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Levin conjecture for group equations of length 9

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Abstract. Levin conjecture states that every group equation is solvable over any torsion free group. The conjecture is shown to hold true for group equation of length seven using weight test and curvature distribution method. Recently, these methods are used to show that Levin conjecture is true for some group equations of length eight and nine modulo some exceptional cases. In this paper, we show that Levin conjecture holds true for a group equation of length nine modulo 2 exceptional cases. In addition, we allude the list of cases that are still open for two more equations of length nine.

2020 Mathematics Subject Classifications: 20F05, 20E06, 57M05

Key Words and Phrases: Group equations, torsion-free groups, relative group presentations, asphericity, weight test, curvature distribution

1. Introduction:

Let G be a non-trivial group and t an element not in G. A *group equation* over G is an equation of the form

$$s(t) = g_1 t^{l_1} g_2 t^{l_2} ... g_n t^{l_n} = 1 \quad (g_i \in G, l_i = \pm 1)$$

such that $l_i + l_{i+1} = 0$ implies $g_{i+1} \neq 1 \in G$ (subscripts modulo n). The non-negative integer n is known as the *length* of equation $s(t) = 1$. The equation $s(t) = 1$ is said to be *solvable* over G if $s(h) = 1$ for some element h of a group H which contain G. Equivalently, $s(t) = 1$ is solvable over G if and only if the natural homomorphism from G to $\frac{G*(t)}{N}$ is injective, in which N is the normal closure of $s(t) = 1$ in the free product $G^*(t)$. The equation $s(t) = 1$ is called *singular* if $\sum_{i=1}^{n} l_i = 0$ and *non-singular* otherwise.

The study of group equations was initiated by Neumann [17] who solved an equation $t^{-1} g_1 t g_2 = 1$ over any torsion free group. Motivated by the solvability of the polynomial

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equations over fields, Levin [16] studied the analogous problem for group equations and proved that the equation $s(t) = 1$ (l_i non-negative and not necessarily 1) is solvable over any group G, for $l_1 + l_2 + ... + l_n = n$. These findings of Neumanm and Levin gave the hope for the conjecture that every equation is solvable over a torsion free group, which is known as *Levin conjecture*.

There has been significant work to verify the Levin conjecture [11, 12, 14, 15] for group equations of length less than or equal to six. Recently, Mairaj and Edjvet [8] proved the Levin conjecture for all group equations of length seven by using weight test and curvature distribution method. By employing the methods used in [8], Mairaj et al. [7] have proved the conjecture for a non-singular equation of length eight modulo one exceptional case. The authors have done some significant work in [4, 5] using weight test which establishes the conjecture up to great extent for length eight. The equations of length nine are considered in [6], where it is proved that there are only three equations of length nine which are open. More recently, Fazeel et al. [3] have investigated the conjecture for a non-singular equation of length nine (one of three) given by

$$s_1(t) = atbtctdtetft^{-1}gthtit^{-1} = 1$$

by applying these methods. Fazeel et al. [4] solved 41 cases of this equation. In this paper we have continued our study of exploring the validity of Levin conjecture for the group equation $s_1(t) = 1$ initiated in [4] and found that the total cases of $s_1(t) = 1$ are 245 in which 183 cases are solved by weight test, 60 cases are solved by curvature distribution method and 2 cases are still open. These findings are formulated in the form of the following Theorem which is the main result of the paper.

Theorem. *The group equation $s_1(t) = 1$ is solvable modulo two exceptional cases:*

(i) $a = g, h = e, d = b, c = b, d = c$;

(ii) $a = g, e = b, e = c, c = b, e = d, d = b, d = c$.

The authors in [2] and [1] have explored the validity of Levin Conjecture by applying weight test and curvature distribution to the remaining two equations of length nine given by

$$s_2(t) = atbtct^{-1}dtetft^{-1}gthtit^{-1} = 1$$

and

$$s_3(t) = atbtctdtet^{-1}ftgthtit^{-1} = 1.$$

They found that Levin conjecture holds true for these equations modulo some exceptional cases. The authors in [2] have found that the total cases of $s_2(t) = 1$ are 318 in which 147 cases are solved by weight test, 117 cases are solved by curvature distribution method and 55 cases are still open.

The authors in [1] have found that the total cases of $s_3(t) = 1$ are 245 in which 136 cases are solved by weight test, 70 cases are solved by curvature distribution method and 39 cases are still open.

2. Methodology:

A presentation is said to be *relative (group) presentation* $\mathcal{P} = \langle G, x \mid r\rangle$ in which r is a set of words which is cyclically reduced belongs to G. All the definitions concerning relative presentations can be seen in [9]. In this paper, we will discuss the equation

$$s_1(t) = atbtctdtetft^{-1}gthtit^{-1} = 1$$

in detail. It is well known that the group equation $s_1(t) = atbtctdtetft^{-1}gthtit^{-1}$ is solvable if the natural homomorphism $\tau : G \to \mathcal{P}(G)$ is injective. The sufficient condition for the injectivity of natural map from G to $\mathcal{P} = \langle G, x \mid r\rangle$ is that the relative presentation is orientable and aspherical [9]. The notion of asphericity is discussed in detail in [9]. In our case $s_1(t) = atbtctdtetft^{-1}gthtit^{-1}$, therefore r is singleton set. As stated in [9], if r is singleton then $\mathcal{P}$ is always orientable, therefore asphericity of $\mathcal{P}$ establishes that $s_1(t) = 1$ has solution. In order to establish the validity of Levin's conjecture, it is only left to prove that the presentation $\mathcal{P}$ is aspherical. So, in this paper we apply two tests for showing asphericity of $\mathcal{P}$: Weight test and curvature distribution method. All the necessary definitions concerning weight test can be found in [9]. The weight test states that if the star graph Γ of $\mathcal{P}$ admits an aspherical weight function θ, then $\mathcal{P}$ is aspherical [9]. All the definitions related to pictures can be found in [14]. The curvature distribution asserts that if K is a reduced picture over $\mathcal{P}$ then by Euler (or Gauss-Bonnet) formula, the sum of the curvature of all regions of K is 4π, that is, K contains regions of positive curvature [14]. Then, if for every region Δ of K of positive curvature $c(\Delta)$, there is a neighbouring region $\widehat{\Delta}$, uniquely associated with Δ, like $c(\widehat{\Delta}) + c(\Delta) \leq 0$, then the sum of the curvature of all regions of K is non-positive, which implies that $\mathcal{P}$ is aspherical [6, 8].

Consider a torsion free group G. By applying the transformation $u = tb$ on $s_1(t) = atbtctdtetft^{-1}gthtit^{-1} = 1$ it can be assumed that $b = 1$. Recall that $\mathcal{P} = \langle G, t \mid s_1(t)\rangle$ in which

$$s_1(t) = atbtctdtetft^{-1}gthtit^{-1} \quad (a, f, g, i \in G \setminus \{1\}, b = 1, c, d, e, h \in G).$$

Moreover, G is not cyclic and $G = \langle a, b, c, d, e, f, g, h, i\rangle$ given in [13]. Suppose that K is a reduced spherical diagram over $\mathcal{P}$. Up to cyclic permutation and inversion, the regions of K are given by Δ as shown in Figure 1(i). The star graph Γ of $\mathcal{P}$ is shown in Figure 1(ii).

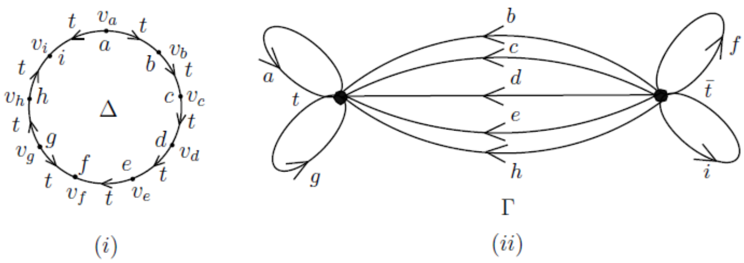

Figure 1: Region Δ of K and star graph Γ of $\mathcal{P}$

Looking at closed paths in star graph Γ, using the fact that G is torsion free and working modulo cyclic permutation and inversion, the possible labels of vertices of degree 2 for a region Δ of K are

$$S = \{ag, ag^{-1}, fi, fi^{-1}, hb^{-1}, hc^{-1}, hd^{-1}, he^{-1}, eb^{-1}, ec^{-1}, ed^{-1}, dc^{-1}, db^{-1}, cb^{-1}\}.$$

We can work modulo equivalence, that is, modulo $t \leftrightarrow t^{-1}$, cyclic permutation, inversion, and

$$a \leftrightarrow f^{-1}, g \leftrightarrow i^{-1}, b \leftrightarrow e^{-1}, c \leftrightarrow d^{-1}, h \leftrightarrow h^{-1}.$$

We will proceed according to the number N of labels in S that are admissible [9] and classify the cases correspondingly [8]. The following remark substantially reduce the number of cases to be considered.

