EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 4, 2020, 830-839
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

On B-ideals in a Topological B-algebra and the Uniform B-topological Space

Katrina E. Belleza[1,*], Jocelyn P. Vilela[2]

[1] *Department of Computer, Information Sciences and Mathematics, School of Arts and Sciences, University of San Carlos, 6000 Cebu City, Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. This paper presents the characterizations and properties of B-ideals in a topological B-algebra and introduces the uniform topology on a B-algebra in terms of its B-ideals. Moreover, this paper shows that a uniform B-topological space is a topological B-algebra.

2020 Mathematics Subject Classifications: 54A05, 54F65, 54H99, 55M99

Key Words and Phrases: B-ideals, topological B-algebra, uniform B-topology

1. Introduction

Y.B. Jun et al. [4] in 1999 introduced topological BCI-algebras, provided some properties on this structure, and characterized a topological BCI-algebra in terms of neighborhoods. In 2002, J.Neggers and H.S. Kim [9] introduced and investigated B-algebras. In 2017, S. Mehrshad and J. Golzarpoor [7] provided some properties of uniform topology and topological BE-algebras. A recent study on topological B-algebras was conducted by N.C. Gonzaga, Jr. [6] in 2019, which characterized a topological B-algebra with respect to neighborhoods.

This paper provides some properties of topological B-algebra, describes the B-ideals in a topological B-algebra, and characterizes uniform B-topology in a B-algebra.

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v13i4.3807

Email addresses: kebelleza@usc.edu.ph (K. Belleza), jocelyn.vilela@g.msuiit.edu.ph (J. Vilela)

2. Preliminaries

An algebra of type (2,0) is an algebra with a binary operation and a constant element.

Definition 1. [9] A *B-algebra* is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms for all x, y, z in X:

$$\text{(B1) } x * x = 0 \quad \text{(B2) } x * 0 = x \quad \text{(B3) } (x * y) * z = x * [z * (0 * y)]$$

Example 1. [8] Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	b	a	c	d	e
a	a	0	b	d	e	c
b	b	a	0	e	c	d
c	c	d	e	0	b	a
d	d	e	c	a	0	b
e	e	c	d	b	a	0

Then $(X, *, 0)$ is a B-algebra.

Definition 2. [6] Let A and B be nonempty subsets of a B-algebra X. The product of A and B, denoted by $A * B$, is given by

$$A * B = \{a * b | a \in A, b \in B\}.$$

Lemma 1. [9] Let $(X, *, 0)$ be a B-algebra. Then for any $x, y \in X$,

(i) $x * y = 0$ implies $x = y$; (ii) $0 * x = 0 * y$ implies $x = y$; (iii) $0 * (0 * x) = x$.

Definition 3. [10] Let $(X, *, 0)$ be a B-algebra. A nonempty subset N of X is called a *subalgebra* of X if $x * y \in N$ for any $x, y \in N$.

Lemma 2. [3] Let X be a B-algebra. If $\{N_\alpha : \alpha \in A\}$ is a nonempty collection of subalgebras of X, then $\bigcap_{\alpha \in A} N_\alpha$ is a subalgebra of X.

Definition 4. [8] Let $(X, *, 0)$ be a B-algebra. A nonempty subset S of X is said to be *normal* in X if for any $x * y, a * b \in S$, $(x * a) * (y * b) \in S$.

Theorem 1. [10] *Let N be a subalgebra of a B-algebra X. Then the following statements are equivalent:*

(*i*) *N is a normal subalgebra*; (*ii*) *if $x \in X$ and $y \in N$, then $x * (x * y) \in N$.*

Suppose $(X, *, 0)$ is a B-algebra and I a normal subalgebra of X. The relation " $\cong^I$ " defined by $x \cong^I y$ if and only if $x * y, y * x \in I$ is a congruence relation on X for any $x, y \in X$. That is, $\cong^I$ is an equivalence relation and for each $a, b, x, y \in X$, if $x \cong^I y$ and $a \cong^I b$, then $a * x \cong^I b * y$. Let $I_x = \{y : y \cong^I x\}$ denote the equivalence class of x and $X/I = \{I_x : x \in X\}$. Then X/I is a B-algebra called the *quotient B-algebra* under the binary operation given by $I_x * I_y = I_{x*y}$ [3].

