists for some families of the following classes of trees: path graphs, star graphs, bistars, centipede graphs, caterpillars, spiders, lobsters, and rooted trees. We also showed results for other common classes of graphs.

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1. Introduction

Graph labeling is an assignment of labels to vertices or edges of a graph. Interesting questions naturally arise from this topic, so there has been a vast amount of literature that aimed to answer these questions while consequently introducing new problems.

In 2014, Acharya *et al* [1] introduced zero ring labeling. In this labeling, each vertex is assigned a unique label from a zero ring such that the sum of any two adjacent vertices is not zero, i.e., the additive identity of the zero ring. It was proved that every graph admits a zero ring labeling with respect to some zero ring.

The zero ring index of a graph, which is the smallest order of a zero ring in which the graph admits a zero ring labeling, was also studied for some well-known graphs. Pranjali *et al* [6] determined a necessary and sufficient condition for a finite graph of order n to attain an optimal zero ring index of n .

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Dela Rosa-Reynera [2] constructed optimal zero ring labelings for some classes of graphs. Included are labeling schemes for trees and cactus graphs. Moreover, the zero ring indices of graphs that result from some graph operations were also determined.

In this paper, we extend the notion of zero ring labeling further. Can a zero ring labeling be constructed for a graph such that the number of distinct sums is equal to some number? In particular, we want to determine a zero ring labeling where this number is as small as possible.

2. Preliminaries

2.1. Basic definitions

We now present some definitions used in this paper. For graph theory, [3, 5] are considered, while [4] is used for abstract algebra.

Definition 1. A **graph** $G = (V, E)$ is a pair of sets V and E such that E is a set of two-element subsets of V . The elements of V are the **vertices** of G , while the elements of E are the **edges** of G .

The vertex set of a graph G is denoted by $V(G)$, while its edge set is denoted by $E(G)$. We say that e is an edge *at* v if $v \in e$. An edge $\{u, v\}$ can be written as uv ; we say that uv *joins* u and v , and u and v are the *ends* of uv . The number of vertices of a graph G is its *order*, denoted by $|G|$.

The *degree* of a vertex v , denoted by $d(v)$, is the number of edges at v . Two vertices u and v in a graph G are *adjacent* if uv is an edge in G .

A graph G of order n is a *complete graph*, denoted by K_n , if for any two distinct vertices u and v in G , uv is an edge in G .

Definition 2. The **maximum degree** of a graph G , denoted by $\Delta(G)$, is the highest degree of a vertex in G .

Definition 3. A **path** in a graph is a subgraph $P = [v_1, v_2, \dots, v_n]$ such that $V(P) = \{v_1, v_2, \dots, v_n\}$ and $E(P) = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$, where v_1, v_2, \dots, v_n are distinct.

The number of edges in a path is its *length*. In a path $P = [v_1, v_2, \dots, v_n]$, the vertices v_1 and v_n are the *endvertices*, while v_2, v_3, \dots, v_{n-1} are the *inner vertices*. Moreover, we say that P is a path *between* v_1 and v_n . The *distance* between two vertices u and v in a graph G is the length of the shortest path between u and v .

Definition 4. A graph G is **connected** if for any vertices u and v in G , there is a path between u and v .

A connected subgraph of a graph G is a *component* of G if it is not a proper subgraph of any other connected subgraph in G .

2.2. Trees

Definition 5. A **tree** is a connected graph with no cycle. The vertices of degree one in a tree are its **leaves**.

Definition 6. A **path graph** P_n , where $n \geq 2$, is a tree with two leaves and $n - 2$ vertices with degree two.

Definition 7. A **star graph** S_n , where $n \geq 1$, is a tree with one vertex of degree $n - 1$ and all others with degree one.

The vertex with the maximum degree in a star graph is its *center*.

Definition 8. A **bistar** B_n , where $n \geq 1$, is a graph formed by joining the centers of two star graphs S_n .

Definition 9. An **n -centipede**, where $n \geq 1$, is a tree with vertex set $A \cup B$, where $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$, and edge set $E = \{a_i b_i : 1 \leq i \leq n\} \cup \{a_i a_{i+1} : 1 \leq i \leq n - 1\}$.

Definition 10. A **caterpillar** is a tree in which all vertices are within distance one of a central path.

We define a *hanging leaf* of a vertex in the central path as a leaf that is adjacent to it but not a vertex in the central path.

Example 1. The graph in Fig. 1 is a caterpillar with respect to any of the following central paths: $[c, d, e, f]$, $[a, c, d, e, f]$, $[g, c, d, e, f]$, $[c, d, e, f, j]$, $[a, c, d, e, f, j]$, or $[g, c, d, e, f, j]$. The hanging leaves of d are i , b , and h .

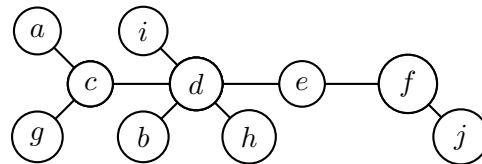


Figure 1: A caterpillar

Definition 11. A **spider** is a tree with a unique vertex of degree at least three and all others with degree at most two.

We define the *head* of a spider as the vertex with the maximum degree and a *leg* as a component of the graph that is obtained by removing the head; that is, a leg is a path.

Definition 12. A **lobster** is a tree in which all vertices are within distance two of a central path.

We define an *elbow* of a vertex in the central path of a lobster as a vertex that is adjacent to it and whose distance from the central path is one. Moreover, we define a *hanging leaf* of an elbow as a vertex that is adjacent to it and whose distance from the central path is two.

Definition 13. A *rooted tree* is a tree in which a particular vertex is designated as the *root*.

The *level* of a vertex in a rooted tree is its distance from the root, and the *height* of a rooted tree is the maximum level of a vertex. A *child* (plural: *children*) of a vertex of level n in a rooted tree is an adjacent vertex whose level is $n + 1$.

2.3. Common classes of graphs

We now define other common classes of graphs that are considered in this paper.

Definition 14. A *cycle graph* $C_n = [v_1, v_2, \dots, v_n, v_1]$ is a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\} \cup \{v_1v_n\}$, where $n \geq 3$.

Definition 15. A *complete bipartite graph* $K_{m,n}$ is a graph with vertex set $V = \{x_1, x_2, \dots, x_m\} \cup \{y_1, y_2, \dots, y_n\}$, where $m \geq 1$ and $n \geq 1$, and edge set $E = \{x_iy_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$.

Definition 16. A *fan graph* F_n is a graph that is obtained by joining all vertices of P_n to another vertex.

Definition 17. A *wheel graph* W_n is a graph that is obtained by joining all vertices of C_n to another vertex.

Definition 18. A *friendship graph* T_n is a graph that consists of n copies of C_3 having exactly one common vertex.

2.4. Efficient zero ring labeling

We now define a vertex labeling of a graph which was introduced by Acharya *et al* [1, 6] called zero ring labeling. Moreover, this paper introduces a variation of this vertex labeling called k -zero ring labeling and efficient zero ring labeling of graphs. In this study, the zero ring that will be used in the vertex labelings is the zero ring $M_2^0(\mathbb{Z}_n)$.

Definition 19. Let R be a ring with additive identity 0. If $ab = 0$ for any $a, b \in R$, then R is a *zero ring*.

Let R be a ring with additive identity 0. We denote by $M_2^0(R)$ the set of all 2×2 matrices of the form

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, \quad a \in R.$$

It can be verified that $M_2^0(R)$ is a ring under matrix addition and matrix multiplication with additive identity

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since for any $a, b \in R$,

$$\begin{bmatrix} a & -a \\ a & -a \end{bmatrix} \begin{bmatrix} b & -b \\ b & -b \end{bmatrix} = \begin{bmatrix} ab - ab & -ab + ab \\ ab - ab & -ab + ab \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

$M_2^0(R)$ is a zero ring. Since \mathbb{Z}_n is a ring, it follows that $M_2^0(\mathbb{Z}_n)$ is a zero ring. We use A_i to denote the matrix

$$\begin{bmatrix} i & -i \\ i & -i \end{bmatrix}, i \in \mathbb{Z}_n.$$

Definition 20. Let G be a graph, and let R^0 be a finite zero ring. An injective function $f : V(G) \rightarrow R^0$ is a **zero ring labeling** of G if $f(u) + f(v) \neq 0$ for every $uv \in E(G)$.

