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On Semitotal k-Fair and Independent k-Fair Domination in Graphs

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Abstract. In this paper, we introduce and investigate the concepts of semitotal k-fair domination and independent k-fair domination, where k is a positive integer. We also characterize the semitotal 1-fair dominating sets and independent k-fair dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs and determine the exact value or sharp bounds of the corresponding semitotal 1-fair domination number and independent k-fair domination number.

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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph and $v \in V(G)$. The *open neighborhood* of v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $X \subseteq V(G)$, the *open neighborhood* of X in G is the set $N_G(X) = N(X) = \bigcup_{v \in X} N_G(v)$ and its *closed neighborhood* is the the set $N_G[X] = N[X] = N(X) \cup X$. A set $D \subseteq V(G)$ is a *dominating set* in G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N[D] = V(G)$. The minimum cardinality of a dominating set in G, denoted by $\gamma(G)$, is the *domination number* of G. Any dominating set in G of cardinality $\gamma(G)$ is referred to as γ-set in G.

The theory of independent domination was formalized by Berge [1] and Ore [9] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetnieme [2]. Let G be a connected graph. A dominating set S in G is

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an *independent dominating set* of G if no two vertices in S are adjacent, that is, S is an independent set. The *independent domination number* $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

A domination parameter called fair domination was introduced by Caro, Hansberg, and Henning [4] in 2012. For an integer $k \geq 1$, a *k-fair dominating set* (*kfd-set*) is a dominating set $S \subseteq V(G)$ such that $|N(u) \cap S| = k$ for every $u \in V(G) \backslash S$. The *k-fair domination number* of G, denoted by $\gamma_{kfd}(G)$, is the minimum cardinality of a kfd-set. Clearly, $k \leq \gamma_{kfd}(G) \leq |V(G)|$.

In 2014, Maravilla, Isla, and Canoy [6] characterized the k-fair dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs and determined the bounds or exact values of the k-fair domination numbers of these graphs. Two variants of k-fair domination, namely connected k-fair domination and neighborhood connected k-fair domination, were studied by Bent-Usman, Gomisong, and Isla [3] in 2018 and by Bent-Usman, Isla, and Canoy [7] in 2019, respectively.

Another domination parameter is the semitotal domination of graphs introduced by Goddard, Henning, and McPillan [5] in 2014. For a graph G with no isolated vertices, a set $S \subseteq V(G)$ is a *semitotal dominating set* in G if S is a dominating set in G such that for every $x \in S$ there exists $y \in S \setminus \{x\}$ such that $d_G(x, y) \leq 2$. In 2019, Aniversario, Canoy, and Jamil [8] characterized the semitotal dominating sets in the join, corona, and lexicographic product of graphs.

Let G be a graph without isolated vertices. A set $S \subseteq V(G)$ is a *semitotal k-fair dominating set* in G, if S is a k-fair dominating set in G and for every $x \in S$, there exists $y \in S \setminus \{x\}$ such that $d(x, y) \leq 2$. The *semitotal k-fair domination number* of G, denoted by $\gamma_{kf}^{t2}(G)$, is the minimum cardinality of a semitotal k-fair dominating set. A semitotal k-fair dominating set of cardinality $\gamma_{kf}^{t2}(G)$ is called a *minimum semitotal k-fair dominating set* or a γ_{kf}^{t2}-set.

Let G be a connected graph. A set $S \subseteq V(G)$ is an *independent k-fair dominating set* in G if S is a k-fair dominating set in G and if no two vertices in S are adjacent. The *independent k-fair domination number* of G, denoted by $\gamma_{kf}^{i}(G)$, is the minimum cardinality of an independent k-fair dominating set. An independent k-fair dominating set of cardinality $\gamma_{kf}^{i}(G)$ is called a *minimum independent k-fair dominating set* or a γ_{kf}^{i}*-set.*

The *join* $G + H$ of two graphs G and H is the graph with vertex-set $V(G + H) = V(G) \cup V(H)$ and edge-set $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$. The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining the i-th vertex of G to every vertex in the i-th copy of H. For every $v \in V(G)$, we denote by H^v the copy of H whose vertices are joined or attached to the vertex v. For each $v \in V(G)$, the subgraph $\langle v \rangle + H^v$ of $G \circ H$ will be denoted by $v + H^v$. The *lexicographic product* of two graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and edge set $E(G[H])$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G[H])$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. The *Cartesian product* of two graphs

G and H, denoted by $G\square H$, is the graph with vertex-set $V(G\square H) = V(G) \times V(H)$ and edge-set $E(G\square H)$ satisfying the following conditions: $(u_1, v_1)(u_2, v_2) \in E(G\square H)$ if and only if either $u_1u_2 \in E(G)$ and $v_1 = v_2$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$.

2. Preliminary Results

Remark 1. *Any semitotal kfd-set is a kfd-set, where k is a positive integer.*

Theorem 1. *Let G be a nontrivial connected graph. Then $\gamma^{t2}_{1f}(G) = 2$ if and only if there exist adjacent vertices a and b such that $N_G(a) \cap N_G(b) = \varnothing$ and $V(G) \setminus N_G[a] = N_G(b) \setminus \{a\}$.*

Proof. Suppose $\gamma^{t2}_{1f}(G) = 2$. Let $S = \{a, b\}$ be a γ^{t2}_{1f}-set of G. Since S is a $1fd$-set, no vertex $v \in V(G) \setminus S$ with $v \in N_G(a) \cap N_G(b)$ exists, i.e., $N_G(a) \cap N_G(b) = \varnothing$. This implies that $ab \in E(G)$ because S is a semitotal dominating set. Moreover, $V(G) \setminus N_G[a] = N_G(b) \setminus \{a\}$ (or $V(G) \setminus N_G[b] = N_G(a) \setminus \{b\}$) because S is a dominating set.