Definition 5. [**1**] Let $(X, *, 0)$ be a B-algebra and I a nonempty subset of X. Then I is called a *B-ideal* of X if it satisfies the following: for any x, y in X,

$$\text{(i) } 0 \in I; \quad \text{(ii) if } x * y \in I \text{ and } y \in I, \text{ then } x \in I.$$

Remark 1. Not every B-ideal is a normal subset of a B-algebra X. Consider Example 1. Then $\{0, c\}$ is a B-ideal but is not normal since $c * a = c, b * d = c \in \{0, c\}$ with $(c * b) * (a * d) = b \notin \{0, c\}$.

Remark 2. Any nonempty normal subset of X is a subalgebra. Hence, if I is a normal B-ideal, then X/I is a B-algebra.

Let X be a set. A *topology* (or topological structure) in X is a family τ of subsets of X that satisfies the following:

(i) Each union of members of τ is also a member of τ;

(ii) Each finite intersection of members of τ is also a member of τ; and

(iii) $\varnothing$ and X are members of τ.

A couple (X, τ) consisting of a set X and a topology τ in X is called a *topological space.* We also say "τ is the topology of the space X". The members of τ are called *open sets* of (X, τ). Let (X, τ_X) and (Y, τ_Y) be topological spaces. A map $f : X \to Y$ is called *continuous* if the inverse image of each open set in Y is open in X (that is, if f^{-1} maps τ_Y into τ_X) [**2**].

Definition 6. [**2**] Let $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be any family of topological spaces. For each $\alpha \in \mathcal{A}$, let τ_α be the topology for Y_α. The *cartersian product topology* in $\prod_\alpha Y_\alpha$ is that having for subbasis all sets $< U_\beta >= \rho_\beta^{-1}(U_\beta)$, where $\rho : \prod_\alpha Y_\alpha \to Y_\alpha$, U_β ranges over all members of τ_β and β over all elements of $\mathcal{A}$.

Definition 7. [**6**] Let X be a B-algebra. A topology τ furnished on X is called a *B-topology* on X. A B-topological space (X, τ) is called a *topological B-algebra* if τ is a B-topology on X and the binary operation $* : X \times X \to X$ is continuous, where $X \times X$ is furnished by the Cartesian product topology.

Let (X, τ) be a topological space and $A \subset X$. By a *neighborhood of an element* x in X (denoted as $U(x)$) is meant any open set (that is, member of τ) containing x. The *interior* *Int*(A) of A is the largest open set contained in A, that is, $\text{Int}(A) = \bigcup\{U | U \in \tau, U \subset A\}$. A point a is an interior point of A if $a \in \text{Int}(A)$, that is, there exists $U(a) \in \tau$ such that $U(a) \subset A$. A is open if and only if $\text{Int}\,(A) = A$. A set $Y \subset X$ is a *closed set* in X if its complement is open. A point $x \in X$ is *adherent* to Y if each neighborhood of x contains at least one point of Y. The set $\overline{Y} = \{x \in X | \forall U(x), U(x) \cap Y \neq \varnothing\}$ of all points in X adherent to Y is called the *closure* of Y [**2**].

Theorem 2. [**6**] *Let $X = (X, *, 0)$ be a B-algebra and τ a B-topology on the set X. Then (X, τ) is a topological B-algebra if and only if for all x, y in X and for every neighborhood W of $x * y$, there are neighborhoods U and V of x and y, respectively, such that $U * V \subseteq W$.*

Throughout this article we will denote a B-topological space (X, τ), topological B-algebra $(X, *, \tau)$, or a B-algebra $(X, *, 0)$ as simply, X.