Given a zero ring labeling, if the order of the zero ring is equal to the order of the graph, we say that the labeling is *optimal*.

Example 2. Figure 2 shows a zero ring labeling of C_5 using $M_2^0(\mathbb{Z}_5)$. Since $|C_5| = |M_2^0(\mathbb{Z}_5)| = 5$, this labeling is optimal.

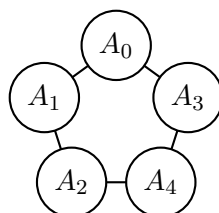


Figure 2: Zero ring labeling of C_5 using $M_2^0(\mathbb{Z}_5)$

Given a graph G with zero ring labeling f , we consider the set $K = \{f(u) + f(v) : uv \in E(G)\}$. Since it was shown in [1] that every graph admits a zero ring labeling, we are interested in finding zero ring labelings for G such that $|K|$ is as small as possible.

Definition 21. Let G be a graph with zero ring labeling f , and let $K = \{f(u) + f(v) : uv \in E(G)\}$. A zero ring labeling f of G is a **k -zero ring labeling** if $|K| = k$. If $|K| = \Delta(G)$, then the zero ring labeling is **efficient**.

Example 3. Figure 3 shows a 4-zero ring labeling of the diamond graph using $M_2^0(\mathbb{Z}_8)$. In this labeling, the set of sums is $K = \{A_1, A_2, A_5, A_7\}$ and thus $|K| = 4$. Figure 4 shows an efficient zero ring labeling of the butterfly graph G using $M_2^0(\mathbb{Z}_{10})$. In this labeling, the set of sums is $K = \{A_2, A_3, A_7, A_9\}$ and thus $|K| = \Delta(G) = 4$.

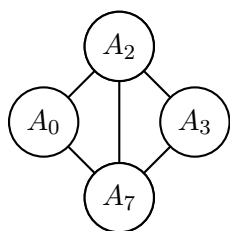


Figure 3: 4-zero ring labeling of the diamond graph using $M_2^0(\mathbb{Z}_8)$

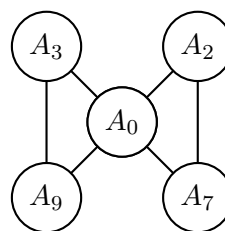


Figure 4: Efficient zero ring labeling of the butterfly graph using $M_2^0(\mathbb{Z}_{10})$

3. Results

Theorem 1. Let G be a graph, and let $f : G \rightarrow R^0$ be a zero ring labeling. If $K = \{f(u) + f(v) : uv \in E(G)\}$, then $\Delta(G) \leq |K| \leq |R^0| - 1$.

Proof. Since f is a zero ring labeling, $0 \notin K$. Furthermore, R^0 contains every sum $f(u) + f(v)$ where $uv \in E(G)$, hence $K \subset R^0$ and $|K| \leq |R^0| - 1$.

Let $v \in V(G)$ and let $N(v) = \{v_1, v_2, \dots, v_m\}$ be the set of vertices that are adjacent to v . Since f is injective, it follows that the number of distinct sums $f(v) + f(v_i)$ for $i = 1, 2, 3, \dots, m$ generated from edges that end at v is equal to $|N(v)| = m$, which is the degree of v . Clearly, the maximum degree of G is the minimum number of elements of K .

Theorem 2. If G is a graph with an efficient zero ring labeling, then any edge-induced subgraph H of G such that $\Delta(G) = \Delta(H)$ has an efficient zero ring labeling.

Proof. Let G be a graph with efficient zero ring labeling $f : V(G) \rightarrow R^0$, and let H be an edge-induced subgraph of G such that $\Delta(G) = \Delta(H)$. Consider the restriction function $f|_H$ of f to the vertex set of H ; that is, $f|_H(v) = f(v)$ for all $v \in V(H)$. Since f is injective, $f|_H$ is injective. Also, $0 \notin K = \{f(u) + f(v) : uv \in E(G)\}$ implies that $0 \notin K_H = \{f|_H(u) + f|_H(v) : uv \in E(H)\}$. It remains to show that $|K_H| = \Delta(H)$.

Since $\Delta(G) = \Delta(H)$, there exists at least one vertex v in $V(G)$ where $d(v) = \Delta(G)$ such that v is also in H where $d(v) = \Delta(G)$. Then $|K_H| = \Delta(G) = \Delta(H)$.

Theorem 3. Let G be a graph with n vertices, and let $f : V(G) \rightarrow R^0$ be a zero ring labeling. If f is optimal and $\Delta(G) = n - 1$, then f is efficient.

Proof. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is efficient, we need to show that $|K| = n - 1$.

Since f is optimal, $|R^0| = n$. By Theorem 1, $\Delta(G) \leq |K| \leq |R^0| - 1$. But $\Delta(G) = |R^0| - 1 = n - 1$. Therefore, $|K| = n - 1$.

We now look at the efficient zero ring labeling of some classes of trees and other common classes of graphs.

Theorem 4. Let G be a caterpillar in which each vertex in the central path has an equal number of hanging leaves. Then G has an efficient zero ring labeling.

Proof. Let $[w_1, w_2, \dots, w_n]$ denote the central path of a caterpillar G and suppose that each w_i , $1 \leq i \leq n$, has r hanging leaves. Denote the hanging leaves of w_i by w_i^j , where $j = 1, 2, \dots, r$.

Case 1: Suppose $n = 1$. In this case, $\Delta(G) = d(w_1) = r$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{r+1})$ such that $f(w_1) = A_0$ and $f(w_1^j) = A_j$.

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = r$ and $A_0 \notin K$. For the set of sums, we obtain

$$f(w_1) + f(w_1^j) = A_0 + A_j = A_j \neq A_0 \tag{2}$$

for $j = 1, 2, \dots, r$. Then $A_0 \notin K$. Moreover,

$$K = \{A_1, A_2, \dots, A_r\} \tag{3}$$

and thus $|K| = r$. Therefore, f is an efficient zero ring labeling of G .

Case 2: Suppose $n = 2$. In this case, $\Delta(G) = d(w_1) = d(w_2) = r + 1$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{2r+2})$ such that $f(w_i) = A_{i(r+1)}$ and $f(w_i^j) = A_{(i-1)(r+1)+j}$.

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = r + 1$ and $A_0 \notin K$. For the set of sums, we obtain

$$f(w_i) + f(w_i^j) = A_{i(r+1)} + A_{(i-1)(r+1)+j} = A_{2ir+2i-r-1+j} = A_{j-r-1} \tag{4}$$

for $j = 1, 2, \dots, r$, and

$$f(w_1) + f(w_2) = A_{r+1} + A_{2r+2} = A_{3r+3} = A_{r+1} \neq A_0 \tag{5}$$

It remains to show that $A_{j-r-1} \neq A_0$ for $j = 1, 2, \dots, r$. By substitution, we obtain the sums $A_{-r}, A_{1-r}, \dots, A_{-1}$, which are equal to $A_{r+2}, A_{r+3}, \dots, A_{2r+1}$, respectively. For $r + 2 \leq m \leq 2r + 1$, we have $0 < m < 2r + 2$, hence $m \not\equiv 0 \pmod{2r + 2}$. Thus, $A_m \neq A_0$ for $m = r + 2, r + 3, \dots, 2r + 1$. Then $A_0 \notin K$. Moreover,

$$K = \{A_{r+1}, A_{r+2}, A_{r+3}, \dots, A_{2r+1}\} \tag{6}$$

and thus $|K| = r + 1$. Therefore, f is an efficient zero ring labeling of G .