The converse is clear. □

Corollary 1. $\gamma^{t2}_{1f}(K_2) = \gamma^{t2}_{1f}(K_1 + (K_1 \cup H)) = \gamma^{t2}_{1f}(K_2 \circ H) = 2$ *for any graph H.*

Lemma 1. [6] Let G be a connected graph of order $n \geq 1$ and let k be a positive integer such that $k \leq n$. Then:

(*i*) $k \leq \gamma_{kfd}(G) \leq n$.

(*ii*) $\gamma_{kfd}(G) = k$ if and only if G has a kfd-set S with $|S| = k$.

(*iii*) If $\gamma_{kfd}(G) = n$, then G has no vertex of degree k.

Theorem 2. *Let G be a connected graph of order $n \geq 2$ and let k be a positive integer with $2 \leq k \leq n$. Then $\gamma^{t2}_{kf}(G) = k$ if and only if $n = k$ or $G = H_1 + H_2$ for some graphs H_1 and H_2 with $|V(H_1)| = k$.*

Proof. Suppose $\gamma^{t2}_{kf}(G) = k$. Suppose further that $k < n$. Let S be a γ^{t2}_{kf}-set of G. Then $|S| = k$. Set $H_1 = \langle S\rangle$ and $H_2 = \langle V(G) \setminus S\rangle$. Since S is a k-fair dominating set of G, it follows that $V(G) \setminus S \subseteq N_G(v)$ for each $v \in S$. Hence, $G = H_1 + H_2$.

For the converse, suppose that $G = H_1 + H_2$ where $|V(H_1)| = k$. Then clearly, $S = V(H_1)$ is a k-fair dominating set of G. Let $x, y \in S$ with $x \neq y$. Suppose $xy \notin E(G)$. Pick any $z \in V(H_2)$. Then $z \in N_G(x) \cap N_G(y)$. This implies that $d_G(x, y) = 2$. Therefore, S is a semitotal kfd-set of G. By Lemma 1 (*ii*), S is a γ^{t2}_{kf}-set of G, that is, $\gamma^{t2}_{kf}(G) = |S| = k$. □

Corollary 2. *Let G be a connected graph of order $n \geq 3$. Then $\gamma^{t2}_{2f}(G) = 2$ if and only if $G = K_2 + H$ or $G = \overline{K_2} + H$ for some graph H.*

Proof. Suppose $\gamma^{t2}_{2f}(G) = 2$, say $S = \{a, b\}$ is a γ^{t2}_{2f}-set of G. Let $H = \langle V(G) \setminus S\rangle$. Since S is a 2-fair dominating set, $V(H) = V(G) \setminus S \subseteq N_G(a) \cap N_G(b)$. Since S is a semitotal 2-fair dominating set, either $ab \in E(G)$ or $d_G(a, b) = 2$. Thus, $G = K_2+H$ or $G = \overline{K_2}+H$.

For the converse, suppose $G = K_2 + H$ or $G = \overline{K_2} + H$. Then clearly, $\gamma^{t2}_{2f}(G) = 2$. □

Theorem 3. *Let G be a connected graph of order $n \geq 2$ and let $k \geq 2$. Then $S \subseteq V(G)$ is a semitotal kfd-set if and only if it is a kfd-set. In particular, $\gamma^{t2}_{kf}(G) = \gamma_{kfd}(G)$.*

Proof. Suppose S is a semitotal kfd-set. Then S is a kfd-set by Remark 1. For the converse, suppose S is a kfd-set. Let $x \in S$. If $N_G(x) \cap S \neq \varnothing$, then there exists $w \in S$ such that $d_G(x, w) = 1$. Suppose $N_G(x) \cap S = \varnothing$. Let $v \in N_G(x)$. Then $v \in V(G) \setminus S$. Since S is a kfd-set and $k \geq 2$, there exists $u \in S \setminus \{x\}$ such that $uv \in E(G)$. Hence, $d_G(x, u) = 2$. Thus, $d_G(x, z) \leq 2$ for some $z \in S$. Therefore, S is a semitotal kfd-set of G. Accordingly, $\gamma^{t2}_{kf}(G) = \gamma_{kfd}(G)$. □

Remark 2. *Not every connected graph of order n admits an independent kfd-set, where k is a positive integer and $1 \leq k \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.*

To see this, consider C_4. $\gamma_{1fd}(C_4) = 2$ but C_4 has no independent $1fd$-set.

Theorem 4. *Let G be a connected graph of order n and let k be a positive integer with $1 \leq k \leq \alpha(G)$. Then G admits an independent kfd-set (and hence, $\gamma^{i}_{kf}(G) = k$) if and only if $G = \overline{K_k} + H$ for some graph H.*

Proof. Suppose G admits an independent kfd-set, say $S = \{a_1, a_2, ..., a_k\}$. Let $H = \langle V(G) \setminus S\rangle$. Since S is a kfd-set, $V(H) = V(G) \setminus S \subseteq N_G(a_i) \cap N_G(a_j)$ for all $i, j = 1, 2, ..., k$ and $i \neq j$. Since S is an independent kfd-set of G, $|N_G(S) \cap S| = \varnothing$. Thus, $G = \overline{K_k} + H$.