3. B-ideals in Topological B-algebras

Example 2. Consider the B-algebra $X = \{0, a, b, c\}$ with the binary operation "$*$" defined on the Cayley table provided. Let $\tau = \{X, \varnothing, \{0, b\}, \{a, c\}\}$. Then τ is a B-topology on X and $(X, *, \tau)$ is a topological B-algebra.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Remark 3. Not every B-ideal of a B-algebra X is either an open or closed set in a topological B-algebra. This remark is illustrated in the next example.

Example 3. Consider the topological B-algebra in Example 2. Let $I = \{0, c\}$. Then I is a B-ideal of X. Observe that $I \notin \tau$ implying that I is not an open set in X. Also, $X\backslash I = \{a, b\} \notin \tau$ implying that I is not a closed set in X.

However if a B-ideal A in a topological B-algebra X is open, it is also a closed B-ideal in X. This is formally stated in the next theorem.

Theorem 3. *If A is an open B-ideal of a topological B-algebra X, then A is also closed.*

Proof. *Suppose A is an open B-ideal of a topological B-algebra X. Let $x \in X\backslash A$. Since A is a B-ideal of X, $x * x = 0 \in A$ by (B1). By Theorem 2, there exists $U(x)$ such that $U(x) * U(x) \subseteq A$. We claim that $U(x) \subseteq X\backslash A$. Assume on the contrary that $U(x) \nsubseteq X\backslash A$, that is, $U(x) \cap A \neq \varnothing$. Then there exists $y \in U(x) \cap A$. Note that for all $z \in U(x)$, $z * y \in U(x) * U(x) \subset A$. Since $y \in A$ and A is a B-ideal, $z \in A$. So, $U(x) \subseteq A$ which implies that $x \in A$, a contradiction. Hence, $X\backslash A$ is open. Therefore, A is closed in X.* □

The next theorem is a characterization of an open set (containing 0) in a topological B-algebra.

Theorem 4. *Let X be a topological B-algebra and $A \subset X$ such that $0 \in A$. Then A is open if and only if 0 is an interior point of A.*

Proof. *Suppose A is open. Since $0 \in A$, 0 is an interior point of A. Conversely, suppose 0 is an interior point of A. Then there exists $U(0)$ such that $U(0) \subseteq A$. Let $y \in A$. By (B1), $y * y = 0 \in U(0)$. Since X is a topological B-algebra, by Theorem 2, there exist $U(y)$ such that $U(y) * U(y) \subseteq U(0)$. It remains to show that $U(y) \subseteq A$. Let $x \in U(y)$. By (B1), $x * x = 0 \in A$. If $x \in A$, we are done. Suppose $x \notin A$. Then $x \notin U(0)$. This implies that $x \notin U(y) * U(y)$. By (B2), $x * 0 = x \notin U(y) * U(y)$. This implies that $x \notin U(y)$ which is a contradiction. Therefore, $U(y) \subseteq A$ and A is open.* □

The following corollary follows from Theorem 4.

Corollary 1. *Let X be a topological B-algebra. Then 0 is an interior point of a B-ideal I if and only if I is open.*

The next example illustrates that an open subset of a topological B-algebra may not be a B-ideal in which the observation is formally stated as a remark.

Example 4. Consider the topological B-algebra in Example 2. Examine the open set $\{a, c\}$. Note that $\{a, c\}$ is not a B-ideal since $0 \notin \{a, c\}$ and $b * c = a$ and $c \in \{a, c\}$ but $b \notin \{a, c\}$.

Remark 4. *Not every open subset of a topological B-algebra X is a B-ideal of X.*

However, if all open sets are neighborhoods of 0, every open subset of a topological B-algebra X is a B-ideal of X. This is formally stated in the next theorem which is a characterization of a B-ideal in a topological B-algebra.