Case 3: Suppose $n \geq 3$. In this case, $\Delta(G) = d(w_i)$, where $i \neq 1$ and $i \neq n$. Then $\Delta(G) = r + 2$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{nr+n})$ such that

$$f(w_i) = \begin{cases} A_{(r+1)(\frac{2n-i-1}{2})} & \text{if } i \text{ is odd} \\ A_{(r+1)(\frac{i-2}{2})} & \text{if } i \text{ is even} \end{cases} \tag{7}$$

and

$$f(w_i^j) = \begin{cases} A_{(r+1)(\frac{i-3}{2})+j} & \text{if } i \text{ is odd} \\ A_{(r+1)(\frac{2n-i-2}{2})+j} & \text{if } i \text{ is even} \end{cases} \tag{8}$$

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = r + 2$ and $A_0 \notin K$. For adjacent vertices in the central path, we obtain the sums

$$f(w_i) + f(w_{i+1}) = A_{(r+1)\left(\frac{2n-i-1}{2}\right)} + A_{(r+1)\left(\frac{(i+1)-2}{2}\right)} = A_{(r+1)(n-1)} \tag{9}$$

if i is odd, and

$$f(w_i) + f(w_{i+1}) = A_{(r+1)\left(\frac{i-2}{2}\right)} + A_{(r+1)\left(\frac{2n-(i+1)-1}{2}\right)} = A_{(r+1)(n-2)} \tag{10}$$

if i is even. Note that $A_{(r+1)(n-1)} \neq A_0$; otherwise, either $r \equiv -1 \pmod{nr + n}$, which is a contradiction by definition of r , or $n \equiv 1 \pmod{nr + n}$, which is also a contradiction by definition of n in this case. Similarly, $A_{(r+1)(n-2)} \neq A_0$.

For pairs of adjacent hanging leaf and vertex in the central path, we obtain the sums

$$f(w_i) + f(w_i^j) = A_{(r+1)\left(\frac{2n-i-1}{2}\right)} + A_{(r+1)\left(\frac{i-3}{2}\right)+j} = A_{(r+1)(n-2)+j} \tag{11}$$

for $j = 1, 2, \dots, r$ if i is odd, and

$$f(w_i) + f(w_i^j) = A_{(r+1)\left(\frac{i-2}{2}\right)} + A_{(r+1)\left(\frac{2n-i-2}{2}\right)+j} = A_{(r+1)(n-2)+j} \tag{12}$$

for $j = 1, 2, \dots, r$ if i is even. We show that $A_{(r+1)(n-2)+j} \neq A_0$. Assume that $A_{(r+1)(n-2)+j} = A_0$ for some j . But this implies that $j \equiv 2r + 2 \pmod{nr + n}$. This is a contradiction since n is at least 3 in this case, and j cannot be greater than r by its definition. Then $A_0 \notin K$. Moreover,

$$K = \{A_{(r+1)(n-2)}, A_{(r+1)(n-2)+1}, \dots, A_{(r+1)(n-2)+r}, A_{(r+1)(n-1)}\} \tag{13}$$

and thus $|K| = r + 2$. Therefore, f is an efficient zero ring labeling of G .

Example 4. Figure 5 shows an efficient zero ring labeling of a caterpillar G with a central path with four vertices, where each vertex in the central path has three hanging leaves, using $M_2^0(\mathbb{Z}_{16})$. In this labeling, the set of sums is $K = \{A_8, A_9, A_{10}, A_{11}, A_{12}\}$ and thus $|K| = \Delta(G) = 5$.

Corollary 1. A path graph has an efficient zero ring labeling.

Proof. Let G be the path graph P_n . Consider G as its own central path. Then G is a caterpillar with a central path with n vertices, where each vertex in the central path has no hanging leaf. By Theorem 4, G has an efficient zero ring labeling.

Example 5. Figure 6 shows an efficient zero ring labeling of P_{10} using $M_2^0(\mathbb{Z}_{10})$. In this labeling, the set of sums is $K = \{A_8, A_9\}$ and thus $|K| = \Delta(P_{10}) = 2$.

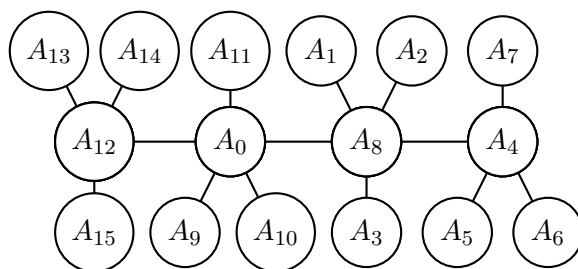


Figure 5: Efficient zero ring labeling of a caterpillar using $M_2^0(\mathbb{Z}_{16})$

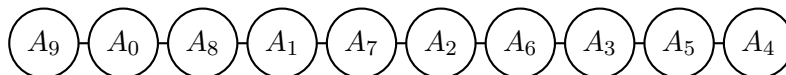


Figure 6: Efficient zero ring labeling of P_{10} using $M_2^0(\mathbb{Z}_{10})$

Corollary 2. *A star graph has an efficient zero ring labeling.*

Proof. Let G be the star graph S_n . Consider the center of G as its central path. Then G is a caterpillar with a central path with one vertex, and this vertex has $n - 1$ hanging leaves. Vacuously, each vertex in the central path of G has an equal number of hanging leaves. By Theorem 4, G has an efficient zero ring labeling.

Example 6. *Figure 7 shows an efficient zero ring labeling of S_9 using $M_2^0(\mathbb{Z}_9)$. In this labeling, the set of sums is $K = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ and thus $|K| = \Delta(S_9) = 8$.*

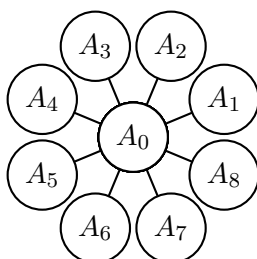


Figure 7: Efficient zero ring labeling of S_9 using $M_2^0(\mathbb{Z}_9)$

Corollary 3. *A bistar has an efficient zero ring labeling.*

Proof. Let G be the bistar B_n . Consider the path joining the centers of the two star graphs S_n in G as its central path. Then G is a caterpillar with a central path with two vertices, where each vertex in the central path has $n - 1$ hanging leaves. By Theorem 4, G has an efficient zero ring labeling.

Example 7. *Figure 8 shows an efficient zero ring labeling of B_9 using $M_2^0(\mathbb{Z}_{18})$. In this labeling, the set of sums is $K = \{A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}\}$ and thus $|K| = \Delta(B_9) = 9$.*

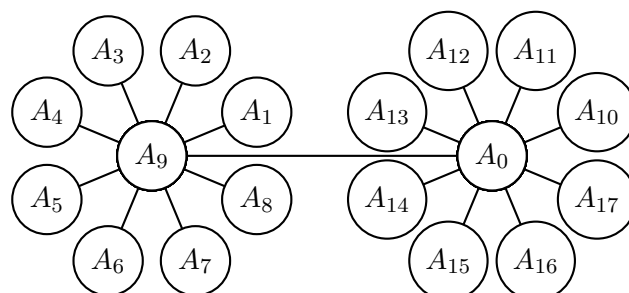


Figure 8: Efficient zero ring labeling of B_9 using $M_2^0(\mathbb{Z}_{18})$

Theorem 5. *A centipede graph has an efficient zero ring labeling.*

Proof. Let G be an n -centipede. By definition of a centipede, we can find a central path in G such that each vertex in the central path has 1 hanging leaf. By Theorem 4, G has an efficient zero ring labeling.

Example 8. *Figure 9 shows an efficient zero ring labeling of a 6-centipede using $M_2^0(\mathbb{Z}_{12})$. In this labeling, the set of sums is $K = \{A_8, A_9, A_{10}\}$ and thus $|K| = 3$, which is the maximum degree of the graph.*

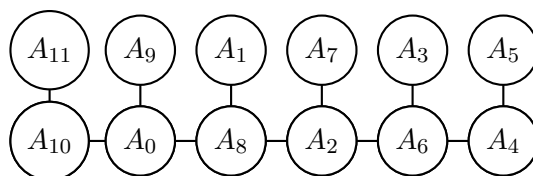


Figure 9: Efficient zero ring labeling of a 6-centipede using $M_2^0(\mathbb{Z}_{12})$

Lemma 1. *Let G be a caterpillar with at least 3 vertices. Then G is a caterpillar with respect to some central path $[w_1, w_2, \dots, w_n]$, where $n \geq 3$ and w_1 and w_n have no hanging leaf.*

Proof. Let G be a caterpillar with at least 3 vertices, and let $[v_1, v_2, \dots, v_m]$ denote its central path. Suppose each vertex v_i has r_i hanging leaves, and let v_i^j , where $j = 1, 2, \dots, r_i$, denote the hanging leaves of v_i .