For the converse, suppose $G = \overline{K_k} + H$ for some graph H. Then clearly, $S = V(\overline{K_k})$ is an independent kfd-set of G and $\gamma^{i}_{kf}(G) = k$. □

Theorem 5. *Let G be a connected graph and suppose G admits an independent $1fd$-set. Then $1 \leq \gamma^{i}_{1f}(G) \leq \alpha(G)$. Moreover,*

(i) $\gamma^{i}_{1f}(G) = 1$ *if and only if $G = K_1 + H$ for some graph H, and*

(ii) $\gamma^{i}_{1f}(G) = \alpha(G) \geq 2$ *if and only if G has a maximum independent set such that $d_G(x, y) \geq 3$ for each pair of vertices $x, y \in S$ with $x \neq y$, and no other independent set satisfies this property.*

Proof. Let S be a γ^{i}_{1f}-set. Since S is an independent set, $1 \leq |S| = \gamma^{i}_{1f}(G) \leq \alpha(G)$.
(i) is an immediate consequence of Theorem 4.
(ii) Suppose $\gamma^{i}_{1f}(G) = \alpha(G) \geq 2$. Let S be a γ^{i}_{1f}-set of G. Then S is a maximum independent set of G. Let $x, y \in S$ with $x \neq y$. Since S is an independent $1fd$-set, $d_G(x, y) \geq 3$.

For the converse, suppose that G has a maximum independent set S such that $d_G(x, y) \geq 3$ for all $x, y \in S$ with $x \neq y$, and that no other independent set satisfies this property. Then S is a dominating set of G. Let $z \in V(G) \setminus S$. Then there exists $v \in S \cap N_G(z)$. Since $d_G(v, y) \geq 3$ for all $y \in S \setminus \{v\}$, $N_G(z) \cap S = 1$. Thus, S is an independent $1fd$-set of G. Hence, $\gamma^i_{1f}(G) \leq |S| = \alpha(G)$. By the additional property, $\gamma^i_{1f}(G) = \alpha(G)$. □

Since $K_n = K_1 + K_{n-1}$ and $\alpha(K_n) = 1$, the next result immediately follows.

Corollary 3. *For any positive integer $n \geq 1$, $\gamma^i_{1f}(K_n) = 1$.*

Remark 3. *Any independent kfd-set is a kfd-set, where k is a positive integer.*

Theorem 6. *Let G be a connected graph with $|V(G)| \geq 4$ and suppose G admits an independent $1fd$-set. Then $\gamma^i_{1f}(G) = 2$ if and only if there exist non-adjacent vertices a and b of G satisfying the following conditions:*

(i) $N_G[a] \cup N_G[b] = V(G)$

(ii) $N_G[a] \cap N_G[b] = \varnothing$

Proof. Suppose $S = \{a, b\}$ is a γ^i_{1f}-set of G. Since S is a dominating set, Condition (i) holds. Suppose there exists $y \in N_G[a] \cap N_G[b]$. Then $d_G(a, b) = 2$, contrary to the fact that $d_G(a, b) \geq 3$ since S is an independent $1fd$-set. Thus, Condition (ii) holds.

For the converse, suppose that $S = \{a, b\}$ satisfies Conditions (i) and (ii). Then clearly, $\gamma^i_{1f}(G) = 2$. □

Theorem 7. *For any positive integer $n \geq 1$, $\gamma^i_{1f}(P_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $P_n = \{v_1, v_2, v_3, ..., v_n\}$. Clearly, $\gamma^i_{1f}(P_1) = \gamma^i_{1f}(P_2) = \gamma^i_{1f}(P_3) = 1$. Let $n > 3$ and consider the following cases:
Case 1: $n = 3r$

Group the first $3r$ vertices of P_n into r disjoint subsets

$$\begin{aligned} S_1 &= \{v_1, v_2, v_3, \} \\ S_2 &= \{v_4, v_5, v_6\} \\ &\vdots \\ S_{r-1} &= \{v_{3r-5}, v_{3r-4}, v_{3r-3}\} \\ S_r &= \{v_{3r-2}, v_{3r-1}, v_{3r}\}. \end{aligned}$$

For every induced subgraph $\langle v_i, v_{i+1}, v_{i+2} \rangle$ of P_n, where $i = 1, 4, ..., 3r-2$, the vertices v_{i+1} form an independent 1-fair dominating set of P_n. Thus, the set $T = \{v_2, v_5, ..., v_{3r-4}, v_{3r-1}\}$ is an independent 1-fair dominating set of P_n. Since $|T| = r$, $\gamma^i_{1f}(P_n) \leq r$. Note that every three adjacent vertices in P_n can be dominated by a single vertex. Thus, every independent 1-fair dominating set of P_n contains at least $\lceil \frac{n}{3} \rceil$ vertices. Hence, $\gamma^i_{1f}(P_n) \geq \lceil \frac{n}{3} \rceil = r$. Thus, $\gamma^i_{1f}(P_n) = \lceil \frac{n}{3} \rceil$.

Case 2: $n = 3r + 2$

In Case 1, the set T is a γ^i_{1f}-set of the induced subgraph $\langle v_1, v_2, ..., v_{3r}\rangle$ of P_n. Since $n = 3r + 2$, the set $T \cup \{v_{3r+2}\}$ is a γ^i_{1f}-set of P_n. Thus, $\gamma^i_{1f}(P_n) = r + 1 = \lceil \frac{3r+2}{3} \rceil = \lceil \frac{n}{3} \rceil$.