Theorem 5. *Let X be a topological B-algebra and I an open subset of X. If $0 \in \bigcap_{U \in \tau} U$, then I is a B-ideal of X.*

Proof. Let $x * y \in I$ where $y \in I$. Since I is open, by Theorem 2, there exist $V(x)$ and $V(y)$ such that $V(x)*V(y) \subseteq U(x*y) \subseteq I$. By (B2), $x = x*0 \in V(x)*V(y) \subseteq I$. Therefore, I is a B-ideal of X. □

Lemma 3. *Let X be a topological B-algebra, $I_0 \subseteq X$ where I_0 contains 0 is such that if $0 \in U$, then $I_0 \subseteq U$ for all $U \in \tau$. Then for any $x \in I_0$ and $U(x) \in \tau, I_0 \subseteq U(x)$.*

Proof. Suppose $x \in I_0$. By (B2), $x * 0 = x \in U(x)$. By Theorem 2, there exist $V(x)$ and $V(0)$ such that $V(x) * V(0) \subseteq U(x)$. By (B1) and the hypothesis, $0 = x * x \in V(x) * I_0 \subseteq V(x) * V(0) \subseteq U(x)$. This implies that $U(x)$ is an open set containing 0. Therefore, $I_0 \subseteq U(x)$. □

The next theorem gives another characterization of a B-ideal.

Theorem 6. *Let X be a topological B-algebra and I an open subset of X containing 0 such that if $0 \in U$, then $I \subseteq U$ for all $U \in \tau$. Then I is a B-ideal of X.*

Proof. Suppose $x*y, y \in I$ for any $x, y \in X$. By Theorem 2, there exist $U(x)$ and $U(y)$ such that $U(x)*U(y) \subset I$. By (B2), the hypothesis, and Lemma 3, $x = x*0 \in U(x)*I \subset U(x)*U(y) \subset I$. Hence, I is a B-ideal of X. □

Note that the converse of Theorem 6 is not always true as shown in the next example.

Example 5. Let X be the topological B-algebra in Example 2. Then the trivial B-ideal X is not the smallest open set containing 0.

However, if the B-ideal I is closed, I is also open but may not be the smallest open set containing 0. This is stated in the next theorem which is the converse of Theorem 3

Theorem 7. *Let X be a topological B-algebra and I a closed B-ideal of X. Then I is also open.*

Proof. Suppose I is a closed B-ideal of X. Assume on the contrary that I is not an open set in X. By Theorem 4, 0 is not an interior point of I. This implies that for all $U(0) \in \tau$, $U(0) \nsubseteq I$. Let I_0 be open with property defined in Lemma 3. Then $I_0 \nsubseteq I$. Hence, $(X\backslash I) \cap I_0 \neq \varnothing$ and so there exists $z \in (X\backslash I) \cap I_0$. Note that $(X\backslash I) \cap I_0$ is an open set containing z. By Lemma 3, $I_0 \subset (X\backslash I)$. This implies that $0 \in X\backslash I$ which is a contradiction. Therefore, I is open. □

The next corollary follows directly from Theorems 3 and 7.

Corollary 2. *Suppose X is a topological B-algebra and I a B-ideal of X. Then I is an open subset of X if and only if I is a closed subset of X.*

Theorem 8. *Let $\mathcal{I}$ be a family of normal B-ideals in a B-algebra X. Then there is a topology $\tau = \{U \subseteq X | \forall x \in U, \exists I \in \mathcal{I}$ such that $I_x \subseteq U\}$ such that $(X, *, \tau)$ is a topological B-algebra.*

Proof. Note that for all $x \in X$, there exists $I \in \mathcal{I}$ such that $I_x \subseteq X$. This implies that $X \in \tau$. Suppose $\varnothing \notin \tau$. Then there exists $x \in \varnothing$ such that for all $I \in \mathcal{I}$, $I_x \nsubseteq U$ which is a contradiction. Hence, $\varnothing \in \tau$. Let $y \in U_1 \cap U_2$, where $U_1, U_2 \in \tau$. Then $y \in U_1$ and $y \in U_2$ which imply that there exist $I_1, I_2 \in \mathcal{I}$ such that $I_{1y} \subseteq U_1$ and $I_{2y} \subseteq U_2$. Let $I = I_1 \cap I_2 \in \mathcal{I}$.