Remark 1. The following observations holds trivially.

(i) If all the admissible cycle has length greater than 2 in region Δ then $c(\Delta) \leq c(3,3,3,3,3,3,3,3,3) = -\pi$.

(ii) If ag and ag^{-1} are admissible then $g^2 = 1$, a contradiction.

(iii) If fi and fi^{-1} are admissible then $i^2 = 1$, a contradiction.

(iv) At most two of ag, ag^{-1}, fi, fi^{-1} are admissible.

(v) If any two of $hb^{-1}, hc^{-1}, cb^{-1}$ are admissible then so is the third.

(vi) If any two of $hb^{-1}, hd^{-1}, db^{-1}$ are admissible then so is the third.

(vii) If any two of $hb^{-1}, he^{-1}, eb^{-1}$ are admissible then so is the third.

(viii) If any two of $hc^{-1}, hd^{-1}, dc^{-1}$ are admissible then so is the third.

(ix) If any two of $hc^{-1}, he^{-1}, ec^{-1}$ are admissible then so is the third.

(x) If any two of $hd^{-1}, he^{-1}, ed^{-1}$ are admissible then so is the third.

(xi) If any two of $eb^{-1}, ec^{-1}, cb^{-1}$ are admissible then so is the third.

(xii) If any two of $eb^{-1}, ed^{-1}, db^{-1}$ are admissible then so is the third.

(xiii) If any two of $ec^{-1}, ed^{-1}, dc^{-1}$ are admissible then so is the third.

(xiv) If any two of $dc^{-1}, db^{-1}, cb^{-1}$ are admissible then so is the third.

The above remark reduces the number of cases to 245 for the group equation $s_1(t) = 1$. From these 245 cases, 41 cases are solved in [4] so there remains 204 cases that needs to be solved. Among these 204 remaining cases, 159 cases are solved by weight test, 43 cases are solved by curvature distribution method and 2 cases are still open.

3. Main Results:

A further 159 cases can be straightforwardly solved using the weight test. For example, consider the case, $a = g^{-1}, f = i^{-1}$ and $h = b$. In this case, the relator is $s(t) = at^2ctdtetft^{-1}a^{-1}t^2f^{-1}t^{-1}$. We put $x = t^{-1}a^{-1}t$ to obtain $v_1 = x^{-1}tctdtetfxtf^{-1}$ and $v_2 = x^{-1}t^{-1}a^{-1}t$.

The presentation $\mathcal{P}$ has star graph Γ which is shown in Figure 2 in which $\mu_1 = c$, $\mu_2 = d$, $\mu_3 = e$, $\mu_4 = 1$, $\mu_5 = f$, $\mu_6 = f^{-1}$, $\mu_7 = 1$; and $\omega_1 = a^{-1}$, $\omega_2 = 1$, $\omega_3 = 1$.

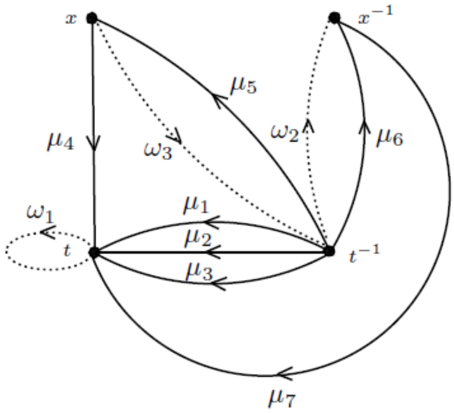

Figure 2: Star graph Γ

We define a weight function θ such that $\theta(\mu_4) = \theta(\mu_6) = \theta(\omega_1) = \theta(\omega_2) = 0$ and $\theta(\mu_1) = \theta(\mu_2) = \theta(\mu_3) = \theta(\mu_5) = \theta(\mu_7) = \theta(\omega_3) = 1$. Then $\Sigma(1-\theta(\mu_i)) = \Sigma(1-\theta(\omega_j)) = 2$ indicates that the first condition of weight test is fulfilled. Moreover, every cycle in Γ of weight smaller than 2 has label a^m, where $m \in \mathbb{Z} \setminus \{0\}$ and $a \in G \setminus \{1\}$, which implies a is torsion element in G, a contradiction, so the second condition of weight test is fulfilled. Furthermore, since θ assigns non-negative weights to each edge, so the third condition of weight test is obviously fulfilled.

A further 24 cases are solved in Lemma 1 by an immediate application of curvature distribution method [10]. In what follows, the vertex labels correspond to the closed paths in the star graph Γ. From now onward, the label and the degree of a vertex v of region Δ will be denoted by $l_\Delta(v)$ and $d_\Delta(v)$ respectively. Furthermore, $l_\Delta \in \{ww_1, \ldots, ww_k\}$ will be indicated by $l_\Delta(v) = \{ww_1, \ldots, ww_k\}$.

Lemma 1. *The presentation $\mathcal{P} = \langle G, t \mid s_1(t) \rangle$ is aspherical if any one of the following holds:*

(i) $a = g$;

(ii) $h = b$;

(iii) $h = c$;

(iv) $e = b$;

(v) $e = c$;

(vi) $e = d$;

(vii) $d = c$;

(viii) $e = b, d = c$;

(ix) $a = g, h = c, d = b$;

(x) $a = g, h = e, d = b$;

(xi) $a = g, h = d, e = b$;

(xii) $a = g, h = d, c = b$;

(xiii) $h = b, h = c, c = b$;

(xiv) $h = b, h = d, d = b$;

(xv) $h = b, h = e, e = b$;

(xvi) $h = c, h = d, d = c$;

(xvii) $e = b, e = c, c = b$;

(xviii) $e = c, e = d, d = c$;

(xix) $a = g, h = c, h = d, d = c$;

(xx) $a = g, h = e, h = d, e = d$;

(xxi) $a = g, e = b, d = b, e = d$;

(xxii) $h = b, h = c, c = b, h = d, d = b, d = c$;

(xxiii) $h = b, h = d, d = b, h = e, e = b, e = d$;

(xxiv) $e = b, e = c, c = b, e = d, d = b, d = c$.

Proof. Here, Δ has at most three vertices of degree 2, so has non-positive curvature for all of these cases. Consider the case,

- $e = b, d = c$.

In this case Δ is given in Figure 3.

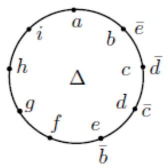

Figure 3: Region Δ

Since $d(v_b) = d(v_c) = 2$ or $d(v_c) = d(v_d) = 2$ or $d(v_d) = d(v_e) = 2$ can not occur together so $c(\Delta) \leq 0$.

A further 19 cases are solved in Lemma 2 by the application of curvature distribution method [10].

Lemma 2. *The presentation $\mathcal{P} = \langle G, t \mid s_1(t) \rangle$ is aspherical if any one of the following holds:*

(i) $a = g, h = c, e = b$;

(ii) $a = g, h = c, e = d$;

(iii) $a = g, h = e, c = b$;

(iv) $a = g, h = e, d = c$;

(v) $a = g, h = d, e = c$;

(vi) $a = g, e = b, d = c$;

(vii) $a = g, e = c, d = b$;

(viii) $a = g, e = d, c = b$;

(ix) $a = g, e = b, e = c, c = b$;

(x) $a = g, e = c, e = d, d = c$;

(xi) $a = g, h = e, h = c, e = c$;

(xii) $a = g, d = b, c = b, d = c$;

(xiii) $a = g, h = c, h = d, d = c, e = b$;

(xiv) $a = g, h = e, h = d, e = d, c = b$;

(xv) $a = g, h = e, h = c, e = c, d = b$;

(xvi) $a = g, h = d, e = b, e = c, c = b$;

(xvii) $a = g, h = c, e = b, d = b, e = d;$

(xviii) $a = g, h = e, h = d, e = d, h = c, e = c, d = c;$

(xix) $e = b, e = c, c = b, e = d, d = b, d = c, h = b, h = e, h = c, h = d.$

Proof.