Case 3: $n = 3r + 1$

Consider the grouping of the first $3r$ vertices of P_n given in Case 1. The set $S = \{v_1, v_4, ..., v_{3r-2}\} \cup \{v_{3r+1}\}$ is an independent 1-fair dominating set of P_n. Thus, $\gamma^i_{1f}(P_n) \leq |S| + 1 = r + 1 = \lceil \frac{3r+1}{3} \rceil = \lceil \frac{n}{3} \rceil$. Note that each of the first $r - 1$ induced subgraph $\langle v_i, v_{i+1}, v_{i+2}\rangle$ can be dominated by a single vertex, while the induced subgraph $\langle v_{3r-2}, v_{3r-1}, v_{3r}, v_{3r+1}\rangle$ can be dominated by the vertices v_{3r-2} and v_{3r+1}. Thus, every independent 1-fair dominating set of P_n contains at least $(r - 1) + 2 = r + 1 = \lceil \frac{3r+1}{3} \rceil = \lceil \frac{n}{3} \rceil$ vertices. Hence, $\gamma^i_{1f}(P_n) \geq \lceil \frac{n}{3} \rceil$. Therefore, $\gamma^i_{1f}(P_n) = \lceil \frac{n}{3} \rceil$. □

Corollary 4. *For any positive integer* $n \equiv 0 \ (mod\ 3)$, $\gamma^i_{1f}(C_n) = \frac{n}{3}$.

Proof. Immediately follows from Case 1 of Theorem 7. □

The following results are used in the succeeding sections.

Theorem 8. [6] *Let G and H be nontrivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{m, n\}$. Then $S \subseteq V(G + H)$ is a kfd-set in $G + H$ if and only if one of the following holds:*

(a) $S = V(G + H)$.

(b) $S \subseteq V(G)$, $|S| = k$ *and S is a kfd-set in G.*

(c) $S \subseteq V(H)$, $|S| = k$ *and S is a kfd-set in H.*

(d) $S = S_G \cup S_H$, *where S_G is a $(k - |S_H|)fd$-set in G and S_H is a $(k - |S_G|)fd$-set in H.*

(e) $S = V(G) \cup T$, *where $|V(G)| = m < k$ and T is a $(k - m)fd$-set in H.*

(f) $S = D \cup V(H)$, *where $|V(H)| = n < k$ and D is a $(k - n)fd$-set in G.*

Theorem 9. [6] *Let G and H be nontrivial connected graphs and let k be a positive integer with $k \leq |V(H)|$. Then $C \subseteq V(G \circ H)$ is a kfd-set in $G \circ H$ if and only if one of the following holds:*

(a) $C = V(G) \cup B$, *where $B = \varnothing$ or* $B = \bigcup_{v \in V(G)} S_v$, *where each S_v is a $(k - 1)fd$-set in H^v.*

(b) $C = \bigcup_{v \in V(G)} S_v$, *where each S_v is a kfd-set in H^v and $|S_v| = k$.*

Theorem 10. [6] *Let G and H be nontrivial connected graphs. Then* $C = \bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ *is a kfd-set in $G[H]$ if and only if the following hold:*

(*i*) *S is a dominating set in G.*

(*ii*) *For each $x \in S \cap N_G(S)$, $T_x = V(H)$ and $|V(H)| = r \leq k$ whenever $C \neq V(G[H])$ or T_x is an rfd-set and $\displaystyle\sum_{z \in N_G(x) \cap S} |T_z| = k - r$.*

(*iii*) *For each $x \in S \backslash N_G(S)$, $T_x = V(H)$ and $|V(H)| \leq k$ or $|T_x| = k$ and T_x is a kfd-set in H.*

(*iv*) *For each $y \in V(G) \backslash S$, $\displaystyle\sum_{v \in N_G(y) \cap S} |T_v| = k$.*

Corollary 5. [6] *Let G and H be nontrivial connected graphs. Then $C = \displaystyle\bigcup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$ is a $1fd$-set in $G[H]$ if and only if S is a $1fd$-set in G, $S \cap N_G(S) = \varnothing$, T_x is a dominating set of H, and $|T_x| = 1$ for each $x \in S$.*

Theorem 11. [6] *Let G and H be nontrivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq min\{m, n\}$. Then $C = \displaystyle\bigcup_{x \in V(G)} [\{x\} \times T_x] \subseteq V(G \Box H)$ is a kfd-set in $G \Box H$ if and only if*

(*i*) *$V(H) \backslash T_x \subseteq N_H(T_x) \cup (\bigcup_{z \in N_G(x)} T_z)$ for each $x \in V(G)$, and*

(*ii*) *For each $x \in V(G)$, $T_x = V(H)$ or for each $a \in V(H) \backslash T_x$, either $|N_H(a) \cap T_x| = k$ and $|\{z : z \in N_G(x), a \in T_z\}| = 0$ or $|N_H(a) \cap T_x| = r < k$ and $a \in \bigcap_{i=1}^{k-r} T_{x_i}$, where $x_i \in N_G(x)$ for $i = 1, 2, ..., k - r$.*

Corollary 6. [6] *Let G and H be nontrivial connected graphs of orders m and n, respectively, and k a positive integer with $k \leq min\{m, n\}$. Then*

$$\gamma_{kfd}(G \Box H) \leq min\{n \cdot \gamma_{kfd}(G), m \cdot \gamma_{kfd}(H)\}.$$

3. Semitotal 1-Fair Domination

In view of Theorem 3, which shows that the concept of semitotal k-fair dominatioin coincides with the notion of k-fair domination when $k \geq 2$, this section investigates semitotal k-fair domination in graphs only for $k = 1$.