Consider v_1, v_2, \dots, v_m as vertices in a central path P , along with v_1^1 if $r_1 \geq 1$, and $v_m^{r_m}$ if $r_m \geq 1$. Thus, P is one of the following: $[v_1, v_2, \dots, v_m]$, $[v_1^1, v_1, v_2, \dots, v_m]$, $[v_1, v_2, \dots, v_m, v_m^{r_m}]$, $[v_1^1, v_1, v_2, \dots, v_m, v_m^{r_m}]$. In any case, the number of vertices in P is at least three, and its endvertices have no hanging leaf. Clearly, each vertex in G is within distance one from P and thus G is a caterpillar with respect to central path P .

Theorem 6. *A caterpillar has an efficient zero ring labeling.*

Proof. Let G be a caterpillar with at least three vertices. By Lemma 1, G is a caterpillar with respect to some central path $[w_1, w_2, \dots, w_n]$, where $n \geq 3$ and w_1 and w_n have no hanging leaf. For $i = 2, 3, \dots, n - 1$, suppose w_i has r_i hanging leaves, and let w_i^j , where $j = 1, 2, \dots, r_i$, denote the hanging leaves of w_i .

Suppose r_k is the maximum number of hanging leaves of a vertex in the central path; that is, $r_k \geq r_i$ for $i = 2, 3, \dots, n - 1$. Then $\Delta(G) = r_k + 2$.

Consider a caterpillar H with a central path with n vertices, and where each vertex in the central path has r_k hanging leaves. Let $[v_1, v_2, \dots, v_n]$ denote its central path, and let v_i^j , where $j = 1, 2, \dots, r_k$, denote the hanging leaves of v_i . By Theorem 4, H has an efficient zero ring labeling h .

Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{nr_k+n-r_k})$ such that $f(w_i) = h(v_i)$ and $f(w_i^j) = h(w_i^j)$. Since h is injective, it follows that f is also injective. G is an edge-induced subgraph of H , so $A_0 \notin K_H = \{h(u) + h(v) : uv \in E(H)\}$ implies that $A_0 \notin K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , it remains to show that $|K| = \Delta(G)$. Using the labeling in the proof of Theorem 4, we obtain

$$K = \{A_{(r_k+1)(n-2)}, A_{(r_k+1)(n-2)+1}, \dots, A_{(r_k+1)(n-2)+r_k}, A_{(r_k+1)(n-1)}\}. \tag{14}$$

Thus, $|K| = r_k + 2 = \Delta(G)$.

Example 9. Figure 10 shows an efficient zero ring labeling of a caterpillar G with maximum degree $\Delta(G) = 6$ using $M_2^0(\mathbb{Z}_{31})$. In this labeling, the set of sums is $K = \{A_{15}, A_{16}, A_{17}, A_{18}, A_{19}, A_{20}\}$ and thus $|K| = \Delta(G) = 6$.

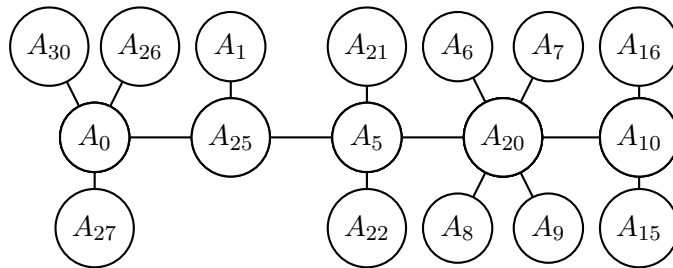


Figure 10: Efficient zero ring labeling of a caterpillar using $M_2^0(\mathbb{Z}_{31})$

Theorem 7. A spider has an efficient zero ring labeling.

Proof. Let G be a spider with n legs, where $n \geq 3$. Then $\Delta(G) = n$. Let the head of G be denoted by w , and let $w_1^1, w_2^1, \dots, w_n^1$ denote the vertices that are adjacent to w . Suppose each leg containing w_i^1 has r_i vertices, and let $[w_i^1, w_i^2, \dots, w_i^{r_i}]$ denote the leg containing w_i^1 .

Suppose $r_k = \max\{r_1, r_2, \dots, r_n\}$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{(n-1)(2r_k+1)})$ such that $f(w) = A_0$,

$$f(w_i^j) = \begin{cases} A_{(n-1)\left(\frac{2r_k-j+1}{2}\right)+i-1} & \text{if } j \text{ is odd} \\ A_{(n-1)\left(\frac{j}{2}\right)-i+1} & \text{if } j \text{ is even} \end{cases} \tag{15}$$

for $i = 1, 2, \dots, n - 1$, and

$$f(w_n^j) = \begin{cases} A_{(n-1)\left(\frac{2r_k+j+1}{2}\right)} & \text{if } j \text{ is odd} \\ A_{(n-1)\left(\frac{4r_k-j+2}{2}\right)} & \text{if } j \text{ is even} \end{cases} . \tag{16}$$

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = n$ and $A_0 \notin K$. For pairs of adjacent head and vertex in a leg, we obtain the sums

$$f(w) + f(w_i^1) = A_0 + A_{(n-1)\left(\frac{2r_k-1+1}{2}\right)+i-1} = A_{(n-1)(r_k)+i-1} \tag{17}$$

for $i = 1, 2, \dots, n - 1$, and

$$f(w) + f(w_n^1) = A_0 + A_{(n-1)(r_k+1)} = A_{(n-1)(r_k+1)} = A_{(n-1)(r_k)+n-1}. \tag{18}$$

Thus, we have the sums

$$A_{(n-1)(r_k)}, A_{(n-1)(r_k)+1}, A_{(n-1)(r_k)+2}, \dots, A_{(n-1)(r_k)+n-2}, A_{(n-1)(r_k)+n-1}.$$

For adjacent vertices in a leg, we obtain the sums

$$\begin{aligned} f(w_i^j) + f(w_i^{j+1}) &= A_{(n-1)\left(\frac{2r_k-j+1}{2}\right)+i-1} + A_{(n-1)\left(\frac{j+1}{2}\right)-i+1} \\ &= A_{(n-1)(r_k+1)} = A_{(n-1)(r_k)+n-1} \end{aligned} \tag{19}$$

if j is odd, and

$$f(w_i^j) + f(w_i^{j+1}) = A_{(n-1)\left(\frac{j}{2}\right)-i+1} + A_{(n-1)\left(\frac{2r_k-(j+1)+1}{2}\right)+i-1} = A_{(n-1)(r_k)} \tag{20}$$

if j is even, for $i = 1, 2, \dots, n - 1$. Moreover, we have

$$\begin{aligned} f(w_n^j) + f(w_n^{j+1}) &= A_{(n-1)\left(\frac{2r_k+j+1}{2}\right)} + A_{(n-1)\left(\frac{4r_k-(j+1)+2}{2}\right)} = A_{(n-1)(3r_k+1)} \\ &= A_{(n-1)(r_k)} \end{aligned} \tag{21}$$

if j is odd, and

$$\begin{aligned} f(w_n^j) + f(w_n^{j+1}) &= A_{(n-1)\left(\frac{4r_k-j+2}{2}\right)} + A_{(n-1)\left(\frac{2r_k+(j+1)+1}{2}\right)} = A_{(n-1)(3r_k+2)} \\ &= A_{(n-1)(r_k+1)} = A_{(n-1)(r_k)+n-1} \end{aligned} \tag{22}$$

if j is even. Then

$$K = \{A_{(n-1)(r_k)}, A_{(n-1)(r_k)+1}, \dots, A_{(n-1)(r_k)+n-2}, A_{(n-1)(r_k)+n-1}\}, \tag{23}$$

hence $|K| = n$. For $m = 0, 1, 2, \dots, n - 1$, it is clear that $(n - 1)(r_k) + m > 0$ and $(n - 1)(r_k) + m < (n - 1)(r_k) + (n - 1)(r_k + 1) = (n - 1)(2r_k + 1)$. Then $A_0 \notin K$. Therefore, f is an efficient zero ring labeling of G .

Example 10. Figure 11 shows an efficient zero ring labeling of a spider with 4 legs using $M_2^0(\mathbb{Z}_{33})$. In this labeling, the set of sums is $K = \{A_{15}, A_{16}, A_{17}, A_{18}\}$ and thus $|K| = 4$.

