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Near ring Multiplications on a Modified Near Module Over a Near ring

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Abstract. We introduce the notion of a modified near module M over a near ring N and explain a method of obtaining near ring multiplications via a special type of maps from M into N called semilinear maps.

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1. Introduction

An interesting question that has attracted the attention of a good number of near ring theorists includes J.R.Clay, R.E.Williams, C.J.Maxson, M.Johnson, K.D.Magill Jr.concerns with finding a near ring multiplication on an algebraic structure over an underlying group. In particular J.R. Clay (1992) [6] proved that a function π on a finite cyclic group $(\mathbb{Z}_{\ltimes}, +)$ generates a multiplication '$*$ 'so that $(\mathbb{Z}_{\ltimes}, +, *)$ is a near ring if $\pi(\pi(p)q) = \pi(p)\pi(q)$. K.D. Magill, Jr. (1995) [4] characterized that any near ring multiplication on a real finite dimensional Euclidean space $\mathbb{R}^n$ is associated with a real-valued function f on $\mathbb{R}^n$ that satisfies $f(f(x)y) = f(x)f(y)$. In this paper we present methods [2], [3] [1] of finding near ring multiplications on some algebraic structures which we call modified near modules. A right near ring [5]is a triple $(N, +, .)$, where $(N, +)$ is a (not necessarily abelian) group, $(N, .)$ is a semigroup satisfying the right distributive law: $(a + b)c = ac + bc$ for all $a, b, c \in N$.

By a near ring we mean a right near ring. When there is no scope for confusion, we write N is a near ring instead of $(N, +, \cdot)$ is a near ring. [6]

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2. Modified near modules

Let $(M, +)$ be a group and let N be a near ring and suppose '$\cdot$' is a mapping of $N \times M$ into M.

Definition 1. *$(M, +, \cdot)$ is called a* **near module** *over N if*

(i) $(n_1 + n_2)m = n_1m + n_2m$ for all $n_1, n_2 \in N$ and $m \in M$;

(ii) $(n_1n_2)m = n_1(n_2m)$ for all $n_1, n_2 \in N$ and $m \in M$.

Remark 1. *Clearly our near module is the N-**group** introduced by Pilz.*

Definition 2. *$(M, +, \cdot)$ is called a* **modified near module** *over N if*

(i) $n(m_1 + m_2) = nm_1 + nm_2$ for all $n \in N$ and $m_1, m_2 \in M$;

(ii) $(n_1n_2)m = n_1(n_2m)$ for all $n_1, n_2 \in N$ and $m \in M$.

Definition 3. *$(M, +, \cdot)$ is called a* **strong near module** *over N if*

(i) $(n_1 + n_2)m = n_1m + n_2m$ for all $n_1, n_2 \in N$ and $m \in M$;

(ii) $n(m_1 + m_2) = nm_1 + nm_2$, for all $m_1, m_2 \in M$ and $n \in N$;

(iii) $(n_1n_2)m = n_1(n_2m)$ for all $n_1, n_2 \in N$ and $m \in M$.

Remark 2. *A strong near module over a field is a vector space if '+' is abelian and $1m = m$ for every m.*

Example 1. *Let $(G, +)$ be a group. Define the function $\cdot$ from $M(G) \times G$ into G by $\cdot(f, x) = f \cdot x = f(x)$ for all $f \in M(G)$ and $x \in G$.*
For any $f, g \in M(G)$ and $x \in G$, $(f \circ g)(x) = f(g(x)) = f(gx)$.
Also $(f + g)(x) = f(x) + g(x) = fx + gx$.
*and $f(x+y) \neq f(x)+f(y)$. Therefore $(G, +, \cdot)$ is a near module over near ring $(M(G), +, \circ)$, but **not** a modified near module.*

Example 2. *Let N be a nontrivial near ring with $ab = a$.*
Let $M = (N, +)$. Define the function $\odot$ from $N \times M$ into M as $\odot(n, m) = n \odot m = m$ for all $n \in N, m \in M$.
For any $n, n_1, n_2 \in N$ and $m, m_1, m_2 \in M$,
(1) *$(n_1n_2) \odot m = n_1 \odot m = m$ and $n_1 \odot (n_2 \odot m) = n_2 \odot m = m$*
(2) *$n \odot (m_1 + m_2) = m_1 + m_2$ and $n \odot m_1 + n \odot m_2 = m_1 + m_2$.*
Therefore $(M, +, \odot)$ is a modified near module over N.
*However $(M, +, \odot)$ is **not** a near module*
since $(n_1 + n_2) \odot m = m$ and $n_1 \odot m + n_2 \odot m = m + m$.