Claim 1: $I_y \subseteq I_{1y}, I_y \subseteq I_{2y}$.

Suppose $x \in I_y$. Then $y \cong^I x$ which implies that $y * x \in I \subseteq I_1$. Hence, $y \cong^{I_1} x$ implying that $x \in I_{1y}$ so that $I_y \subseteq I_{1_y}$. Similarly, $I_y \subseteq I_{2y}$. This proves claim 1.

Since $I_{1y} \subseteq U_1$ and $I_{2y} \subseteq U_2$, it follows that $I_y \subseteq (U_1 \cap U_2)$. Hence, $U_1 \cap U_2 \in \tau$. Let $y \in \bigcup_{\alpha\in\mathcal{A}} U_\alpha$ where $U_\alpha \in \tau$ for all $\alpha \in \mathcal{A}$. Then $y \in U_\beta$ for some $\beta \in \mathcal{A}$. This implies that there exists $I_\beta \in \mathcal{I}$ such that $I_{\beta_y} \subseteq U_\beta \subseteq \bigcup_{\alpha\in\mathcal{A}} U_\alpha$. Hence, $\underset{\alpha\in\mathcal{A}}{U_\alpha} \in \tau$. This implies that τ is a B-topology.

Claim 2: For any $I \in \mathcal{I}$ and $x \in X$, $I_x \in \tau$

Let $y \in I_x$. Then $y \cong^I x$. We will show that $I_y \subseteq I_x$. Let $z \in I_y$. Then $z \cong^I y$. By transitivity, $z \cong^I x$. Hence, $z \in I_x$ so that $I_y \subseteq I_x$. This proves claim 2.

Suppose $x * y \in U \in \tau$. Then there exists $I \in \mathcal{I}$ such that $I_{x*y} \subseteq U$. Note that I_x and I_y are open sets containing x and y, respectively. Then $I_x * I_y = I_{x*y} \subseteq U$. This implies that $*$ is continuous. Therefore, $(X, *, \tau)$ is a topological B-algebra by Theorem 2. □

4. Uniform Topology on B-Algebras

Throughout this section, all B-ideals of a B-algebra X are normal B-ideals of X. The following definitions are parallel to that of [5], page 340-341.

Suppose X is a B-algebra and $U, V \subseteq X \times X$, consider the following notations:

(i) $U^{-1} = \{(y,x)|(x,y) \in U\}$;

(ii) $U \circ V = \{(x,z)|\exists y \in X, (x,y) \in V, (y,z) \in U\}$;

(iii) $U[[x]] = \{y|(x,y) \in U\}$;

(iv) $\Delta = \{(x,x)|x \in X\}$.

Suppose Ω is an arbitrary family of B-ideals in a B-algebra X and $A \subseteq X$. Consider the following notations:

(i) $U_I = \{(x,y) \in X \times X | x \cong^I y\}$;

(ii) $\mathcal{K}^\star = \{U_I : I \in \Omega\}$;

(iii) $\mathcal{K} = \{U \subseteq X \times X | U_I \subseteq U, \exists U_I \in \mathcal{K}^\star\}$;

(iv) $U_I[[A]] = \bigcup_{a \in A} U_I[[a]]$.

Remark 5. $\mathcal{K}^\star \subseteq \mathcal{K}$.

Definition 8. By a *uniformity* on a B-algebra X, we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions for any $U, V \in \mathcal{K}$:

(i) $\Delta \subseteq U$;

(ii) $U^{-1} \in \mathcal{K}$;

(iii) $W \circ W \subseteq U$, for some $W \in \mathcal{K}$;

(iv) $U \cap V \in \mathcal{K}$; and

(v) If $U \subseteq W \subseteq X \times X$ then $W \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform B-structure*.