1. In this case Δ is given in Figure 4(i).

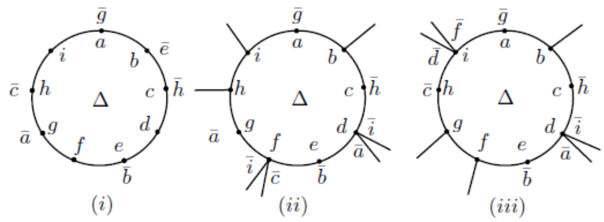

Figure 4: Region Δ

The subcases which are to be examined are given below:

(a) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2;$

(b) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2.$

(a) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ch^{-1}$, $l(v_e) = eb^{-1}$, and $l_\Delta(v_g) = ga^{-1}$, as given in Figure 4(ii). Notice that $l_\Delta(v_c) = ch^{-1}$ and $l_\Delta(v_e) = eb^{-1}$ implies that $l_\Delta(v_d) = i^{-1}da^{-1}w$ in which $w \in \{b^{-1}, c^{-1}, e^{-1}, h^{-1}\}$ which implies $d_\Delta(v_d) > 3$. Similarly notice that $l_\Delta(v_g) = ga^{-1}$ and $l_\Delta(v_e) = eb^{-1}$ implies that $l_\Delta(v_f) = fi^{-1}c^{-1}w$ in which $w \in \{b, d, e, h\}$ which implies $d_\Delta(v_f) > 3$. Since $d_\Delta(v_d) > 3$ and $d_\Delta(v_f) > 3$ so $c(\Delta) \leq 0$.

(b) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ch^{-1}$, $l(v_e) = eb^{-1}$, and $l_\Delta(v_h) = hc^{-1}$, as given in Figure 4(iii). Notice that $l_\Delta(v_c) = ch^{-1}$ and $l_\Delta(v_e) = eb^{-1}$ implies that $l_\Delta(v_d) = i^{-1}da^{-1}w$ in which $w \in \{b^{-1}, c^{-1}, e^{-1}, h^{-1}\}$ which implies $d_\Delta(v_d) > 3$. Similarly notice that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_h) = hc^{-1}$ implies that $l_\Delta(v_i) = if^{-1}d^{-1}w$ in which $w \in \{b, c, e, h\}$ which implies $d_\Delta(v_i) > 3$. Since $d_\Delta(v_d) > 3$ and $d_\Delta(v_i) > 3$ so $c(\Delta) \leq 0$.

2. In this case Δ is given in Figure 5(i).

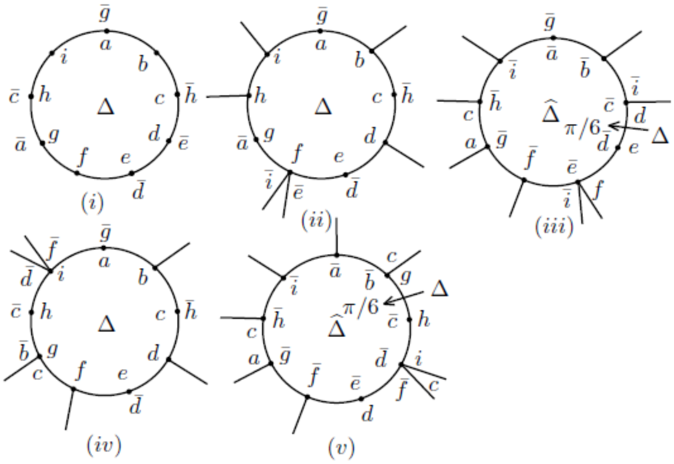

Figure 5: Regions Δ and $\widehat{\Delta}$

The subcases which are to be examined are given below:

(a) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$.

(a) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ch^{-1}$, $l_\Delta(v_e) = ed^{-1}$, $l_\Delta(v_g) = ga^{-1}$, as shown in Figure 5(ii). Notice that $l_\Delta(v_g) = ga^{-1}$ and $l_\Delta(v_e) = ed^{-1}$ implies that $l_\Delta(v_f) = fi^{-1}e^{-1}w$ in which $w \in \{b, c, d, h\}$ which implies $d_\Delta(v_f) > 3$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\widehat{\Delta})$ is given by Figure 5(iii). Notice that $d_{\widehat{\Delta}}(v_{a^{-1}}) = d_{\widehat{\Delta}}(v_{e^{-1}}) = 2$. Similarly notice that either $d_{\widehat{\Delta}}(v_{g^{-1}}) = 2$ or $d_{\widehat{\Delta}}(v_{h^{-1}}) = 2$ otherwise contradiction occur and all other vertices have degree atleast 3. Therefore $c(\widehat{\Delta}) \leq c(2, 2, 2, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{6}$.

(b) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ch^{-1}$, $l_\Delta(v_e) = ed^{-1}$, $l_\Delta(v_h) = hc^{-1}$, as shown in Figure 5(iv). Notice that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_h) = hc^{-1}$ implies that $l_\Delta(v_i) = if^{-1}d^{-1}w$ in which $w \in \{b, c, e, h\}$ which implies $d_\Delta(v_i) > 3$. Add $c(\Delta) \leq \frac{\pi}{6}$ to $c(\widehat{\Delta})$ is given by Figure 5(v). Notice that $d_{\widehat{\Delta}}(v_{c^{-1}}) = d_{\widehat{\Delta}}(v_{e^{-1}}) = 2$. Similarly notice that either $d_{\widehat{\Delta}}(v_{g^{-1}}) = 2$ or $d_{\widehat{\Delta}}(v_{h^{-1}}) = 2$ otherwise contradiction occur. Observe that either $d_{\widehat{\Delta}}(v_{b^{-1}}) = 3$ or $d_{\widehat{\Delta}}(v_{a^{-1}}) = 2$ since $d_{\widehat{\Delta}}(v_{b^{-1}}) = 3$ already present so $d_{\widehat{\Delta}}(v_{a^{-1}}) > 2$ otherwise contradiction occur and all other vertices have degree atleast 3. Therefore $c(\widehat{\Delta}) \leq c(2, 2, 2, 3, 3, 3, 3, 3, 4) = \frac{-\pi}{3}$.

3. In this case Δ is given in Figure 6(i).

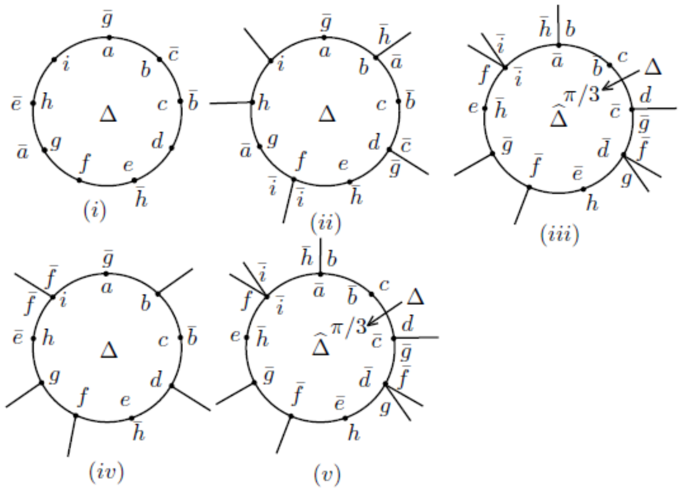

Figure 6: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

4. In this case Δ is given in Figure 7(i).

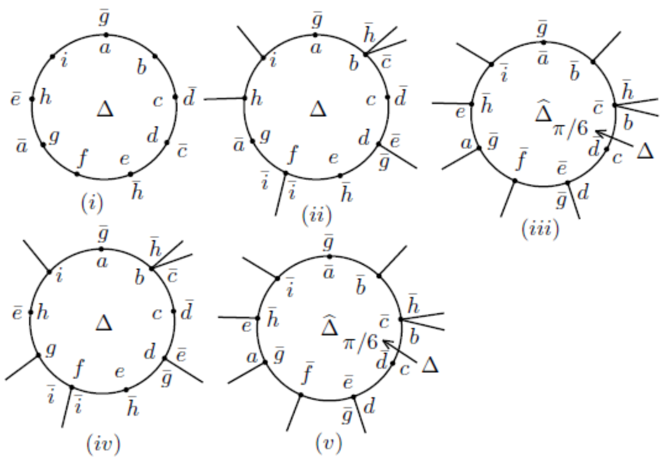

Figure 7: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

5. In this case Δ is given in Figure 8(i).

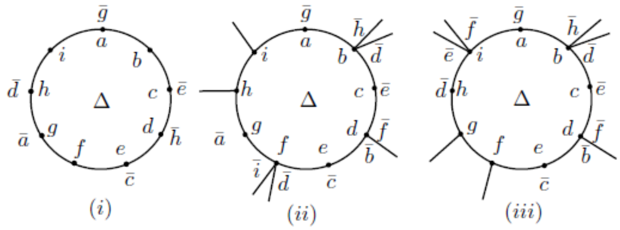

Figure 8: Region Δ

The subcases which are to be examined are given below:

(a) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$;

(b) $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$.