Example 3. *Let $R = (\mathbb{R}, +, \cdot)$.*
Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(r) = r^2$. Define '$\odot$': $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ as $\odot(r, m) = r \odot m = r^2 m$ for $r, m \in \mathbb{R}$.
Then $(R, +)$ is a modified near module over the near ring $(\mathbb{R}, +, \cdot)$ but not a near module. That $(R, +)$ is a modified near module can be verified easily. However R is not a near module as is evident from the following:
Take $r_1 = 1, r_2 = 1, m = 2$. Then
$(r_1 + r_2) \odot m = (1 + 1) \odot 2 = 2 \odot 2 = 2^2 2 = 8$ and
$r_1 \odot m + r_2 \odot m = 1 \odot 2 + 1 \odot 2 = 1^2 2 + 1^2 2 = 2 + 2 = 4$.

Infact the above example is a special case of the following theorem:

Theorem 1. *Let $(R, +, \cdot)$ and $(S, +_1, \cdot_1)$ be near rings and let $\phi : R \to S$ be a mapping such that $\phi(r_1 \cdot r_2) = \phi(r_1) \cdot_1 \phi(r_2)$ for all $r_1, r_2 \in R$. Let $(M, \oplus, \odot)$ be a left S-module. Therefore $(M, \oplus, \odot_1)$ is a modified near module over the near ring $(R, +, \cdot)$ when $\odot_1$ is defined by $r \odot_1 m = \phi(r) \odot m$ for all $r \in R$ and $m \in M$.*

Proof. Since $(M, \oplus, \odot)$ is a left S-module,

(i) $s \odot (m_1 \oplus m_2) = s \odot m_1 \oplus s \odot m_2$;

(ii) $(s_1 +_1 s_2) \odot m = s_1 \odot m \oplus s_2 \odot m$;

(iii) $s_1 \odot (s_2 \odot m) = (s_1 \cdot_1 s_2) \odot m$
for all $s, s_1, s_2 \in S$ and $m, m_1, m_2 \in M$.

For any $r, r_1, r_2 \in R$ and $m, m_1, m_2 \in M$,
(1) $r_1 \odot_1 (r_2 \odot_1 m) = \phi(r_1) \odot (r_2 \odot_1 m) = \phi(r_1) \odot [\phi(r_2) \odot m]$
$= [\phi(r_1) \cdot_1 \phi(r_2)] \odot m = \phi(r_1 r_2) \odot m = (r_1 r_2) \odot_1 m$ and
(2) $r \odot_1 (m_1 \oplus m_2) = \phi(r) \odot [m_1 \oplus m_2] = \phi(r) \odot m_1 \oplus \phi(r) \odot m_2$
$= r \odot_1 m_1 \oplus r \odot_1 m_2$.
Therefore $(M, +, \cdot)$ is a modified near module over $(R, +, \cdot)$.

Definition 4. *Let $(M, +, \cdot)$ be a modified near module over N. A normal subgroup I of M is called an* **ideal** *of M if*

$$n(m + i) - nm \in I \text{ for all } n \in N, i \in I \text{ and } m \in M.$$

Definition 5. *Let $(M_1, +_1, \cdot_1)$ and $(M_2, +_2, \cdot_2)$ be modified near modules over N. A mapping $\phi : M_1 \to M_2$ is called a* **modified near module homomorphism** *if*

(i) $\phi(m +_1 m') = \phi(m) +_2 \phi(m')$;

(ii) $\phi(n \cdot_1 m) = n \cdot_2 \phi(m)$ *for all $m, m' \in M_1$ and $n \in N$.*

The proofs of the following theorems are similar to those of their counterparts in near ring theory [5], hence omitted.