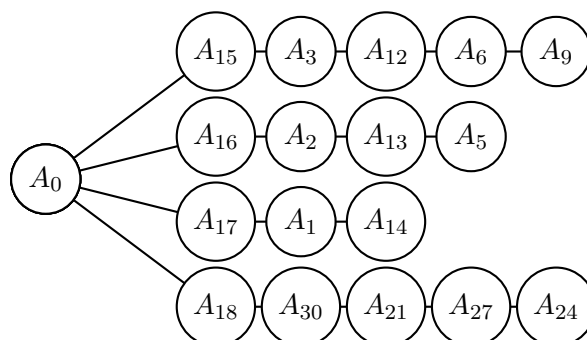


Figure 11: Efficient zero ring labeling of a spider using $M_2^0(\mathbb{Z}_{33})$

Lemma 2. Let G be a lobster with at least three vertices. Then G is a lobster with respect to some central path $[v_1, v_2, \dots, v_m]$, where $m \geq 3$ and v_1 and v_m have no elbow.

Proof. Let G be a lobster with at least three vertices, and let $[w_1, w_2, \dots, w_n]$ denote its central path. Suppose each vertex w_i has r_i elbows, and let $w_{i,j}$, where $j = 1, 2, \dots, r_i$, denote the elbows of w_i . Suppose each elbow $w_{i,j}$ has $s_{i,j}$ hanging leaves, and let $w_{i,j,k}$, where $k = 1, 2, \dots, s_{i,j}$, denote the hanging leaves of $w_{i,j}$.

Consider w_1, w_2, \dots, w_n as vertices in a central path P , along with $w_{1,1}$ if $r_1 \geq 1$, $w_{1,1,1}$ if $s_{1,1} \geq 1$, $w_{n,1}$ if $r_n \geq 1$, and $w_{n,1,1}$ if $s_{n,1} \geq 1$. Thus, P is one of the following:

- $[w_1, w_2, \dots, w_n]$
- $[w_{1,1}, w_1, w_2, \dots, w_n]$
- $[w_{1,1,1}, w_{1,1}, w_1, w_2, \dots, w_n]$
- $[w_1, w_2, \dots, w_n, w_{n,1}]$
- $[w_{1,1}, w_1, w_2, \dots, w_n, w_{n,1}]$
- $[w_{1,1,1}, w_{1,1}, w_1, w_2, \dots, w_n, w_{n,1}]$
- $[w_1, w_2, \dots, w_n, w_{n,1}, w_{n,1,1}]$

- $[w_{1,1}, w_1, w_2, \dots, w_n, w_{n,1}, w_{n,1,1}]$
- $[w_{1,1,1}, w_{1,1}, w_1, w_2, \dots, w_n, w_{n,1}, w_{n,1,1}]$

In any case, the number of vertices in P is at least three, and its endvertices have no elbow. Clearly, each vertex in G is within distance two from P and thus G is a lobster with respect to central path P .

Theorem 8. *Let G be a lobster in which each elbow has at most one hanging leaf. Then G has an efficient zero ring labeling.*

Proof. Let G be a lobster with at least three vertices in which each elbow has at most one hanging leaf. By Lemma 2, G is a lobster with respect to some central path $[w_1, w_2, \dots, w_n]$, where $n \geq 3$ and w_1 and w_n have no elbow. Clearly, each elbow has at most one hanging leaf with respect to this central path.

Suppose each vertex w_i has r_i elbows, and let $w_{i,j}$, where $j = 1, 2, \dots, r_i$, denote the elbows of w_i . If it exists, let $w_{i,j,1}$ denote the hanging leaf of $w_{i,j}$.

Suppose r_k is the maximum number of elbows of a vertex in the central path; that is, $r_k \geq r_i$ for $i = 1, 2, \dots, n$. Then $\Delta(G) = r_k + 2$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{2nr_k+n-2r_k})$ such that

$$f(w_i) = \begin{cases} A_{(2r_k+1)\binom{2n-i-1}{2}} & \text{if } i \text{ is odd} \\ A_{(2r_k+1)\binom{i-2}{2}} & \text{if } i \text{ is even} \end{cases}, \tag{24}$$

$$f(w_{i,j}) = \begin{cases} A_{(2r_k+1)\binom{i-3}{2}+j} & \text{if } i \text{ is odd} \\ A_{(2r_k+1)\binom{2n-i-2}{2}+j} & \text{if } i \text{ is even.} \end{cases}, \tag{25}$$

and

$$f(w_{i,j,1}) = \begin{cases} A_{(2r_k+1)\binom{2n-i+1}{2}-j} & \text{if } i \text{ is odd} \\ A_{(2r_k+1)\binom{i}{2}-j} & \text{if } i \text{ is even.} \end{cases}. \tag{26}$$

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = r_k + 2$ and $A_0 \notin K$. For adjacent vertices in the central path, we obtain the sums

$$f(w_i) + f(w_{i+1}) = A_{(2r_k+1)\binom{2n-i-1}{2}} + A_{(2r_k+1)\binom{(i+1)-2}{2}} = A_{(2r_k+1)(n-1)} \tag{27}$$

if i is odd, and

$$f(w_i) + f(w_{i+1}) = A_{(2r_k+1)\binom{2n-i-1}{2}} + A_{(2r_k+1)\binom{2n-(i+1)-1}{2}} = A_{(2r_k+1)(n-2)} \tag{28}$$

if i is even. Thus, there are only two distinct sums obtained for adjacent vertices in the central path. For pairs of adjacent elbow and vertex in the central path, we obtain the sums

$$f(w_i) + f(w_{i,j}) = A_{(2r_k+1)\binom{i-2}{2}} + A_{(2r_k+1)\binom{i-3}{2}+j} = A_{(2r_k+1)(n-2)+j} \tag{29}$$

for $j = 1, 2, \dots, r_i$ if i is odd, and

$$f(w_i) + f(w_{i,j}) = A_{(2r_k+1)\binom{i-2}{2}} + A_{(2r_k+1)\binom{i-3}{2}+j} = A_{(2r_k+1)(n-2)+j} \tag{30}$$

for $j = 1, 2, \dots, r_i$ if i is even. Since r_k is the maximum number of elbows of a vertex in the central path, the obtained sums for pairs of adjacent elbow and vertex in the central path are $A_{(2r_k+1)(n-2)+1}, A_{(2r_k+1)(n-2)+2}, \dots, A_{(2r_k+1)(n-2)+r_k}$. For pairs of adjacent hanging leaf and elbow, we obtain the sums

$$f(w_{i,j}) + f(w_{i,j,1}) = A_{(2r_k+1)\binom{i-3}{2}+j} + A_{(2r_k+1)\binom{2n-i+1}{2}-j} = A_{(2r_k+1)(n-1)} \tag{31}$$

if i is odd, and

$$f(w_{i,j}) + f(w_{i,j,1}) = A_{(2r_k+1)\binom{2n-i-2}{2}+j} + A_{(2r_k+1)\binom{i}{2}-j} = A_{(2r_k+1)(n-1)} \tag{32}$$

if i is even. Thus, the sum of any pair of adjacent elbow and hanging leaf is equal. Then

$$K = \{A_{(2r_k+1)(n-2)}, A_{(2r_k+1)(n-2)+1}, \dots, A_{(2r_k+1)(n-2)+r_k}, A_{(2r_k+1)(n-1)}\}, \tag{33}$$

hence $|K| = r_k + 2$.

To show that $A_0 \notin K$, it is sufficient to show that $(2r_k + 1)(n - 2), (2r_k + 1)(n - 2) + 1, (2r_k + 1)(n - 2) + 2, \dots, (2r_k + 1)(n - 2) + r_k, (2r_k + 1)(n - 1)$ are greater than 0 but less than $2nr_k + n - 2r_k$. The central path has at least 3 vertices, so $n \geq 3$, and r_k is at least zero by its definition. Then $(2r_k + 1)(n - 2) + m$ for $m = 0, 1, \dots, r_k$ and $(2r_k + 1)(n - 1)$ are greater than zero. Moreover, $(2r_k + 1)(n - 2) + m < (2r_k + 1)(n - 2) + (2r_k + 1) = (2r_k + 1)(n - 1)$ for $m = 0, 1, \dots, r_k$. Since $(2r_k + 1)(n - 1) = (2r_k + 1)(n) - 2r_k - 1 < (2r_k + 1)(n) - 2r_k = 2nr_k + n - 2r_k$, it follows that $A_0 \notin K$.