The following remark is an immediate consequence of Remark 1 for $k = 1$.

Remark 4. *For any connected graph G of order $n \geq 2$, $\gamma_{1fd}(G) \leq \gamma_{1f}^{t2}(G)$ and $\gamma_{1f}^{t2}(G) \geq 2$.*

The succeeding two results are easy to verify.

Proposition 1. *Let n and r be positive integers where $n \geq 2$ and $r \geq 1$. Then*

$$\gamma_{1f}^{t2}(P_n) = \begin{cases} 2, & 2 \leq n \leq 4 \\ 2r, & n = 4r \\ 2r+1, & n = 4r+1 \\ 2r+2, & n = 4r+2,\ 4r+3. \end{cases}$$

Proposition 2. *Let n and r be positive integers where $n \geq 3$ and $r \geq 1$. Then*

$$\gamma_{1f}^{t2}(C_n) = \begin{cases} 3, & n = 3 \\ 2r, & n = 4r \\ 2r+1, & n = 4r+1 \\ 2r+2, & n = 4r+2 \\ 2r+3, & n = 4r+3. \end{cases}$$

Theorem 12. *Let a and b be positive integers such that $2 \leq a \leq b$. Then there exists a connected graph G such that $\gamma_{1fd}(G) = a$ and $\gamma_{1f}^{t2}(G) = b$.*

Proof. Consider the following cases:
Case 1. $a = b$
Let $G = G_1$ be the graph shown in Figure 1.

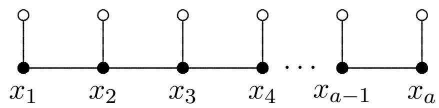

Figure 1: A graph G with $\gamma_{1fd}(G) = a = \gamma_{1f}^{t2}(G) = b$

It is clear that the set $A = \{x_1, x_2, ..., x_{a-1}, x_a\}$ is both a γ_{1fd}-set and a γ_{1f}^{t2}- set in G_1. It follows that $\gamma_{1fd}(G) = a = \gamma_{1f}^{t2}(G) = b$.
Case 2. $a < b$
Let $G = G_2$ be the graph shown in Figure 2.

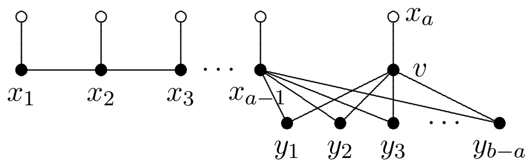

Figure 2: A graph G with $\gamma_{1fd}(G) = a < \gamma_{1f}^{t2}(G) = b$

Let $A = \{x_1, x_2, ..., x_a\}$. It is clear that the set A is a γ_{1fd}-set and the set $B = (A \setminus \{x_a\}) \cup \{v\} \cup \{y_1, y_2, y_3, ..., y_{b-a}\}$ is a γ_{1f}^{t2}-set in G. It follows that $\gamma_{1fd}(G) = |A| = a$ and $\gamma_{1f}^{t2}(G) = |B| = (a-1) + 1 + (b-a) = b$. □

Corollary 7. *$\gamma_{1f}^{t2} - \gamma_{1fd}$ can be made arbitrarily large.*

We now characterize the semitotal 1-fair dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs in this section. We also establish the exact value or sharp bounds of the corresponding semitotal 1-fair domination number.

Theorem 13. *Let G and H be any two graphs of orders m and n, respectively. A set $C \subseteq V(G+H)$ is a semitotal $1fd$-set of $G+H$ if and only if $C = V(G+H)$ or $C = \{v, w\}$ for some isolated vertices v and w of G and H, respectively.*

Proof. Immediately follows from $\gamma_{1f}^{t2}(G+H) \geq 2$ and Theorem 8. □

Corollary 8. *Let G and H be any graphs of orders m and n, respectively. Then $\gamma_{1f}^{t2}(G+H) = 2$ if G and H both contain isolated vertices and $\gamma_{1f}^{t2}(G+H) = m+n$ otherwise.*

Theorem 14. *Let G be a nontrivial connected graph and H be any graph. Then $C \subseteq V(G \circ H)$ is a semitotal $1fd$-set in $G \circ H$ if and only if $C = V(G)$ or $C = V(G \circ H)$.*

Proof. Suppose $C \subseteq V(G \circ H)$ is a semitotal $1fd$-set in $G \circ H$. Then C is a $1fd$-set in $G \circ H$. It now follows by Theorem 9 that $C = V(G)$ or $C = V(G \circ H)$.
The converse is obvious. □

Corollary 9. *Let G be a nontrivial connected graph and H be any graph. Then*

$$\gamma_{1f}^{t2}(G \circ H) = |V(G)|.$$

Theorem 15. *Let G and H be nontrivial connected graphs. A set $C \subseteq V(G[H])$ is a semitotal $1fd$-set of $G[H]$ if and only if $C = V(G[H])$.*