Theorem 2. *Let M_1, M_2 be modified near modules over N and let $\phi : M_1 \to M_2$ be a modified near module homomorphism. Then $\ker\phi$ is an ideal of M_1 and $\frac{M_1}{\ker\phi} \simeq \phi(M_1)$.*

Theorem 3. *The intersection of any family of ideals of a modified near module M is ideal of M.*

Proposition 1. *Let M be a modified near module over N and let I be an ideal of M. Let $\frac{M}{I} = \{m + I | m \in M\}$. Then $(\frac{M}{I}, \oplus, \odot)$ is a modified near module when $\oplus$ and $\odot$ are defined as*

$$(m + I) \oplus (m' + I) = (m + m') + I \text{ and}$$
$$n \odot (m + I) = nm + I \text{ for all } m + I, m' + I \in \frac{M}{I} \text{ and } n \in N$$

and the natural projection map $\pi : M \to \frac{M}{I}$ defined by $\pi(m) = m + I$ is a modified near module homomorphism with kernel I.

3. Near ring Multiplication On a Modified Near Module

The following theorem explains a method of obtaining a near ring multiplications on a modified near module over N via semilinear map from M into N.

Definition 6. *Let $(M, +, \cdot)$ be a modified near module over N. We call a mapping $f : M \to N$ a semilinear if $f(f(m_1)m_2) = f(m_1)f(m_2)$ for all $m_1, m_2 \in M$.*

Theorem 4. *Let $(M, +, \cdot)$ be a modified near module over a near ring $(N, +, \cdot)$. Let f be a semilinear map from M into N. Define the binary operation $*$ on M as $m_1 * m_2 = f(m_2)m_1$ for all $m_1, m_2 \in M$. Then $(M, +, *)$ is a near ring.*

Proof. For any $m_1, m_2, m_3 \in M$,
$m_1 * (m_2 * m_3) = f(m_2 * m_3)m_1 = f(f(m_3)m_2)m_1 = [f(m_3)f(m_2)]m_1$ and
$(m_1 * m_2) * m_3 = f(m_3)(m_1 * m_2) = f(m_3)[f(m_2)m_1] = [f(m_3)f(m_2)]m_1$.
So the binary operation $*$ is associative.
Now $(m_1 + m_2) * m_3 = f(m_3)(m_1 + m_2) = f(m_3)m_1 + f(m_3)m_2 = m_1 * m_3 + m_2 * m_3$.
So the binary operation $*$ is right distributive and hence $(M, +, *)$ is a near ring.

Examples 3.3 through 3.7 illustrate the technique of defining a near ring multiplication on $(M, +)$

Example 4. *Let $M = \{f | f : \mathbb{R} \to \mathbb{R}\}$ and $N = (End(\mathbb{R}, +), +, \circ)$*
. Then $(M, +, \circ)$ is a modified near module over $(N, +, \circ)$.
Define $\alpha : M \to N$ by $\alpha(f) = f'$
where

$$f'(x) = \begin{cases} 0 \text{ if } x = 0 \\ f(x) \text{ if } x \neq 0. \end{cases}$$

Note that $f(x) = 0$ implies $f'(x) = 0$ for $x \in \mathbb{R}$.
We claim that α is semilinear.

Let $f, g \in M$ and $x \in \mathbb{R}$.
Case(i): $x \neq 0$. *Now* $\alpha(\alpha(f)og)(x) = \alpha(f'og)(x)$
$= (f' \circ g)'(x) = (f' \circ g)(x) = f'(g(x))$
$= f'(g'(x)) = (f' \circ g')(x) = (\alpha(f) \circ \alpha(g))(x)$ *implies* $\alpha(\alpha(f) \circ g) = \alpha(f) \circ \alpha(g)$.
Case(ii): $x = 0$.

$$\begin{aligned} \text{Now } \alpha(\alpha(f) \circ g)(0) &= \alpha(f' \circ g)(0) \\ &= (f' \circ g)'(0) = 0. \end{aligned}$$

Also $[\alpha(f) \circ \alpha(g)](0) = (f' \circ g')(0) = f'(g'(0)) = f'(0) = 0$.
So $\alpha(\alpha(f) \circ g) = \alpha(f) \circ \alpha(g)$ *when* $x = 0$.
Hence α is semilinear; therefore $(M, +, *)$ is a near ring with $*$ defined by $f*g = \alpha(g) \circ f = g' \circ f$.