(a) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ce^{-1}$, $l(v_e) = ec^{-1}$, and $l_\Delta(v_g) = ga^{-1}$, as shown in Figure 8(ii). Notice that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_c) = ce^{-1}$ implies that $l_\Delta(v_b) = bd^{-1}h^{-1}w$ in which $w \in \{b, c, d, e\}$ which implies $d_\Delta(v_b) > 3$. Similarly notice that $l_\Delta(v_g) = ga^{-1}$ and $l_\Delta(v_e) = ec^{-1}$ implies that $l_\Delta(v_f) = fi^{-1}d^{-1}w$ in which $w \in \{b, c, e, h\}$ which implies $d_\Delta(v_f) > 3$. Since $d_\Delta(v_b) > 3$ and $d_\Delta(v_f) > 3$ so $c(\Delta) \leq 0$.

(b) Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_h) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = ce^{-1}$, $l(v_e) = ec^{-1}$, and $l_\Delta(v_h) = hd^{-1}$, as shown in Figure 8(iii). Notice that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_c) = ce^{-1}$ implies that $l_\Delta(v_b) = bd^{-1}h^{-1}w$ in which $w \in \{b, c, d, e\}$ which implies $d_\Delta(v_b) > 3$. Similarly notice that $l_\Delta(v_a) = ag^{-1}$ and $l_\Delta(v_h) = hd^{-1}$ implies that $l_\Delta(v_i) = if^{-1}e^{-1}w$ in which $w \in \{b, c, d, h\}$ which implies $d_\Delta(v_i) > 3$. Since $d_\Delta(v_b) > 3$ and $d_\Delta(v_i) > 3$ so $c(\Delta) \leq 0$.

6. In this case Δ is given in Figure 9(i).

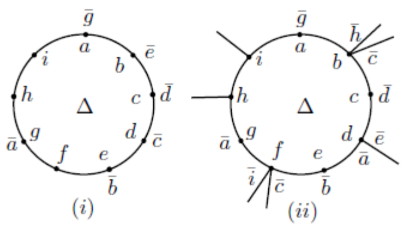

Figure 9: Region Δ

Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = cd^{-1}$, $l(v_e) = eb^{-1}$, and $l_\Delta(v_g) = ga^{-1}$, as shown in Figure 9(ii). Notice that $l_\Delta(v_a) =$

ag^{-1} and $l_\Delta(v_c) = cd^{-1}$ implies that $l_\Delta(v_b) = bc^{-1}h^{-1}w$ in which $w \in \{b, c, d, e\}$ which implies $d_\Delta(v_b) > 3$. Similarly notice that $l_\Delta(v_g) = ga^{-1}$ and $l_\Delta(v_e) = eb^{-1}$ implies that $l_\Delta(v_f) = fi^{-1}c^{-1}w$ in which $w \in \{b, d, e, h\}$ which implies $d_\Delta(v_f) > 3$. Since $d_\Delta(v_b) > 3$ and $d_\Delta(v_f) > 3$ so $c(\Delta) \leq 0$.

7. In this case Δ is given in Figure 10(i).

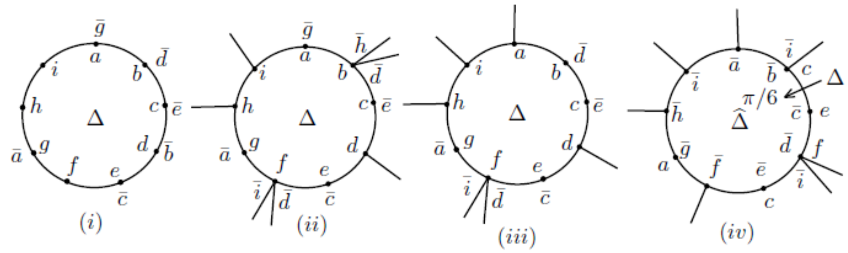

Figure 10: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

8. In this case Δ is given in Figure 11(i).

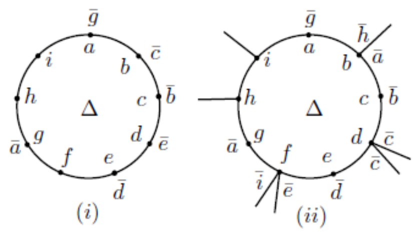

Figure 11: Region Δ

Here $d_\Delta(v_a) = d_\Delta(v_c) = d_\Delta(v_e) = d_\Delta(v_g) = 2$ which implies $l_\Delta(v_a) = ag^{-1}$, $l_\Delta(v_c) = cb^{-1}$, $l(v_e) = ed^{-1}$, and $l_\Delta(v_g) = ga^{-1}$, as shown in Figure 11(ii). Notice that $l_\Delta(v_e) = ed^{-1}$ and $l_\Delta(v_c) = cb^{-1}$ implies that $l_\Delta(v_d) = dc^{-1}c^{-1}w$ in which $w \in \{b, d, e, h\}$ which implies $d_\Delta(v_d) > 3$. Similarly notice that $l_\Delta(v_g) = ga^{-1}$ and $l_\Delta(v_e) = ed^{-1}$ implies that $l_\Delta(v_f) = fi^{-1}e^{-1}w$ in which $w \in \{b, c, d, h\}$ which implies $d_\Delta(v_f) > 3$. Since $d_\Delta(v_d) > 3$ and $d_\Delta(v_f) > 3$ so $c(\Delta) \leq 0$.