Proof. Suppose $C = \bigcup_{x \in S} (\{x\} \times T_x)$ is a semitotal $1fd$-set of $G[H]$. Then S is a $1fd$-set of G, $S \cap N_G(S) = \varnothing$, T_x is a dominating set of H, and $|T_x| = 1$ for each $x \in S$, by Corollary 5. Suppose $C \neq V(G[H])$, say there exists $(y, a) \in V(G[H]) \setminus C$. If $y \notin S$, then $|N_G(y) \cap S| = 1$ because S is a $1fd$-set of G. Let $N_G(y) \cap S = \{z\}$ and let $T_z = \{b\}$. Since C is a semitotal $1fd$-set of $G[H]$, there exists $(w, c) \in C$ such that $d_{G[H]}((z, b), (w, c)) \leq 2$. Now, since $S \cap N_G(S) = \varnothing$ and $w \in S \setminus \{z\}$, it follows that $d_G(z, w) = 2$ (that is, $d_{G[H]}((z, b), (w, c)) = 2$). Let $u \in N_G(z) \cap N_G(w)$. Then $u \in V(G) \setminus S$. Since $z, w \in N_G(u) \cap S$, S is not a $1fd$-set, a contradiction. Suppose $y \in S$. Then $|T_y| = 1$. Again, since C is a semitotal $1fd$-set of $G[H]$, $S \cap N_G(S) = \varnothing$, and $|T_y| = 1$, there exists $p \in N_G(y) \cap S$ such that $d_G(y, p) = 2$. This implies that there exists $q \in V(G) \setminus S$ ($q \in N_G(y) \cap N_G(p)$) such that $|N_G(q) \cap S| \geq 2$, contrary to the fact that S is a $1fd$-set of G. Thus, $C = V(G[H])$.
The converse is clear. □

Corollary 10. *Let G and H be nontrivial connected graphs of orders m and n, respectively. Then $\gamma_{1f}^{t2}(G[H]) = m \cdot n$.*

Theorem 16. *Let G and H be nontrivial connected graphs. Then* $C = \bigcup_{x\in V(G)} [\{x\}\times T_x] \subseteq V(G\Box H)$ *is a semitotal* $1fd$*-set of* $G\Box H$ *if and only if:*

(*i*) $V(H)\backslash T_x \subseteq N_H(T_x) \cup (\bigcup_{z\in N_G(x)} T_z)$ *for each* $x \in V(G)$*;*

(*ii*) *for each* $x \in V(G)$*,* $T_x = V(H)$ *or for each* $a \in V(H)\backslash T_x$*, either* $|N_H(a)\cap T_x| = 1$ *and* $\{z : z \in N_G(x), a \in T_x\} = \varnothing$*, or* $N_H(a) \cap T_x = \varnothing$ *and* $a \in T_y$ *for exactly one* $y \in N_G(x)$*; and*

(*iii*) *for each* $x \in V(G)$ *and for each* $a \in T_x$*, there exists* $b \in T_x$ *such that* $ab \in E(H)$ *or there exists* $y \in N_G(x)$ *such that* $a \in T_y$*.*

Proof. Suppose $C = \bigcup_{x\in V(G)} [\{x\} \times T_x] \subseteq V(G\Box H)$ is a semitotal $1fd$-set in $G\Box H$. Since C is a $1fd$-set in $G\Box H$, Conditions (i) and (ii) hold by Theorem 11. Let $x \in V(G)$. Suppose there exists $a \in T_x$ such that for all $b \in T_x$, $ab \notin E(H)$ and for all $y \in N_G(x)$, $a \notin T_y$. Since C is a semitotal dominating set, there exists $(x, c) \in C$ such that $d_{G\Box H}((x, a), (x, c)) = 2$ or there exists $(z, a) \in C$ such that $d_{G\Box H}((x, a), (z, a)) = 2$ or there exist $y \in N_G(x)$ and $b \in T_y$ such that $d_{G\Box H}((x, a), (y, b)) = 2$, where $(y, b) \in C$. However, in each of these cases, there exists $(w, d) \in V(G\Box H)\backslash C$ such that $|N_{G\Box H}(w, d)\cap C| \geq 2$, contrary to the assumption that C is a $1fd$-set. Hence, Condition (iii) must be satisfied.

For the converse, suppose Conditions (i), (ii), and (iii) hold. By Theorem 11, (i) and (ii) imply that C is a $1fd$-set in $G\Box H$, while (iii) implies that C is a semitotal dominating set. Thus, C is a semitotal $1fd$-set in $G\Box H$. □

Corollary 11. *Let G and H be nontrivial connected graphs. Then* $C_1 = S_1 \times V(H)$ *and* $C_2 = V(G) \times S_2$ *are semitotal* $1fd$*-sets in* $G\Box H$ *if and only if* S_1 *and* S_2 *are* $1fd$*-sets in* G *and* H*, respectively.*

The following result is an immediate consequence of Corollary 11.

Corollary 12. *Let G and H be nontrivial connected graphs. Then*

$$\gamma^{t2}_{1f}(G\Box H) \leq min\{|V(H)| \cdot \gamma_{1fd}(G), |V(G)| \cdot \gamma_{1fd}(H)\}.$$

Remark 5. *The bound given in Corollary* 12 *is sharp.*

To see this, consider the graph shown in Figure 3. The shaded vertices in $P_4\Box P_6$ form a γ^{t2}_{1f}-set. Thus, $\gamma^{t2}_{1f}(P_4\Box P_6) = 8 = min\{6 \cdot 2, 4 \cdot 2\} = min\{|V(P_6)| \cdot \gamma_{1fd}(P_4), |V(P_4)| \cdot \gamma_{1fd}(P_6)\} = |V(P_4)| \cdot \gamma_{1fd}(P_6)$.