Example 5. *Let M be the abelian group of all $n \times n$ circulant matrices with real entries. Then $(M, +)$ is a modified near module over $N = (\mathbb{R}, +, \cdot)$, if we define kA as the matrix obtained by multiplying each entry of A by k. Define $\alpha : M \to N$ by $\alpha(A) = specA$ for all $A \in M$*
where $specA = max\{|\lambda_i||\lambda_i$ is an eigen value of $A\}$.

$$\begin{aligned} \text{Now } \alpha(\alpha(A)B) &= spec(\alpha(A)B) \\ &= \alpha(A) specB \\ &= \alpha(A)\alpha(B) \text{ for all } A, B \in M. \end{aligned}$$

Hence α is semilinear; therefore $(M, +, *)$ is a near ring with $*$ defined by $A*B = \alpha(B)A = specB\ A$.

Example 6. *Let $(G, +)$ be a (not necessarily abelian) group. Let $N = (End(G), +, \circ)$. For f in N and a in G, define $fa = f(a)$. Then G is a modified near module over N.*
Define $\alpha : G \to N$ by $\alpha(a) = L_a$
where L_a is the left addition by a: $L_a(x) = a + x$ for all $x \in G$.
Then $\alpha : M \to N$ is semilinear.

Example 7. *Let $(\mathbb{C}, +)$ be the module of complex numbers over the real field $(\mathbb{R}, +, \cdot)$ with usual product.* (*i*) *Define $f : \mathbb{C} \to \mathbb{R}$ by $f(x + iy) = (x^2 + y^2)^{\frac{1}{2}}$ for all $x + iy \in \mathbb{C}$.*
For any $x_1 + iy_1, x_2 + iy_2 \in \mathbb{C}$,
$f(f(x_1 + iy_1)(x_2 + iy_2)) = f(({x_1}^2 + {y_1}^2)^{\frac{1}{2}}(x_2 + iy_2))$
$= f(({x_1}^2 + {y_1}^2)^{\frac{1}{2}} x_2 + i({x_1}^2 + {y_1}^2)^{\frac{1}{2}} y_2)$
$= [(({x_1}^2 + {y_1}^2)^{\frac{1}{2}} x_2)^2 + (({x_1}^2 + {y_1}^2)^{\frac{1}{2}} y_2)^2]^{\frac{1}{2}}$
$= ({x_1}^2 + {y_1}^2)^{\frac{1}{2}} ({x_2}^2 + {y_2}^2)^{\frac{1}{2}}$
$= f(x_1 + iy_1) f(x_2 + iy_2)$. *Hence f is semilinear; therefore $(\mathbb{C}, +, *)$ is a near ring with $*$ defined by*

$$\begin{aligned} (x_1 + iy_1) * (x_2 + iy_2) &= f(x_2 + iy_2)(x_1 + iy_1) \\ &= ({x_2}^2 + {y_2}^2)^{\frac{1}{2}} (x_1 + iy_1). \end{aligned}$$

(ii) Define $f : \mathbb{C} \to \mathbb{R}$ *by* $f(x+iy) = |x|$ *for all* $x+iy \in \mathbb{C}$.
For any $x_1+iy_1, x_2+iy_2 \in \mathbb{C}$,

$$\begin{aligned} f(f(x_1+iy_1)(x_2+iy_2)) &= f(|x_1|(x_2+iy_2)) \\ &= f(|x_1|x_2 + i|x_1|y_2) \\ &= ||x_1|x_2| = |x_1||x_2| \\ &= f(x_1+iy_1)f(x_2+iy_2). \end{aligned}$$

Hence f *is semilinear; therefore* $(\mathbb{C}, +, *)$ *is a near ring with* $*$ *defined by*

$$\begin{aligned} (x_1+iy_1) * (x_2+iy_2) &= f(x_2+iy_2)(x_1+iy_1) \\ &= |x_2|(x_1+iy_1). \end{aligned}$$

Example 8. *Let* $M = \{a+bi+cj+dk | a,b,c,d \in \mathbb{R}\}$ *be the ring of real quaternions. Then* M *is a modified near module over the real number field* $(\mathbb{R}, +, \cdot)$.
Define $f : M \to \mathbb{R}$ *by* $f(a+bi+cj+dk) = a^2+b^2+c^2+d^2$ *for all* $a+bi+cj+dk \in M$.
For any $a_1+b_1i+c_1j+d_1k, a_2+b_2i+c_2j+d_2k \in M$,
$f(f(a_1+b_1i+c_1j+d_1k)(a_2+b_2i+c_2j+d_2k))$
$= f(({a_1}^2+{b_1}^2+{c_1}^2+{d_1}^2)(a_2+b_2i+c_2j+d_2k))$
$= f(a_1+b_1i+c_1j+d_1k)f(a_2+b_2i+c_2j+d_2k)$.
Hence f *is semilinear and therefore* $(M, +, *)$ *is a near ring with*
$(a_1+b_1i+c_1j+d_1k) * (a_2+b_2i+c_2j+d_2k)$
$= f(a_2+b_2i+c_2j+d_2k)(a_1+b_1i+c_1j+d_1k)$
$= ({a_2}^2+{b_2}^2+{c_2}^2+{d_2}^2)(a_1+b_1i+c_1j+d_1k)$.