Example 6. Consider the B-algebra $X = \{0,a,b,c,d,e\}$ in Example 1. The normal B-ideals of X are $\{X, I\}$ where $I = \{0,a,b\}$. By routine calculations, $(X,\mathcal{K})$ is a uniform B-structure where $\mathcal{K}^\star = \{U_X, U_I\}$, $U_X[[0]] = U_X[[a]] = U_X[[b]] = U_X[[c]] = U_X[[d]] = U_X[[e]] = \{0,a,b,c,d,e\} = X$, $U_I[[0]] = U_I[[a]] = U_I[[b]] = \{0,a,b\}, U_I[[c]] = U_I[[d]], = U_I[[e]] = \{c,d,e\}$.

Remark 6. $(X, \mathcal{K}^*)$ is not a uniform B-structure as shown in the next example.

Example 7. Consider the B-algebra $X = \{0,a,b,c,d,e\}$ in Example 1 and the B-ideal $I = \{0,a,b\}$ in Example 6. By Theorem 1, X and I are normal B-ideals of X. Hence, $\mathcal{K}^\star = \{U_X, U_I\}$ where $U_X = \{(x,y) \in X \times X | x * y, y * x \in X\} = X \times X$ and $U_I = \{(0,0),(a,a),(b,b),(c,c),(d,d),(e,e),$
$(0,b),(b,0),(a,0),(0,a),(b,a),(a,b),(c,e),(e,c),(d,c),(c,d),(e,d),(d,e)\}$. Let $M = I \cup \{0,e\} = \{0,a,b,e\}$. Then $U_M = U_I \cup \{(0,e),(e,0),(a,d),(d,a),(b,c),$
$(c,b)\}$. Note that $U_I \subseteq U_M \subseteq X \times X$. Moreover, $M \notin \Omega$ since $d * a = e, a \in M$ but $d \notin M$. Hence, $U_M \notin \mathcal{K}^\star$. This implies that $\mathcal{K}^\star$ does not satisfy condition (v) of Definition 8.

However, in view of Remark 5, the next theorem states that the pair $(X,\mathcal{K})$ is a uniform B-structure.

Theorem 9. *Let Ω be an arbitrary family of B-ideals in a B-algebra X. Then $(X,\mathcal{K})$ is a uniform B-structure.*

Proof. Suppose Ω is an arbitrary family of B-ideals in a B-algebra X and $U, V \in \mathcal{K}$. Then there exist $U_I, U_J \in \mathcal{K}^\star$ such that $U_I \subseteq U$ and $U_J \subseteq V$, respectively. (i) Let $(x, x) \in \Delta$. Since $x \cong^I x$, it follows that $(x, x) \in U_I$. Hence, $(x, x) \in U$ so that $\Delta \subseteq U$. (ii) Let $(x, y) \in U_I$. Then $x \cong^I y$ and $y \cong^I x$. This implies that $(y, x) \in U_I$. Hence, $(y, x) \in U$. It follows that $(x, y) \in U^{-1}$ with $U_I \subseteq U^{-1}$ so that $U^{-1} \in \mathcal{K}$. (iii) Consider $U_I \in \mathcal{K}$ and $(x, z) \in U_I \circ U_I$. Then there exists $y \in X$ such that (x, y), $(y, z) \in U_I$. This implies that $x \cong^I y$ and $y \cong^I z$. Hence, $x \cong^I z$. It follows that $(x, z) \in U_I \subseteq U$ so that $U_I \circ U_I \subseteq U$. (iv) Let $U_I, U_J \in \mathcal{K}^\star$. Note that a B-ideal is a subalgebra of X. By Lemma 2, Ω is closed under finite intersections.

Claim: $U_I \cap U_J = U_{I \cap J} \in \mathcal{K}^\star$.