9. In this case Δ is given in Figure 12(i).

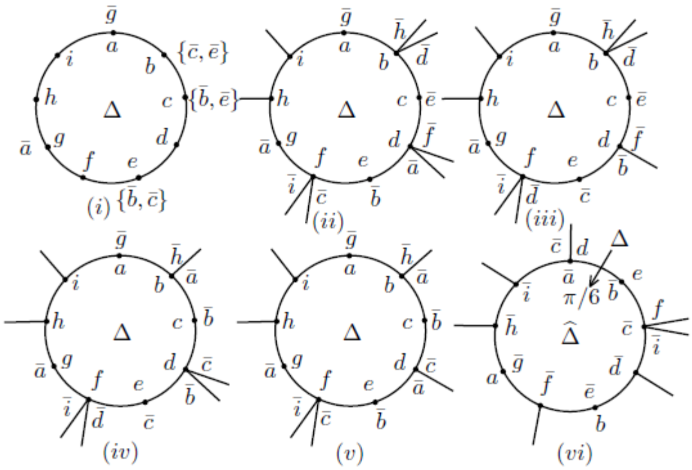

Figure 12: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

10. In this case Δ is given in Figure 13(i).

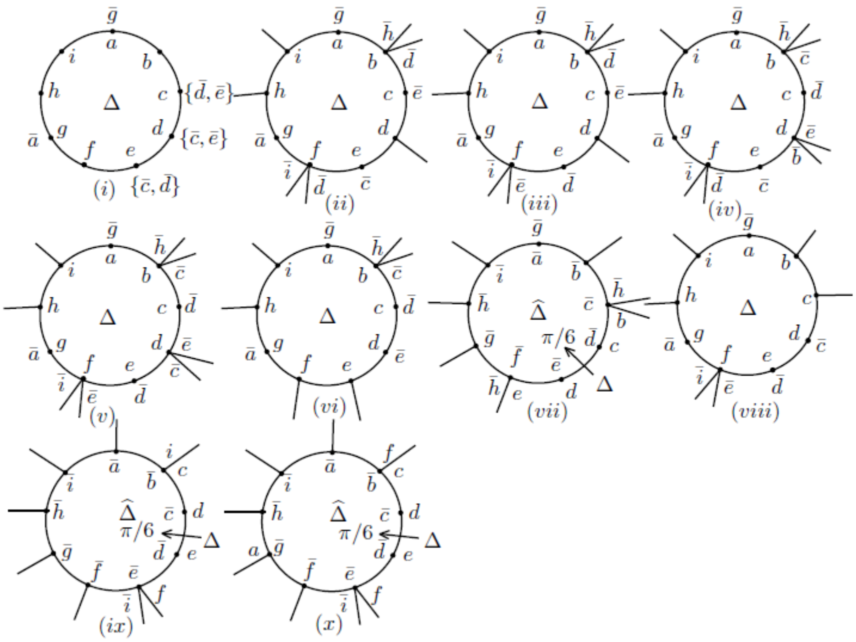

Figure 13: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

11. In this case Δ is given in Figure 14(i).

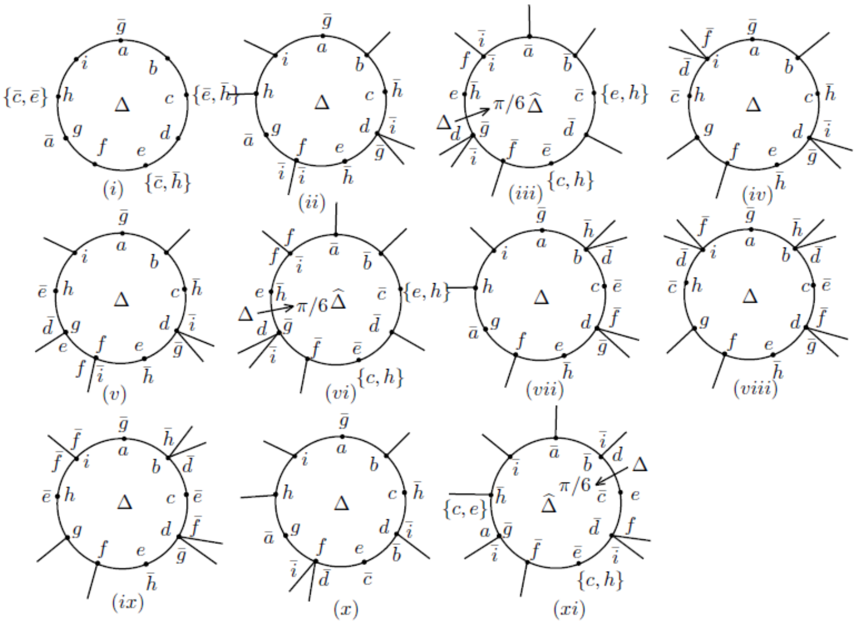

Figure 14: Regions Δ and $\widehat{\Delta}$

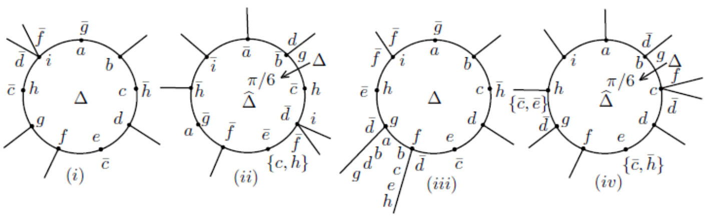

Figure 15: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

12. In this case Δ is given in Figure 16(i).

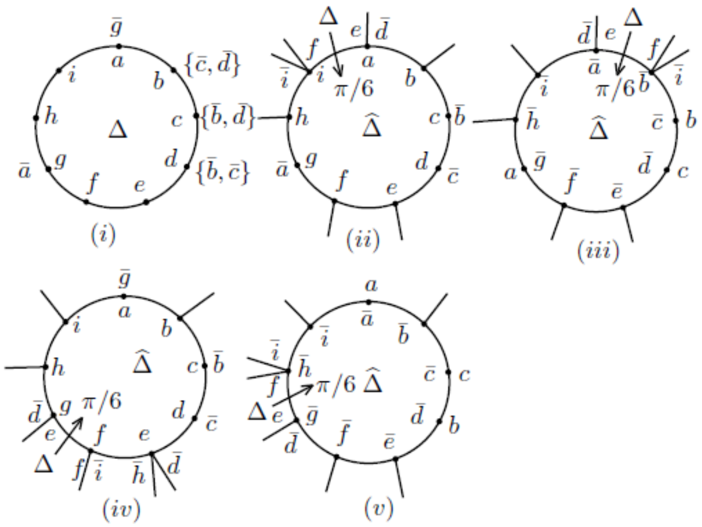

Figure 16: Region Δ

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

13. In this case $c(\Delta) \leq 0$ for all of its subcases as shown in Figure 17.

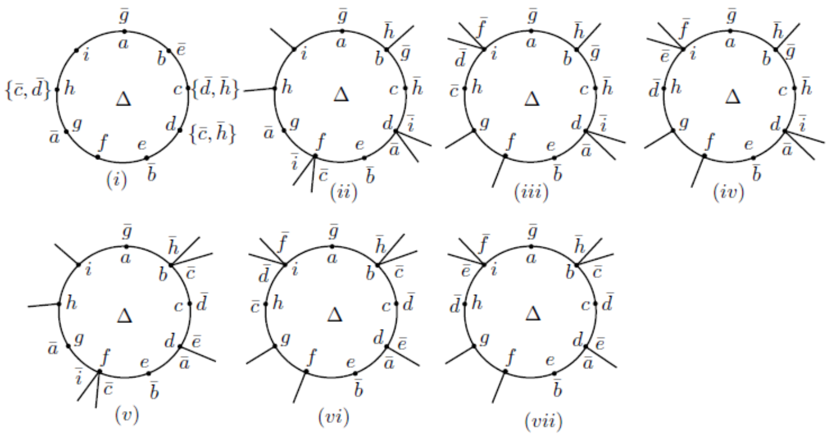

Figure 17: Region Δ

14. In this case Δ as given in Figure 18(i).

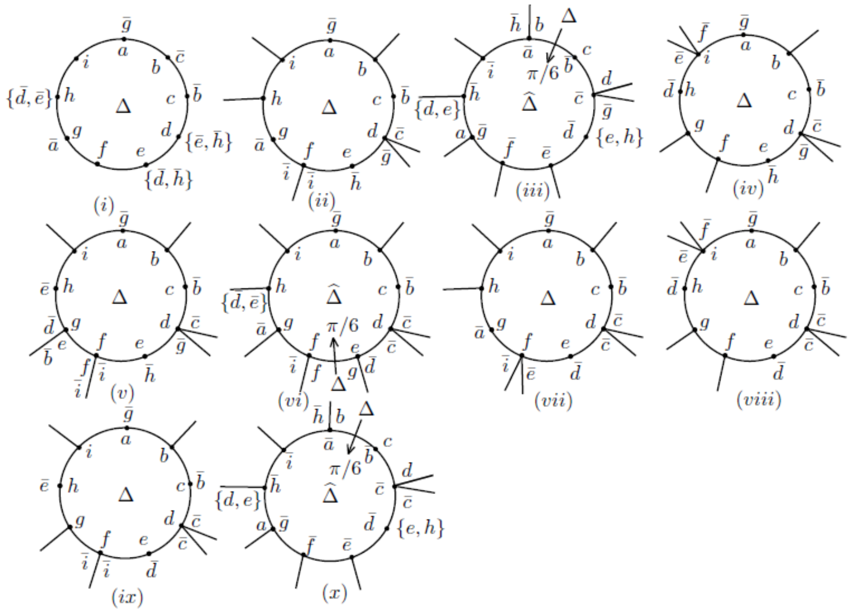

Figure 18: Region Δ

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

15. In this case Δ is given in Figure 19(i).

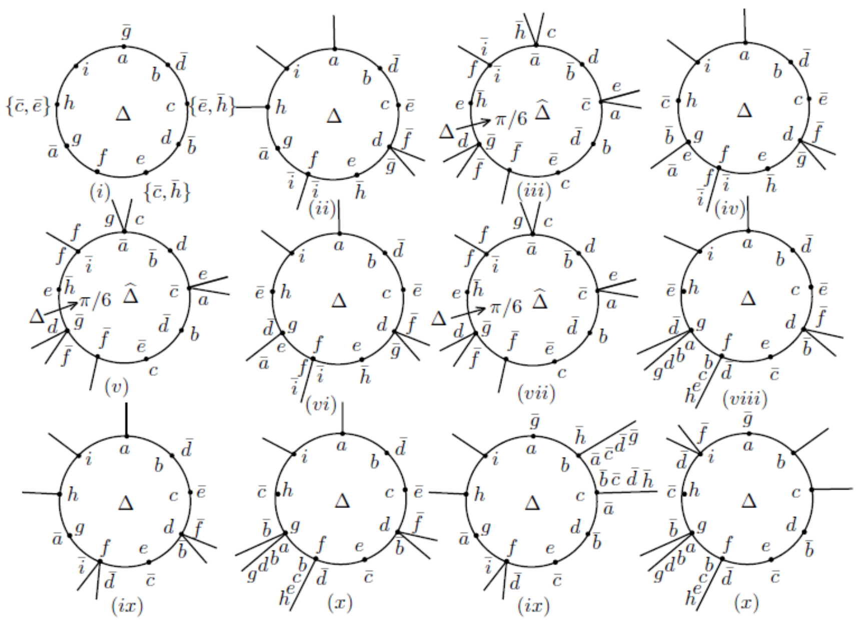

Figure 19: Regions Δ and $\widehat{\Delta}$

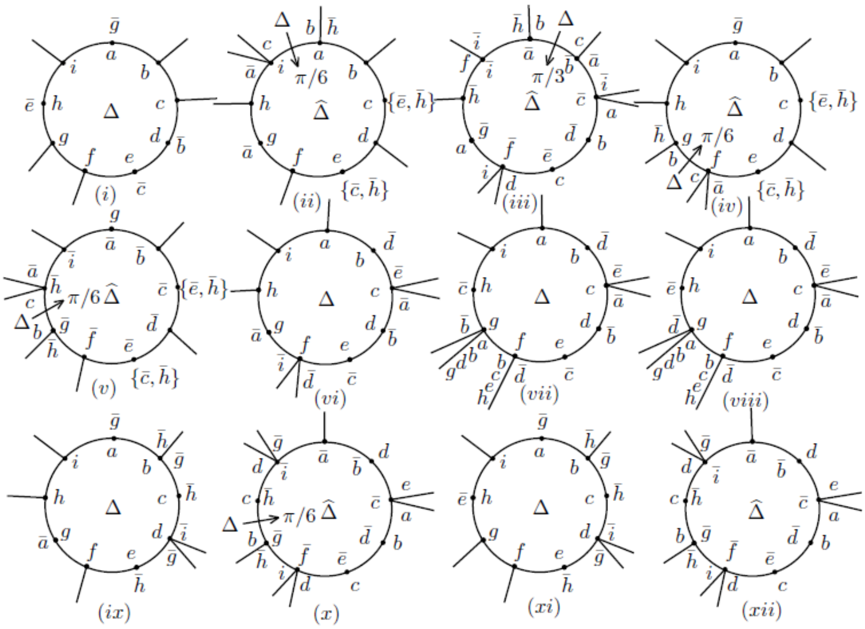

Figure 20: Regions Δ and $\widehat{\Delta}$

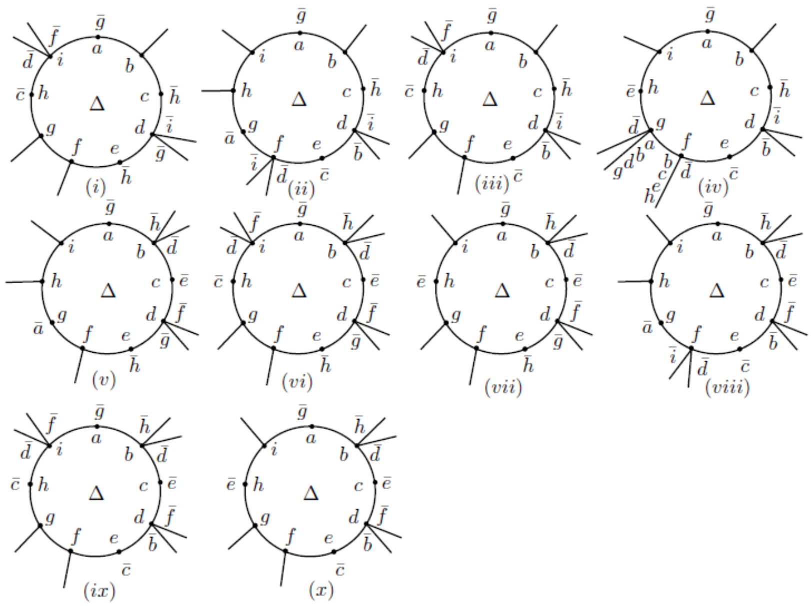

Figure 21: Regions Δ and $\widehat{\Delta}$

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

16. In this case $c(\Delta) \leq 0$ for all of its subcases as shown in Figure 22(i).

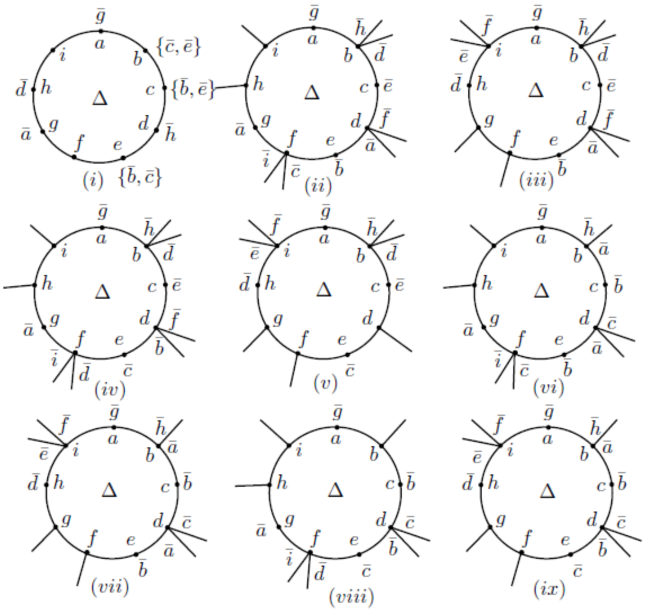