Example 9. *Let* M *be the set of all* $n \times n$ *real matrices. Then* $(M, +, \cdot)$ *is a strong near module over the real number field* $(\mathbb{R}, +, \cdot)$.
Define $f : M \to \mathbb{R}$ *by* $f(A) = \sum_{1 \le i,j \le n} (a_{ij})^2$. *Then* f *is a semilinear map and hence* $(M, +, *)$ *is a near ring with* $A * B = f(B)A$.

Theorem 5. *Let* $(M, +, \cdot)$ *be a modified near module over* N. *Define the function* $\odot$ *from* $N \times M$ *into* M *as* $\odot(n, m) = n \odot m = f(n)m$ *for all* $m \in M$ *and* $n \in N$. *Then* $(M, +, \odot)$ *is a modified near module over* N_f, *where* N_f *is a near ring induced by the semilinear map* f.

Proof. For any $n \in N$ and $m_1, m_2 \in M$,
$n \odot (m_1+m_2) = f(n)(m_1+m_2) = f(n)m_1 + f(n)m_2 = n \odot m_1 + n \odot m_2$.
For any $n_1, n_2 \in N$ and $m \in M$,
$(n_1 * n_2) \odot m = f(n_1 * n_2)m = f(n_1 f(n_2))m = [f(n_1)f(n_2)]m$ and
$n_1 \odot (n_2 \odot m) = f(n_1)(n_2 \odot m) = f(n_1)[f(n_2)m] = [f(n_1)f(n_2)]m$. Therefore $(M, +, \odot)$ is a modified near module over N_f.

Theorem 6. *Let* M_1, M_2 *be modified near modules over* N *and* $f : M_1 \to N$ *be a semilinear map and* $\phi : M_2 \to M_1$ *be a near module homomorphism. Then* $fo\phi$ *is a semilinear map.*

Proof. Let $g = f \circ \phi$.
For any $m_2, m_2' \in M_2$, $g(g(m_2)m_2') = g([(f \circ \phi)(m_2)]m_2') = g((f(\phi(m_2))(m_2'))$
$= (f \circ \phi)[f(\phi(m_2))m_2'] = f[\phi[f(\phi(m_2))m_2']]$
$= f[f(\phi(m_2))\phi(m_2')] = f(\phi(m_2))f(\phi(m_2'))$
$= g(m_2)g(m_2')$. Therefore g is a semilinear map.

Remark 3. *Suppose a modified near module $(M, +, \cdot)$ over a near ring $(N, +, \cdot)$ is made into a near ring $(M, +, *)$ with the help of a semilinear map f. Then we know that M^k, the k-fold product of $(M, +, *)$ is also a near ring. It may be hoped that the near ring module $(M^k, \oplus, \cdot)$ can be made into the near ring $(M^k, \oplus, \otimes)$ directly by employing a suitable semilinear map from M^k into N. The following example warns that not every modified near module comes through a semilinear map.*

As an illustration we present the following:

Example 10. *Define $x \cdot y = 2xy$ for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, +, \cdot)$ is a modified near module over $\mathbb{R}$.*
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(m) = 2m$ for all $m \in \mathbb{R}$.
Then $f(f(a) \cdot b) = f(a) \cdot f(b) = 8ab$.
So that f is semilinear.

$$\text{Now } m_1 * m_2 = f(m_2) \cdot m_1 = 2f(m_2)m_1 = 2(2m_2)m_1 = 4m_2m_1.$$

Consider $(\mathbb{R}^2, \oplus, \otimes)$, the product of the near ring $(\mathbb{R}, +, *)$ with itself.
Suppose if possible there is a semilinear map $g : \mathbb{R}^2 \to \mathbb{R}$ such that '$\otimes$' is induced by g.