Let $(x, y) \in U_I \cap U_J$. Then $x \cong^I y$ and $x \cong^J y$. These imply that $x * y, y * x \in I, J$. Hence, $x * y, y * x \in I \cap J$ implying that $x \cong^{I \cap J} y$ and so $(x, y) \in U_{I \cap J}$. Therefore, $U_I \cap U_J \subseteq U_{I \cap J}$. The converse is similar. This proves the claim.

Let $(x, y) \in U_{I \cap J} = U_I \cap U_J$. Then $x \cong^I y$ and $x \cong^J y$. Thus, $(x, y) \in U_I$ and $(x, y) \in U_J$. Hence, $(x, y) \in U$ and $(x, y) \in V$. Therefore, $(x, y) \in U \cap V$ so that $U_{I \cap J} \subseteq U \cap V$. Consequently, $U \cap V \in \mathcal{K}$. It remains to show that $\mathcal{K}$ satisfies condition (v). Let $U \in \mathcal{K}$ such that $U \subseteq V \subseteq X \times X$. Then there exists $U_I \in \mathcal{K}^\star$ such that $U_I \subseteq U \subseteq V$. Hence, $V \in \mathcal{K}$. Therefore, $\mathcal{K}$ satisfies condition (v) and $(X, \mathcal{K})$ is a uniform B-structure. □

Definition 9. Let (X, K) be a uniform B-structure. If τ is a topology on X, then τ is called a *uniform B-topology* and the pair (X, τ) is called a *uniform B-topological space.*

Example 8. Consider the uniform structure $(X, \mathcal{K})$ in Example 6. Then the family $\tau = \{X, \varnothing, \{0, a, b\}, \{c, d, e\}\}$ is a uniform B-topology on X. Thus, (X, τ) is a uniform B-topological space.

Theorem 10. *Suppose $(X, \mathcal{K})$ is a uniform B-structure. Then $\tau = \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[[x]] \subseteq G\}$ is a uniform B-topology on X.*

Proof. Suppose $(X, \mathcal{K})$ is a uniform B-structure. Note that for all $x \in X$ and $U \in \mathcal{K}$, $U[[x]] \subseteq X$. Hence, $X \in \tau$. Also, $\varnothing \in \tau$ by definition. Let $x \in \bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i$. Then there exists $j \in \mathcal{A}$ such that $x \in G_j$. Since $G_j \in \tau$, there exists $U_j \in \mathcal{K}$ such that $U_j[[x]] \subseteq G_j$. This implies that $U_j[[x]] \subseteq \bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i$. Hence, $\bigcup_{G_i \in \tau, i \in \mathcal{A}} G_i \in \tau$. Suppose $G, H \in \tau$ such that $x \in G \cap H$. Then there exist $U, V \in \mathcal{K}$ such that $U[[x]] \subseteq G$ and $V[[x]] \subseteq H$. Let $W = U \cap V$. By Definition 8(iv), $W \in \mathcal{K}$.

Claim: $W[[x]] \subseteq U[[x]] \cap V[[x]]$.

Let $y \in W[[x]]$. Then $(x, y) \in U$ and $(x, y) \in V$. This implies that $y \in U[[x]]$ and $y \in V[[x]]$. Hence, $W[[x]] \subseteq U[[x]] \cap V[[x]]$. This proves the claim.

By the claim, $W[[x]] \subseteq U[[x]] \subseteq G$ and $W[[x]] \subseteq V[[x]] \subseteq H$. Hence, $W[[x]] \subseteq G \cap H$. This implies that $G \cap H \in \tau$. Therefore, τ is a B-topology on X. □

The next remark follows from Definition 9 and Theorem 10.

Remark 7. Suppose X is a B-topological space.

(i) Then (X, τ) in Theorem 10 is a uniform B-topological space.

(ii) For any $U_I \in \mathcal{K}^*$ and $x \in X$, $x \in U_I[[x]]$ and $U_I[[x]] \in \tau$, that is, $U_I[[x]]$ is a neighborhood of x.