Figure 22: Region Δ

17. In this case Δ is given in Figure 23(i).

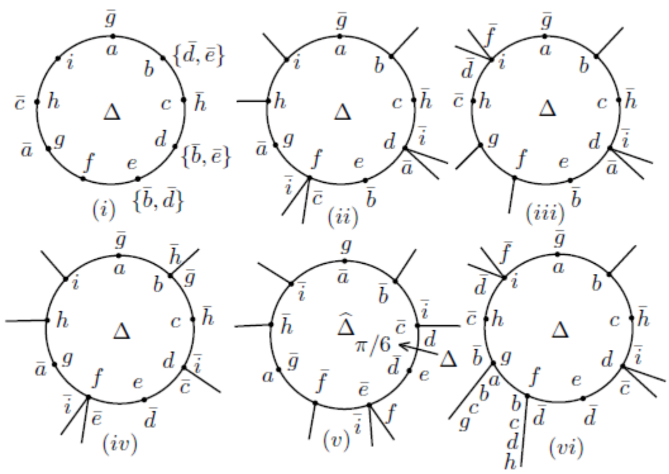

Figure 23: Region Δ

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

18. In this case Δ is given in Figure 24(i).

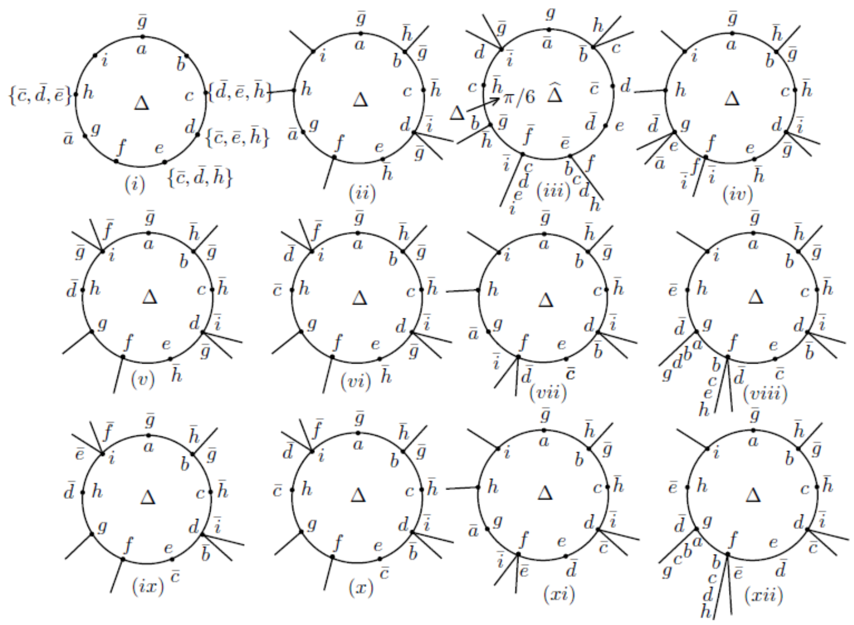

Figure 24: Region Δ

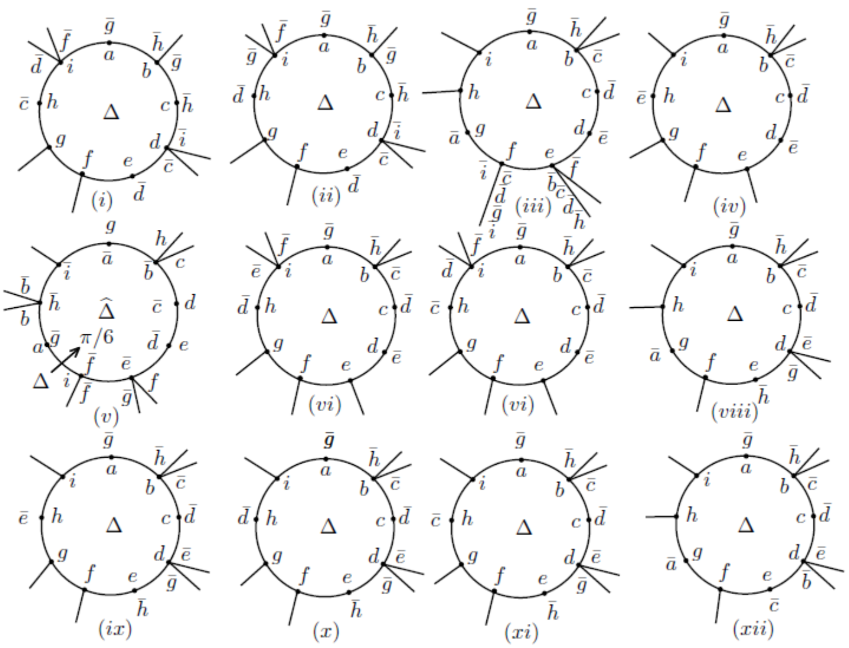

Figure 25: Region Δ

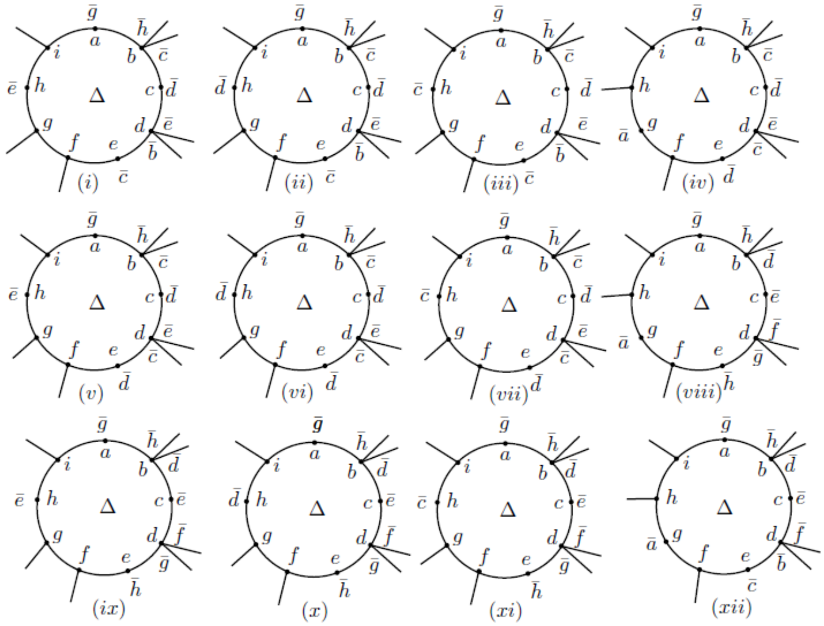

Figure 26: Region Δ

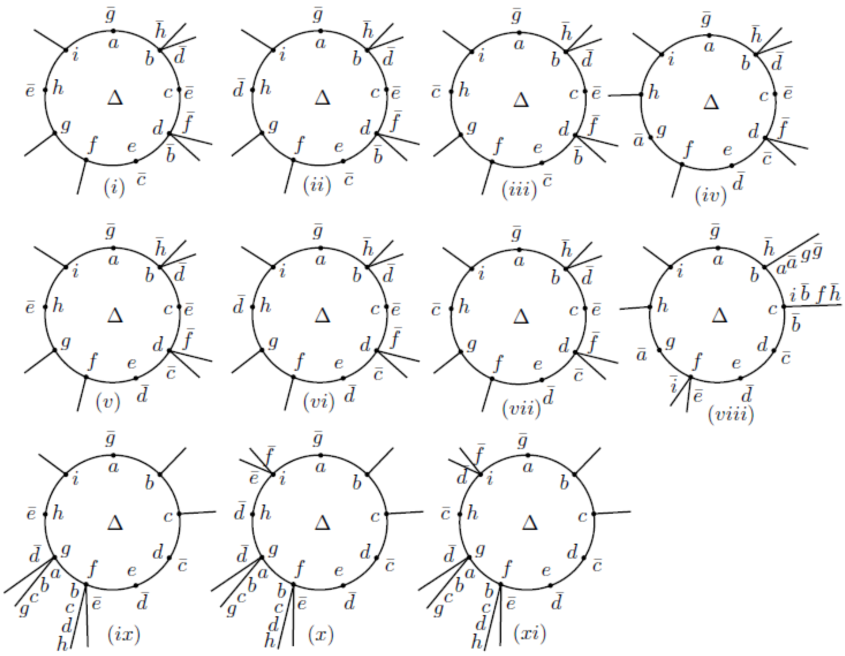

Figure 27: Region Δ

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

19. In this case Δ is given in Figure 28(i).

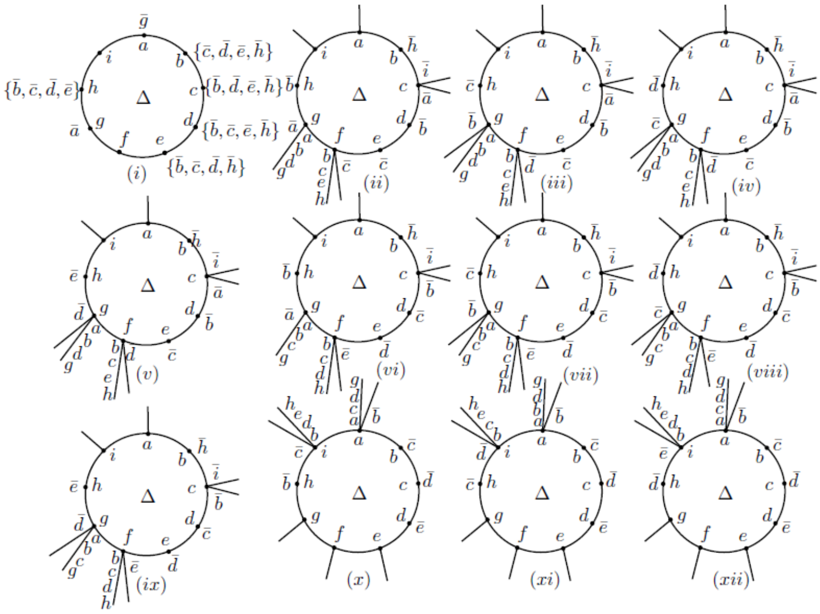

Figure 28: Region Δ

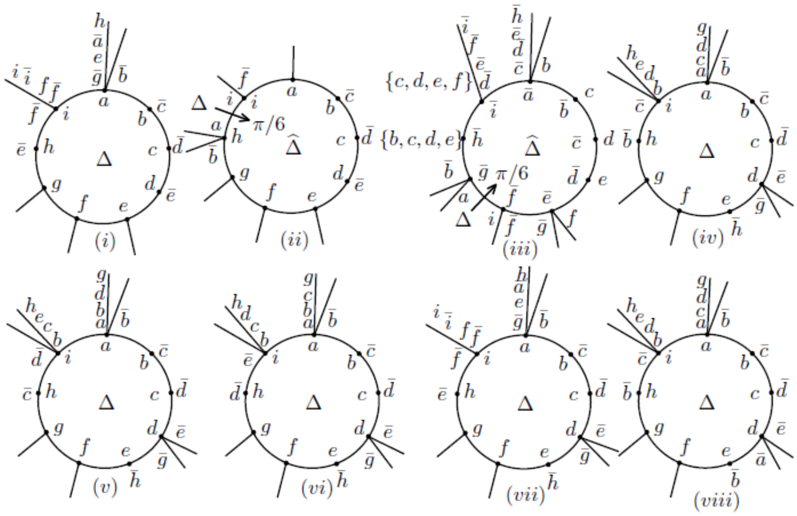

Figure 29: Region Δ

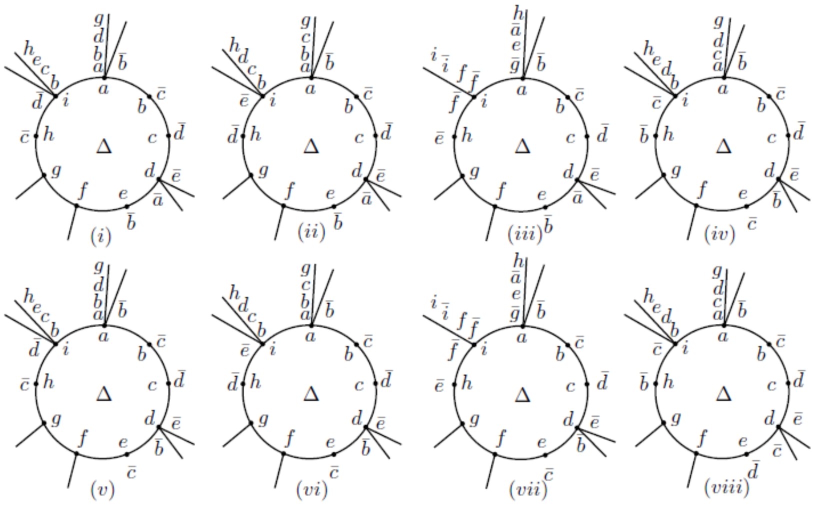

Figure 30: Region Δ

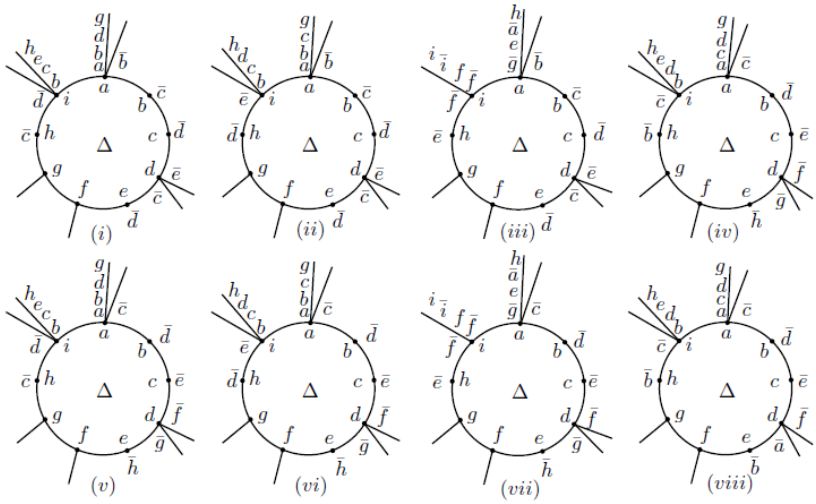

Figure 31: Region Δ

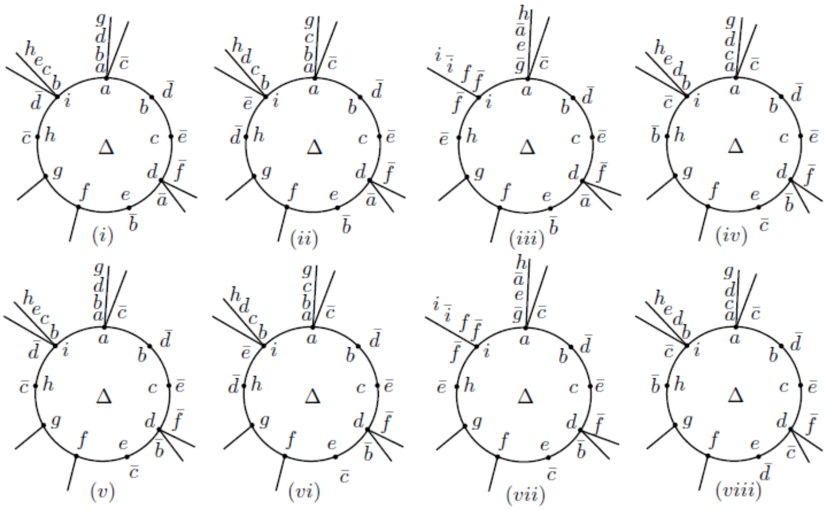

Figure 32: Region Δ

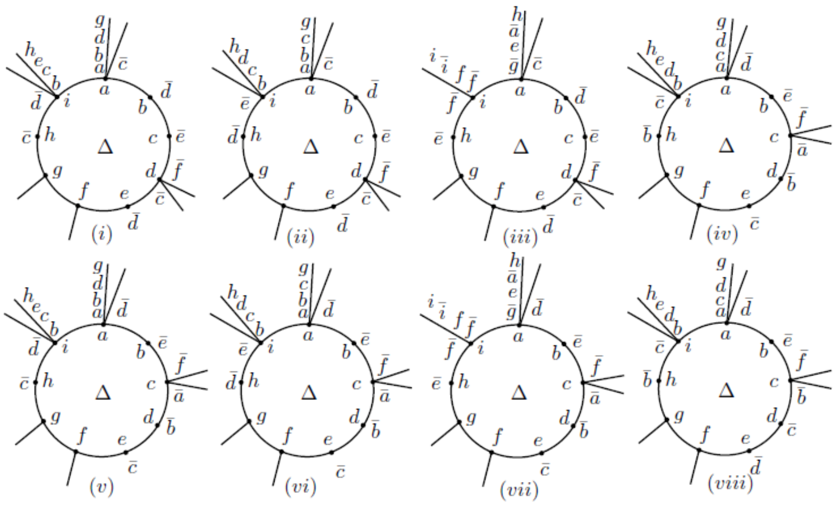

Figure 33: Region Δ

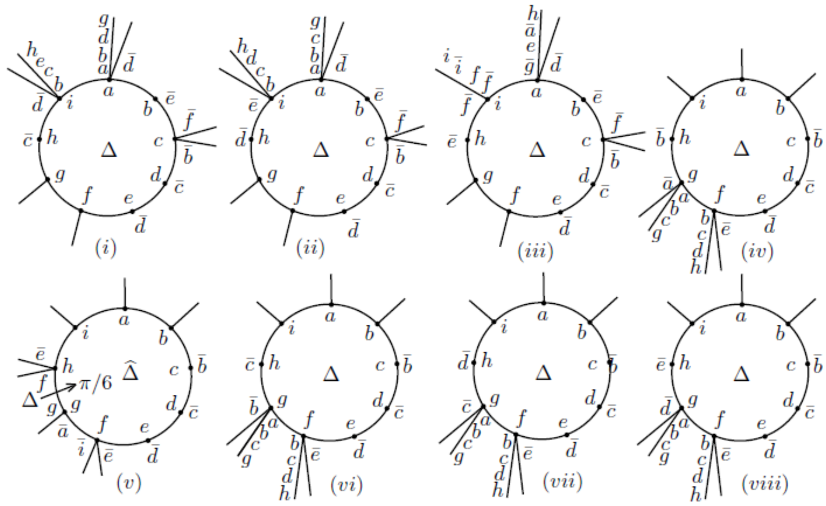

Figure 34: Region Δ

By adding $c(\Delta)$ to $c(\widehat{\Delta})$ we get $c(\Delta) \leq 0$.

There remains only 2 cases given in Theorem that still needs to be solved. In fact, weight test and curvature distribution method can not be applied to these cases. The weight test can not be applied to these cases as it is not possible to find a weight function θ that satisfies all the three conditions of weight test simultaneously, whereas curvature test can not be applied to these cases as it is impossible to find any neighbouring region $\widehat{\Delta}$ in the neighbourhood of region Δ that cancels the curvatures of region Δ.