Example 11. *Figure 12 shows an efficient zero ring labeling of a lobster G in which each elbow has at most one hanging leaf. In this labeling, the set of sums is $K = \{A_{54}, A_{55}, A_{56}, A_{57}, A_{58}, A_{63}\}$ and thus $|K| = 6 = \Delta(G)$.*

Theorem 9. *Let G be a rooted tree whose root has the maximum degree. If the height of G is at most two, then G has an efficient zero ring labeling.*

Proof. Let G be a rooted tree of height two and with root w such that $d(w) = n = \Delta(G) = n$. If n is one or two, then G is a caterpillar. Thus, by Theorem 6, G has an efficient zero ring labeling.

Suppose n is at least three, and there exists at least one vertex of level two. Denote the children of w by w_1, w_2, \dots, w_n . Since w has n children and $\Delta(G) = n$, every child of w has at most $n - 1$ children. Suppose w_i has r_i children, and let $w_{i,j}$, where $j = 1, 2, \dots, r_i$ denote the children of w_i .

Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{2^{n+1}-1})$ such that $f(w) = A_0, f(w_i) = A_{2^{i+1}-3}$, and

$$f(w_{i,j}) = \begin{cases} A_{2^{i+j+1}-2^{i+1}} & \text{if } i + j < n + 1 \\ A_{2^{n+1}-2^{i+1}+3} & \text{if } i + j = n + 1 \\ A_{2^{n+1}-2^{i+1}+2^{i+j}-n+1-1} & \text{if } i + j > n + 1 \end{cases} \tag{34}$$

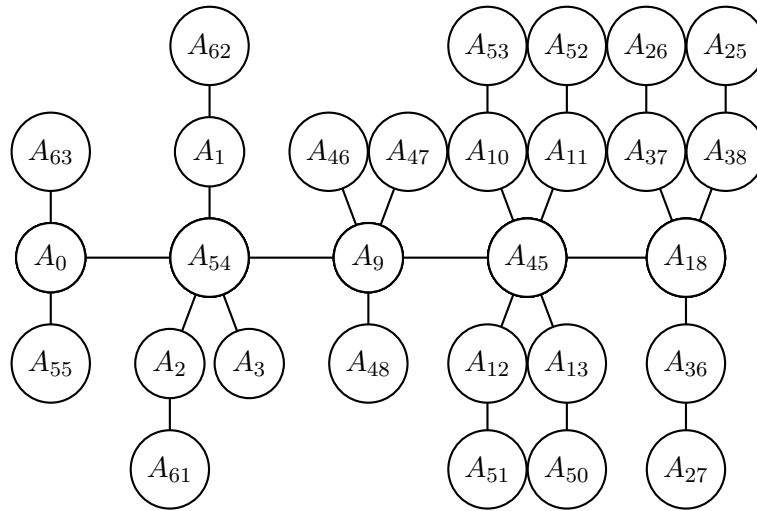


Figure 12: Efficient zero ring labeling of a lobster using $M_2^0(\mathbb{Z}_{64})$

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = \Delta(G) = n$ and $A_0 \notin K$.

For pairs of adjacent root and vertex of level one, we obtain

$$f(w) + f(w_i) = A_0 + A_{2^{i+1}-3} = A_{2^{i+1}-3} \tag{35}$$

for $i = 1, 2, \dots, n$. Then we have the sums $A_1, A_5, \dots, A_{2^{n+1}-3}$. For pairs of adjacent vertex of level one and vertex of level two, we have three cases.

Case 1: Suppose $i + j < n + 1$. Then we obtain

$$f(w_i) + f(w_j^i) = A_{2^{i+1}-3} + A_{2^{i+j+1}-2^{i+1}} = A_{2^{i+j+1}-3}. \tag{36}$$

Since each of i and j are at least one, we have the sums $A_{2^{m+1}-3}$ for $m = 2, 3, \dots, n$. These are $A_5, A_{13}, \dots, A_{2^{n+1}-3}$. Note that these sums are same with those from pairs of adjacent root and vertex of level one, only that A_1 is not included.

Case 2: Suppose $i + j = n + 1$. Then we obtain

$$f(w_i) + f(w_j^i) = A_{2^{i+1}-3} + A_{2^{n+1}-2^{i+1}+3} = A_{2^{n+1}} = A_1. \tag{37}$$

Thus, in this case, there is only one possible sum, which is A_1 .

Case 3: Suppose $i + j > n + 1$. Then we obtain

$$\begin{aligned} f(w_i) + f(w_j^i) &= A_{2^{i+1}-3} + A_{2^{n+1}-2^{i+1}+2^{i+j-n+1}-1} = A_{2^{n+1}+2^{i+j-n+1}-4} \\ &= A_{2^{i+j-n+1}-3}. \end{aligned} \tag{38}$$

Since $i + j > n + 1$, it follows that $i + j$ is at least $n+2$. Also, i is at most n and j is at most $n - 1$, so $i + j$ is at most $2n - 1$. Hence, $2^{i+j-n+1} - 3 \geq 2^{(n+2)-n+1} - 3 = 2^3 - 3 = 5$ and $2^{i+j-n+1} - 3 \leq 2^{(2n-1)-n+1} - 3 = 2^n - 3$. Then we have the sums $A_5, A_{13}, \dots, A_{2^n-3}$.

It can be observed that these sums are same with those from pairs of adjacent root and vertex of level one, only that A_1 and $A_{2^{n+1}-3}$ are not included.

Therefore,

$$K = \{A_1, A_5, \dots, A_{2^{n+1}-3}\}, \tag{39}$$

hence $|K| = n$. For $m = 1, 5, \dots, 2^{n+1} - 3$, it is also clear that $m > 0$ and $m < 2^{n+1} - 1$. Thus, $A_0 \notin K$.

Example 12. Figure 13 shows an efficient zero ring labeling of a rooted tree G with a height of two. In this labeling, the set of sums is $K = \{A_1, A_5, A_{13}, A_{29}\}$ and thus $|K| = 4 = \Delta(G)$.

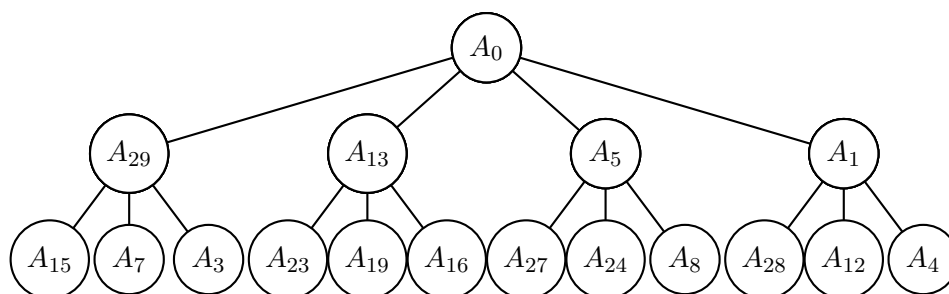


Figure 13: Efficient zero ring labeling of a rooted tree with a height of two using $M_2^0(\mathbb{Z}_{31})$

Theorem 10. Let G be a disjoint union of a finite number of caterpillars. Then G has an efficient zero ring labeling.

Proof. Let G be a disjoint union of p caterpillars. We show that G has an efficient zero ring labeling.

Case 1: The order of each caterpillar is one or two; that is, each caterpillar is either the trivial graph or P_2 .

If all p caterpillars are trivial, then $\Delta(G) = 0$. We label the vertices by $A_0, A_1, A_2, \dots, A_{p-1}$. Clearly, the labeling is injective. Moreover, the set of sums is empty and thus $A_0 \notin K$. We are done.

Suppose at least one caterpillar is not trivial. Then $\Delta(G) = 1$. Let G_1, G_2, \dots, G_m denote the caterpillars with two vertices, and let $G_{m+1}, G_{m+2}, \dots, G_p$ denote the trivial caterpillars, where $1 \leq m \leq p$. Let $V(G_i) = \{w_{i-1}, w_{2m-i+1}\}$ for $i = 1, 2, \dots, m$, and let $V(G_i) = \{w_{i+m}\}$ for $i = m + 1, m + 2, \dots, p$.

Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{m+p+1})$ such that $f(w_i) = A_i$. Let $K = \{f(u) + f(v) : uv \in E(G)\}$.

Clearly, f is injective. To show that f is an efficient zero ring labeling of G , we need to show that $|K| = \Delta(G) = 1$ and $A_0 \notin K$.