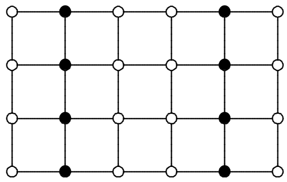

Figure 3: The graph $P_4 \Box P_6$, with $\gamma_{1f}^{t2}(P_4 \Box P_6) = 8$.

4. Independent k-Fair Domination

We characterize the independent k-fair dominating sets in the join, corona, lexicographic product, and Cartesian product of graphs in this section. We also determine the exact value or sharp bounds of the corresponding independent k-fair domination number.

Theorem 17. *Let G and H be nontrivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$. Then $G+H$ admits an independent kfd-set if and only if G or H admits an independent kfd-set. Moreover, $S \subsetneq V(G+H)$ is an independent kfd-set in $G+H$ if and only if one of the following holds:*

(i) $S \subsetneq V(G)$, $|S| = k$ and S is an independent kfd-set in G.

(ii) $S \subsetneq V(H)$, $|S| = k$ and S is an independent kfd-set in H.

Proof. Suppose $G+H$ admits an independent kfd-set, where $1 \leq k \leq max\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$. Suppose further that $S \subsetneq V(G+H)$ is an independent kfd-set in $G+H$. If there exist $u, x \in S$ such that $u \in V(G)$ and $x \in V(H)$, then $ux \in E(G+H)$, contrary to the assumption that S is an independent set in $G+H$. Hence, either $S \subsetneq V(G)$ or $S \subsetneq V(H)$. Assume that $S \subsetneq V(G)$. Let $x \in V(H)$. Then $|N_{G+H}(x) \cap S| = |S| = k$. Since S is a kfd-set of $G+H$, $|N_G(v) \cap S| = |N_{G+H}(v) \cap S| = k$ for every $v \in V(G) \setminus S$. Hence, S is a kfd-set in G. Since S is an independent set by assumption, Statement (i) holds. Similarly, if $S \subsetneq V(H)$, then S is an independent kfd-set in H, showing that Statement (ii) holds. Therefore, G or H admits an independent kfd-set.

Conversely, suppose that Statement (i) or (ii) holds. Assume that Statement (i) is true. Then $|N_{G+H}(v) \cap S| = |N_G(v) \cap S| = k$ for each $v \in V(G) \setminus S$. Moreover, $|N_{G+H}(x) \cap S| = |S| = k$ for every vertex $x \in V(H)$. Thus. S is a kfd-set in $G+H$. Therefore, $S \subsetneq V(G+H)$ is an independent kfd-set in $G+H$. The same conclusion similarly follows if Statement (ii) holds. □

The next result is an immediate consequence of Theorem 17.

Corollary 13. *Let G and H be connected nontrivial graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq max\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$. If G or H has an independent kfd-set S with $|S| = k$, then $\gamma_{kf}^{i}(G+H) = k$.*

Theorem 18. *Let G and H be nontrivial connected graphs, and let k be a positive integer with $k \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$. Then $G \circ H$ admits an independent kfd-set if and only if H admits an independent kfd-set consisting of k vertices.*

Proof. Suppose $G \circ H$ admits an independent kfd-set, where $k \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$. Suppose further that $C \subsetneq V(G \circ H)$ is an independent kfd-set in $G \circ H$. Suppose $C \cap V(G) \neq \varnothing$, say $v \in C \cap V(G)$. Since C is an independent set, $V(H^v) \cap C = \varnothing$ and $|N_{G \circ H}(x) \cap C| = 1$ for all $x \in V(H^v)$. This implies that $k = 1$. Now let $w \in N_G(v)$. Then $w \notin C$. Since $|N_{G \circ H}(w) \cap C| = 1$, $V(H^w) \cap C = \varnothing$. Hence, $V(H^w) \cap N_{G \circ H}[C] = \varnothing$, a contradiction (since C is a dominating set). Thus, $C \cap V(G) = \varnothing$. Then by Theorem 9, $C = \bigcup_{v \in V(G)} S_v$, where each S_v is an independent kfd-set in H^v and $|S_v| = k$ for each $v \in V(G)$. Therefore, H admits an independent kfd-set consisting of k vertices.

Conversely, suppose H admits an independent kfd-set consisting of k vertices. Let $C = \bigcup_{v \in V(G)} S_v$, where each S_v is an independent kfd-set in H^v and $|S_v| = k$. Then C is an independent kfd-set in $G \circ H$ by Theorem 9. □

The next result is an immediate consequence of Theorem 18.

Corollary 14. *Let G and H be nontrivial connected graphs of orders m and n, respectively, and let k be a positive integer with $k \leq \left\lceil \frac{n}{2} \right\rceil$. If H has an independent kfd-set S with $|S| = k$, then $\gamma^i_{kf}(G \circ H) = mk$.*

Theorem 19. *Let G and H be nontrivial connected graphs and let k be a positive integer with $1 \leq k \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$. If $G[H]$ admits an independent kfd-set, then $C = \bigcup_{x \in S} (\{x\} \times T_x) \subsetneq V(G[H])$ is an independent kfd-set in $G[H]$ if and only if the following hold:*

(i) S is an independent $1fd$-set in G,

(ii) for each $x \in S$, $|T_x| = k$ and T_x is an independent kfd-set in H.