Remark 2. We remark that the list of exceptional cases given in Theorem in section 1 is open for the equation $s_1(t)$. Since weight test and curvature distribution can not be applied to these cases to prove Levin conjecture, therefore, some new methods needs to be developed to establish the validity of the remaining cases of Levin conjecture for these group equations of length 9.

References

[1] Muhammad Saeed Akram and Maira Amjid. Solving a group equation of length nine. *arXiv preprint arXiv:2008.09508*, 2020.

[2] Muhammad Saeed Akram and Khawar Hussain. Levin's conjecture for an equation of length nine. *arXiv preprint arXiv:2008.06846*, 2020.

[3] M Fazeel Anwar, Mairaj Bibi, and Muhammad Saeed Akram. On a nonsingular equation of length 9 over torsion free groups. *European Journal of Pure and Applied Mathematics*, 12(2):590–604, 2019.

[4] Muhammad Fazeel Anwar, Mairaj Bibi, and Muhammad Saeed Akram. On solvability of certain equations of arbitrary length over torsion-free group. *Preprint*, 2018.

[5] Muhammad Fazeel Anwar, Mairaj Bibi, and Sohail Iqbal. On certain equations of arbitrary length over torsion-free groups. *Preprint*, 2019.

[6] Mairaj Bibi. *Equations of length seven over torsion free groups*. PhD thesis, PhD thesis, University of Notingham, 2015.

[7] Mairaj Bibi, Muhammad Fazee Anwar, Sohail Iqbal, and Muhammad Saeed Akram. Solution of a non-singular equation of length 8 over torsion free groups. *Preprint*, 2019.