We obtain the sums

$$f(w_{i-1}) + f(w_{2m-i+1}) = A_{i-1} + A_{2m-i+1} = A_{2m} \tag{40}$$

for $i = 1, 2, \dots, m$. Then $K = \{A_{2m}\}$ and thus $|K| = 1$. Also, $A_{2m} \neq A_0$ since $m > 0$ and $m < m + p + 1 \leq m + m + 1 = 2m + 1$.

Case 2: The order of at least one caterpillar is at least three. Applying Lemma 1, we denote the vertices of each caterpillar such that the endvertices of the central path have no hanging leaf.

Let G_1, G_2, \dots, G_p denote the caterpillars. Suppose the length of the central path of G_i has s_i vertices. Denote the central path of G_1 by $[w_1, w_2, \dots, w_{s_1}]$, and the central path of G_i by $[w_{q+1}, w_{q+2}, \dots, w_{q+s_i}]$ for $i = 2, 3, \dots, p$, where $q = s_1 + s_2 + \dots + s_{i-1}$. Suppose the number of hanging leaves of w_i is r_i , and denote the hanging leaves of w_i by w_i^j , where $j = 1, 2, \dots, r_i$.

Suppose the maximum number of hanging leaves of a vertex in G is r_k . Since the endvertices in the central path of each caterpillar in G have no hanging leaf and there is at least one caterpillar with at least three vertices, it follows that $\Delta(G) = r_k + 2$.

Consider a caterpillar H such that $V(H) = V(G)$ and $E(H) = E(G) \cup \{\{w_{q_i}, w_{q_i} + 1\} : i = 1, 2, \dots, p - 1\}$, where $q_i = s_1 + s_2 + \dots + s_i$. By Theorem 6, H has an efficient zero ring labeling h .

Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{nr_k+n-r_k})$ such that $f(a) = h(a)$ for all $a \in V(G)$. Since h is injective, it follows that f is also injective. G is an edge-induced subgraph of H , so $A_0 \notin K_H = \{h(u) + h(v) : uv \in E(H)\}$ implies that $A_0 \notin K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , it remains to show that $|K| = \Delta(G) = r_k + 2$.

Using the labeling in the proof of Theorem 4, we obtain

$$K = \{A_{(r_k+1)(q-2)}, A_{(r_k+1)(q-2)+1}, \dots, A_{(r_k+1)(q-2)+r_k}, A_{(r_k+1)(q-1)}\}, \tag{41}$$

where $q = s_1 + s_2 + \dots + s_p$. Thus, $|K| = r_k + 2 = \Delta(G)$.

Example 13. Figure 14 shows an efficient zero ring labeling of a disjoint union of three caterpillars using $M_2^0(\mathbb{Z}_{66})$. In this labeling, the set of sums is $K = \{A_{60}, A_{61}, A_{62}, A_{63}, A_{64}, A_{65}\}$ and thus $|K| = 6$, which is the maximum degree of the graph.

Theorem 11. Let G be a disjoint union of a finite number of lobsters in which each elbow has at most one hanging leaf. Then G has an efficient zero ring labeling.

Proof. Let G be a disjoint union of p lobsters. We show that G has an efficient zero ring labeling.

If the order of each lobster is one or two, then by Theorem 10, G has an efficient zero ring labeling. Suppose the order of at least one lobster is at least three. Applying Lemma 2, we denote the vertices of each lobster such that the endvertices of the central path have no hanging leaf.

Let G_1, G_2, \dots, G_p denote the disjoint lobster graphs. Suppose the length of the central path of G_i has s_i vertices. Denote the central path of G_1 by $[w_1, w_2, \dots, w_{s_1}]$, and the central path of G_i by $[w_{q+1}, w_{q+2}, \dots, w_{q+s_i}]$ for $i = 2, 3, \dots, p$, where $q = s_1 + s_2 + \dots + s_{i-1}$.

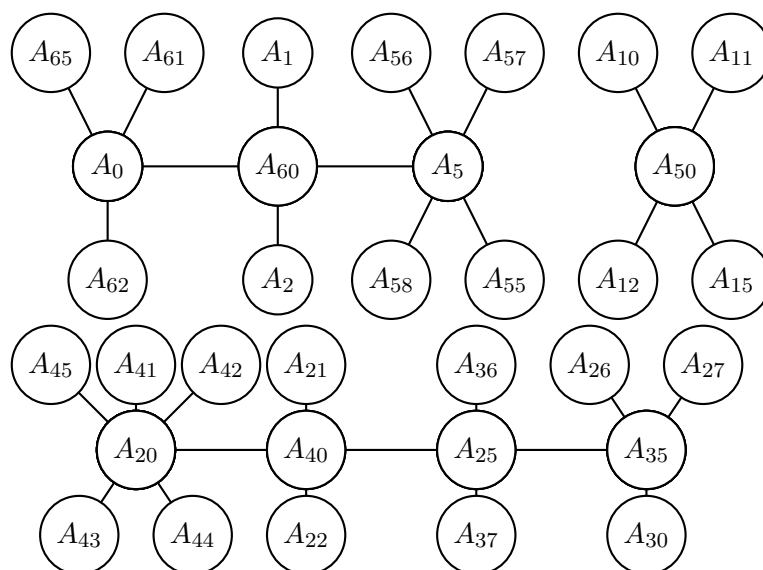


Figure 14: Efficient zero ring labeling of a disjoint union of three caterpillars using $M_2^0(\mathbb{Z}_{66})$

Suppose the number of elbows of w_i is r_i , and denote the elbows of w_i by $w_{i,j}$, where $j = 1, 2, \dots, r_i$. If it exists, denote the hanging leaf of $w_{i,j}$ by $w_{i,j,1}$.

Suppose the maximum number of elbows of a vertex in G is r_k . Since the endvertices in the central path of each lobster in G have no elbow and there is at least one lobster with at least three vertices, it follows that $\Delta(G) = r_k + 2$.

Consider a lobster H such that $V(H) = V(G)$ and $E(H) = E(G) \cup \{\{w_{q_i}, w_{q_i + 1}\} : i = 1, 2, \dots, p - 1\}$, where $q_i = s_1 + s_2 + \dots + s_i$. By Theorem 8, H has an efficient zero ring labeling h .

Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{2nr_k+n-2r_k})$ such that $f(a) = h(a)$ for all $a \in V(G)$. Since h is injective, it follows that f is also injective. G is an edge-induced subgraph of H , so $A_0 \notin K_H = \{h(u) + h(v) : uv \in E(H)\}$ implies that $A_0 \notin K = \{f(u) + f(v) : uv \in E(G)\}$. To show that f is an efficient zero ring labeling of G , it remains to show that $|K| = \Delta(G) = r_k + 2$.

Using the labeling in the proof of Theorem 8, we obtain

$$K = \{A_{(2r_k+1)(q-2)}, A_{(2r_k+1)(q-2)+1}, \dots, A_{(2r_k+1)(q-2)+r_k}, A_{(2r_k+1)(q-1)}\}, \quad (42)$$

where $q = s_1 + s_2 + \dots + s_p$. Thus, $|K| = r_k + 2 = \Delta(G)$.

Example 14. Figure 15 shows an efficient zero ring labeling of a disjoint union of two lobsters using $M_2^0(\mathbb{Z}_{57})$. In this labeling, the set of sums is $K = \{A_{49}, A_{50}, A_{51}, A_{52}, A_{56}\}$ and thus $|K| = 5$, which is the maximum degree of the graph.

Theorem 12. A cycle graph with n vertices has an efficient zero ring labeling if n is even.