Lemma 4. *Suppose X is a B-algebra such that $U \subseteq V$ for any $U, V \in \mathcal{K}$. Then $U[[x]] \subseteq V[[x]]$ for all $x \in X$.*

Proof. Let $U \subseteq V$ for any $U, V \in \mathcal{K}$ and $x \in X$. Suppose $a \in U[[x]]$. Then $(x, a) \in U \subseteq V$. This implies that $(x, a) \in V$. Therefore, $a \in V[[x]]$. □

Theorem 11. *Suppose X is a uniform B-topological space. Then X is a topological B-algebra.*

Proof. Let $(X, \mathcal{K})$ be a uniform structure. By Theorem 10 and Remark 7(i), there is a uniform B-topology $\tau = \{G \subseteq X | \forall x \in G, \exists U \in \mathcal{K}, U[[x]] \subseteq G\}$. Suppose $x * y \in U(x * y)$ where $x, y \in X$. By Theorem 10, there exists $G \in \mathcal{K}$ such that $G[[x * y]] \subseteq U(x * y)$. Then there exists $G_I \in \mathcal{K}^\star$ such that $G_I \subseteq G$ for some B-ideal I of X. By Lemma 4, $G_I[[x * y]] \subseteq G[[x * y]]$. Note that $G_I[[x]]$ and $G_I[[y]]$ are open neighborhoods of x and y, respectively by Remark 7(ii).
Claim: $G_I[[x]] * G_I[[y]] \subseteq G_I[[x * y]]$.
Suppose $a * b \in G_I[[x]] * G_I[[y]]$. Then (x, a), $(y, b) \in G_I$. This implies that $x \cong^I a$ and $y \cong^I b$. Hence, $x * y \cong^I a * b$. It follows that $a * b \in G_I[[x * y]]$. This proves the claim.
Hence, $G_I[[x]] * G_I[[y]] \subseteq U(x * y)$. By Theorem 2, X is a topological B-algebra. □

The converse of Theorem 11 follows directly from Definition 9 provided that the B-topology is a uniform B-topology. This is formally stated in the next corollary.

Corollary 3. *Suppose X is a topological B-algebra. If τ is a uniform B-topology, then X is a uniform B-topological space.*

Acknowledgements

This research is funded by the Commission on Higher Education (CHED) and Mindanao State University-Iligan Institute of Technology, Philippines.

References

[1] D. Al-Kadi. Anti Fuzzy Ideals of B-algebra. *International Journal of Pure and Applied Mathematics*, 117(3):437–445, 2017.

[2] J. Dugunji. *Topology.* Allyn and Bacon, Inc., Boston, 1996.

[3] J. Endam and J. Vilela. The Second Isomorphism Theorem for B-algebras. *Applied Mathematical Sciences*, 8(38):1865–1872, 2014.

[4] Y.B. Jun et al. On Topological BCI-Algebras. *Information Sciences*, 116:253–261, 1999.

[5] K.D. Joshi. *Introduction to General Topology.* New Age International Limited Publishers, India, 1997.

[6] N.C. Gonzaga Jr. Analyzing Some Structural Properties of Topological B-Algebra. *International Journal of Mathematics and Mathematical Sciences*, 2019:1–7, 2019.

[7] S. Mehrshad and J. Golzarpoor. On Topological BE-Algebras. *Mathematica Moravica*, 21(2):1–13, 2017.

[8] J. Neggers and H. Kim. A Fundamental Theorem of B-homomorphism for B-algebras. *Inter.Math.J.*, 2:207–214, 2002.

[9] J. Neggers and H. Kim. On B-algebras. *Mat. Vesnik*, 54:21–29, 2002.

[10] A. Walendziak. A Note on Normal Subalgebras in B-algebras. *Scientiae Mathematicae Japonicae Online*, pages 49–53, 2005.