$$\begin{aligned}
\text{Now } (m_1 \cdot m_3, m_2 \cdot m_4) &= (m_1, m_2) \otimes (m_3, m_4) \\
&= g(m_3, m_4) \cdot (m_1, m_2) \\
&= (\alpha \cdot m_1, \alpha \cdot m_2) \text{ where } \alpha = g(m_3, m_4) \\
\Rightarrow m_1 \cdot m_3 &= \alpha \cdot m_1 \text{ and} \\
m_2 \cdot m_4 &= \alpha \cdot m_2 \\
\Rightarrow 2m_1m_3 &= 2\alpha m_1 \text{ and} \\
2m_2m_4 &= 2\alpha m_2 \text{ for all } m_1, m_2, m_3, m_4 \in M.
\end{aligned}$$

Taking $m_1 = m_3 = 1, m_2 = m_4 = 2$, we get $2 = 2\alpha$ and $8 = 4\alpha$
$\Rightarrow \alpha = 1$ and $\alpha = 2$, which is a contradiction.

Theorem 7. *Let M be a modified near module over $(\mathbb{R}, +)$ and $f : M \to \mathbb{R}$ be a semilinear map.*

(i) If f is one-one, then $(M_f, +, *)$ is commutative.

(ii) Suppose $M = (\mathbb{R}^k, +)$. Then $(M_f, +, *)$ is commutative if and only if either $M = \{0\}$ or $(M_f, +, *) \simeq (\mathbb{R}, +, \cdot)$, where M_f is a near ring induced by the semilinear map f.

Proof. (1) For any $m_1, m_2 \in M$, $m_1 * m_2 = f(m_2)m_1$ and $m_2 * m_1 = f(m_1)m_2$.

$$\begin{aligned} \text{Now } f(m_1 * m_2) &= f(f(m_2)m_1) \\ &= f(m_2)f(m_1). \\ \text{Also } f(m_2 * m_1) &= f(f(m_1)m_2) \\ &= f(m_1)f(m_2). \end{aligned}$$

Since $(\mathbb{R}, \cdot)$ is commutative, we have $f(m_1 * m_2) = f(m_2 * m_1)$.
Since f is one-one, we have $m_1 * m_2 = m_2 * m_1$.
So '* ' is commutative on M and hence $(M_f, +, *)$ is commutative.
(2) Suppose $(M_f, +, *)$ is commutative. Then

$$\begin{aligned} & m_1 * m_2 = m_2 * m_1 \\ & \quad \Rightarrow f(m_2)m_1 = f(m_1)m_2 \\ & \quad \Rightarrow \text{The vectors } m_1 \text{ and } m_2 \text{ are parallel} \\ & \quad \Rightarrow k = 0 \text{ or } k = 1. \end{aligned}$$

<u>When k=1</u>:
Now $m_1 * m_2 = f(m_2)m_1$ and $m_2 * m_1 = f(m_1)m_2 \Rightarrow f(m_2)m_1 = f(m_1)m_2$ for all $m_1, m_2 \in M$.
This equality is true for $m_1 = 1$, we get $f(m_2) = f(1)m_2$.
Put $f(1) = \lambda \Rightarrow f(m_2) = \lambda m_2$ for some constant.
Therefore f is linear.

$$\begin{aligned} \text{Now } m_1 * (m_2 + m_3) &= f(m_2 + m_3)m_1 \\ &= [f(m_2) + f(m_3)]m_1 \\ &= f(m_2)m_1 + f(m_3)m_1 \\ &= m_1 * m_2 + m_1 * m_3. \end{aligned}$$

Therefore $(M_f, +, *)$ is a commutative ring.
Let $0 \neq m \in \mathbb{R}$, then $m * m_1 = f(m_1)m = \lambda m_1 m = \lambda m m_1$.
Put $m_1 = \frac{1}{\lambda m}$. Then $m * m_1 = 1$.
Define $\psi : (M, +, *) \to (\mathbb{R}, +, \cdot)$ by $\psi(m) = \lambda m$ for all $m \in M$. Then $(M, +, *) \simeq (\mathbb{R}, +, \cdot)$.
Conversely suppose that $M = \{0\}$ or $(M_f, +, *) \simeq (\mathbb{R}, +, \cdot)$.
Since the ring $\{0\}$ is commutative and since any ring isomorphic to $(\mathbb{R}, +, \cdot)$ is commutative, the converse is clear.

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