[8] Mairaj Bibi and Martin Edjvet. Solving equations of length seven over torsion-free groups. *Journal of Group Theory*, 21(1):147–164, 2018.

[9] William A Bogley and Stephen J Pride. Aspherical relative presentations. *Proceedings of the Edinburgh Mathematical Society*, 35(1):1–39, 1992.

[10] Martin Edjvet. On the asphericity of one-relator relative presentations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 124(4):713–728, 1994.

[11] Martin Edjvet and James Howie. The solution of length four equations over groups. *Transactions of the American Mathematical Society*, 326(1):345–369, 1991.

[12] Anastasia Evangelidou. The solution of length five equations over groups. *Communications in Algebra®*, 35(6):1914–1948, 2007.

[13] James Howie. On pairs of 2-complexes and systems of equations over groups. *Journal für die reine und angewandte Mathematik*, 1981(324):165–174, 1981.

[14] James Howie. The solution of length three equations over groups. *Proceedings of the Edinburgh Mathematical Society*, 26(2):89–96, 1983.

[15] Sergey V Ivanov and Anton A Klyachko. Solving equations of length at most six over torsion-free groups. *Journal of Group Theory*, 3(3):329–337, 2000.

[16] F Levin. Solutions of equations over groups. *Bulletin of American Mathematical Society*, 68:603–604, 1962.

[17] Bernhard Hermann Neumann. Adjunction of elements to groups. *Journal of the London Mathematical Society*, 18:4–11, 1943.