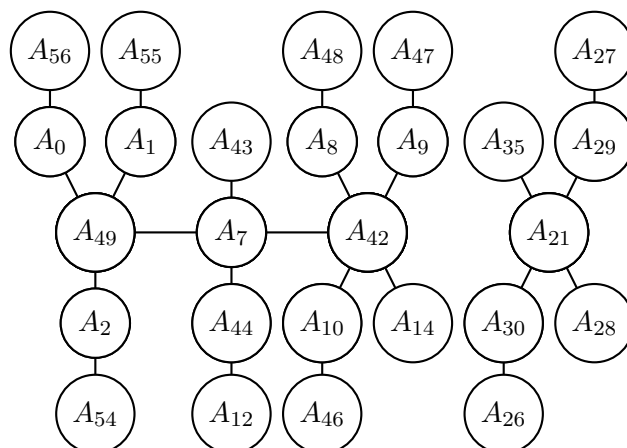


Figure 15: Efficient zero ring labeling of a disjoint union of two lobsters using $M_2^0(\mathbb{Z}_{57})$

Proof. Let $C_n = [w_1, w_2, \dots, w_n, w_1]$, where n is even. Clearly, $\Delta(C_n) = 2$. Define a function $f : V(C_n) \rightarrow M_2^0(\mathbb{Z}_n)$ such that

$$f(w_i) = \begin{cases} A_{i-1} & \text{if } i \text{ is odd} \\ A_{n-i+3} & \text{if } i \text{ is even} \end{cases} \tag{43}$$

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(C_n)\}$. We obtain

$$f(w_i) + f(w_{i+1}) = A_{i-1} + A_{n-(i+1)+3} = A_{n+1} = A_1 \neq A_0 \tag{44}$$

if i is odd, and

$$f(w_i) + f(w_{i+1}) = A_{n-i+3} + A_{(i+1)-1} = A_{n+3} = A_3 \neq A_0 \tag{45}$$

if i is even. Thus, $K = \{A_1, A_3\}$, hence $|K| = 2 = \Delta(C_n)$. Therefore, f is an efficient zero ring labeling of C_n .

Example 15. Figure 16 shows an efficient zero ring labeling of C_8 using $M_2^0(\mathbb{Z}_8)$. In this labeling, the set of sums is $K = \{A_1, A_3\}$ and thus $|K| = 2 = \Delta(C_8)$.

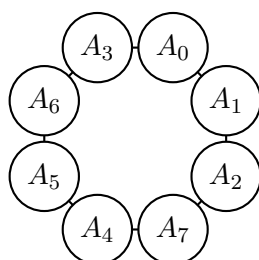


Figure 16: Efficient zero ring labeling of C_8 using $M_2^0(\mathbb{Z}_8)$

Theorem 13. *A cycle graph with n vertices has no efficient zero ring labeling if n is odd.*

Proof. Let $C_n = [w_1, w_2, \dots, w_n, w_1]$, where n is odd. Clearly, $\Delta(C_n) = 2$. Assume that C_n has an efficient zero ring labeling $f : V(C_n) \rightarrow R^0$.

Since w_1 and w_2 are adjacent, $f(w_1) + f(w_2) \in K = \{f(u) + f(v) : uv \in E(C_n)\}$. It follows that $f(w_2) + f(w_3) \neq f(w_1) + f(w_2)$; otherwise, $f(w_1) = f(w_3)$, which is not possible since f is injective. K has only two elements, so $K = \{f(w_1) + f(w_2), f(w_2) + f(w_3)\}$.

Similarly, $f(w_3) + f(w_4) \neq f(w_2) + f(w_3)$. Then $f(w_3) + f(w_4) = f(w_1) + f(w_2)$. Continuing in this manner, $f(w_n) + f(w_1)$ would be equal to $f(w_1) + f(w_2)$ since n odd. But this implies that $f(w_2) = f(w_n)$. This is a contradiction since f is injective.

Theorem 14. *A complete bipartite graph has an efficient zero ring labeling.*

Proof. Let $G = K_{m,n}$ be a complete bipartite graph with vertex set $\{x_1, x_2, \dots, x_m\} \cup \{y_1, y_2, \dots, y_n\}$ and edge set $\{x_i y_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$. Without loss of generality, suppose $m \geq n$. Then $\Delta(G) = m$. Define a function $f : V(G) \rightarrow M_2^0(\mathbb{Z}_{2m})$ such that $f(x_i) = A_{2i-2}$ and $f(y_j) = A_{2j-1}$.

Clearly, f is injective. Let $K = \{f(u) + f(v) : uv \in E(G)\}$. We obtain

$$f(x_i) + f(y_j) = A_{2i-2} + A_{2j-1} = A_{2i+2j-3} \tag{46}$$

for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then $K = \{A_1, A_3, \dots, A_{2m-1}\}$, hence $|K| = m$. Moreover, since m is at least one, $A_0 \notin K$. Thus, f is an efficient zero ring labeling of G .

Example 16. *Figure 17 shows an efficient zero ring labeling of $K_{5,4}$ using $M_2^0(\mathbb{Z}_{10})$. In this labeling, the set of sums is $K = \{A_1, A_3, A_5, A_7, A_9\}$ and thus $|K| = 5 = \Delta(K_{5,4})$.*

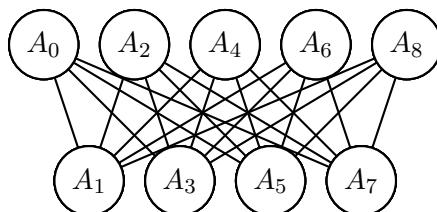


Figure 17: Efficient zero ring labeling of $K_{5,4}$ using $M_2^0(\mathbb{Z}_{10})$

Theorem 15. *A fan graph F_n has an efficient zero ring labeling for $n \geq 3$.*

Proof. It was shown in [2] that F_n has an optimal zero ring labeling for $n \geq 3$. By definition of F_n , it has a vertex with degree $|F_n| - 1$. Thus, by Theorem 3, F_n has an efficient zero ring labeling for $n \geq 3$.

Theorem 16. *A wheel graph W_n has an efficient zero ring labeling for $n \geq 3$.*

Proof. It was shown in [2] that W_n has an optimal zero ring labeling for $n \geq 3$. By definition of W_n , it has a vertex with degree $|W_n| - 1$. Thus, by Theorem 3, W_n has an efficient zero ring labeling for $n \geq 3$.

Theorem 17. *A friendship graph T_n has an efficient zero ring labeling for $n \geq 2$.*

Proof. It was shown in [2] that T_n has an optimal zero ring labeling for $n \geq 2$. By definition of T_n , it has a vertex with degree $|T_n| - 1$. Thus, by Theorem 3, T_n has an efficient zero ring labeling for $n \geq 2$.

Example 17. *Figure 18 shows an efficient zero ring labeling of F_5 using $M_2^0(\mathbb{Z}_6)$. Figure 19 shows an efficient zero ring labeling of W_8 using $M_2^0(\mathbb{Z}_8)$. Figure 20 shows an efficient zero ring labeling of T_4 using $M_2^0(\mathbb{Z}_9)$. In these labelings, the set of sums is the difference of the zero ring used and $\{A_0\}$.*

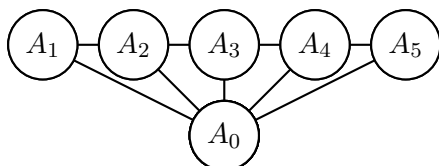


Figure 18: Efficient zero ring labeling of F_5 using $M_2^0(\mathbb{Z}_6)$

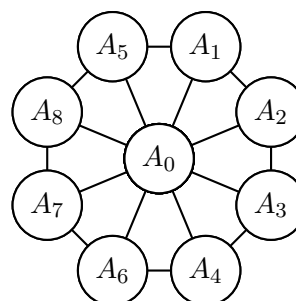


Figure 19: Efficient zero ring labeling of W_8 using $M_2^0(\mathbb{Z}_9)$

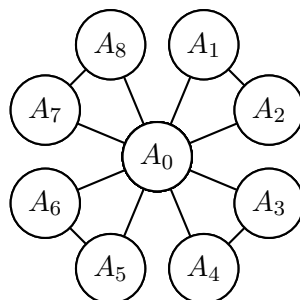


Figure 20: Efficient zero ring labeling of T_4 using $M_2^0(\mathbb{Z}_9)$

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References

- [1] Mukti Acharya, Pranjali, and Purnima Gupta. Zero ring labeling of graphs. *Electronic Notes in Discrete Mathematics*, 48:65–72, July 2015.
- [2] Michelle Dela Rosa-Reynera. *On Graphs of Minimum Zero Ring Index*. PhD thesis, August 2018.
- [3] Reinhard Diestel. *Graph Theory*. Springer, 4th edition, 2010.
- [4] Joseph A. Gallian. *Contemporary Abstract Algebra*. Brooks/Cole, 8th edition, 2013.
- [5] Frank Harary. *Graph Theory*. Addison-Wesley, 1969.
- [6] Pranjali, Mukti Acharya, and Purnima Gupta. Further results on zero ring labeling of graphs. *Bulletin of the International Mathematical Virtual Institute*, 5:205–210, 2014.