Proof. Suppose $C = \bigcup_{x \in S} (\{x\} \times T_x) \subsetneq V(G[H])$ is an independent kfd-set in $G[H]$. Then C is a kfd-set in $G[H]$ and by Theorem 10, S is a dominating set in G. Moreover, since C is an independent set, $S \cap N_G(S) = \varnothing$. Finally, from Statement (iv) of Theorem 10, for each $y \in V(G) \backslash S$, $\sum_{v \in N_G(y) \cap S} |T_v| = k$, hence $|N_G(y) \cap S| = 1$. Thus, S is an independent $1fd$-set in G and Statement (i) holds. Furthermore, for each $x \in (S \backslash N_G(S)) = S$, $|T_x| = k$ and T_x is a kfd-set in H by Statement (iii) of Theorem 10, where $k \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$ since C is an independent set. Suppose there is a vertex $a \in T_x$ which is adjacent to a vertex $b \in T_x$. Then (x, a) is adjacent to (x, b) in C, contrary to assumption. Hence, T_x is an independent kfd-set in H and Statement (ii) holds.

Conversely, suppose Statements (i) and (ii) hold. Then T_x is a kfd-set in H for each $x \in S$, and $\sum_{v \in N_G(y) \cap S} |T_v| = k$ for each $y \in V(G) \backslash S$. Thus, C is a kfd-set in $G[H]$ by Theorem 10. Let $(x, a) \in C$. Then $x \in S$ and $a \in T_x$. Suppose there exists $(x, b) \in C$ such that $(x, a)(x, b) \in E(G[H])$. Then $b \in T_x$ and $ab \in E(H)$, contrary to Statement (ii) that

T_x is an independent set. Hence, (x, a) is not adjacent to any $(x, b) \in C$. Next, suppose there exists $(y, d) \in C$, $y \neq x$, such that $(x, a)(y, d) \in E(G[H])$. Then $y \in N_G(x) \cap S$, contrary to the fact that $S \cap N_G(S) = \varnothing$. Hence, (x, a) is not adjacent to any $(y, d) \in C$. Therefore, C is an independent kfd-set in $G[H]$. □

Corollary 15. *Let G and H be nontrivial connected graphs with $\gamma^i_{kf}(H) = k \leq \left\lceil \frac{|V(H)|}{2} \right\rceil$. If $G[H]$ admits an independent kfd-set, then*

$$\gamma^i_{kf}(G[H]) = k \cdot \gamma^i_{1f}(G).$$

Proof. Let S be a γ^i_{1f}-set of G and let $\{a_1, ..., a_k\}$ be a γ^i_{kf}-set of H. Let $T_x = \{a_1, ..., a_k\}$ for each $x \in S$. Then $C = \bigcup_{x \in S} (\{x\} \times T_x)$ is an independent kfd-set in $G[H]$ by Theorem 19. Hence, $\gamma^i_{kf}(G[H]) \leq |C| = k \cdot \gamma^i_{1f}(G)$.

Now, let C_0 be a γ^i_{kf}-set of $G[H]$. By Theorem 19, $C_0 = \bigcup_{x \in S_0} [\{x\} \times Q_x]$, where S_0 is an independent $1fd$-set and Q_x is an independent kfd-set of H with $|Q_x| = k$ for each $x \in S_0$. Hence, $\gamma^i_{kf}(G[H]) = |C_0| = k|S_0| \geq k \cdot \gamma^i_{1f}(G)$. This establishes the desired equality. □

Theorem 20. *Let G and H be nontrivial connected graphs of orders m and n, respectively, and let k be a positive integer with $1 \leq k \leq min\{m, n\}$. If $G \square H$ admits an independent kfd-set, then $C = \bigcup_{x \in V(G)} (\{x\} \times T_x) \subsetneq V(G \square H)$ is an independent kfd-set in $G \square H$ if and only if:*

(i) T_x is an independent set in H for each $x \in V(G)$,

(ii) for each $x \in V(G)$ and each $a \in T_x$, $|\{z \in V(G) : z \in N_G(x), a \in T_z\}| = 0$,

(iii) $V(H) \backslash T_x \subseteq N_H(T_x) \bigcup (\bigcup_{z \in N_G(x)} T_z)$ for each $x \in V(G)$, and

(iv) for each $b \in V(H) \setminus T_x$, either $|N_H(b) \cap T_x| = k$ and $|\{z : z \in N_G(x), b \in T_z\}| = 0$ or $|N_H(b) \cap T_x| = r < k$ and $b \in \bigcap_{i=1}^{k-r} T_{x_i}$, where $x_i \in N_G(x)$ for $i = 1, 2, ..., k - r$.

Proof. Suppose $C = \bigcup_{x \in V(G)} (\{x\} \times T_x) \subsetneq V(G \square H)$ is an independent kfd-set in $G \square H$. Then by Theorem 11, (iii) and (iv) hold. Suppose there is a vertex $a \in T_x$ which is adjacent to some vertex b in T_x. Then (x, a) is adjacent to (x, b) in C, contrary to assumption. Hence, T_x is an independent set in H and (i) holds. Finally, suppose there is a vertex $a \in T_x$ such that for some vertex $z \in N_G(x)$, $a \in T_z$. Then $(z, a) \in C$ and (x, a) is adjacent to (z, a) in C, contrary to assumption. Hence, (ii) holds.

Conversely, suppose (i) to (iv) hold. From (iii) and (iv), C is a kfd-set by Theorem

11. By (i) and (ii), C is an independent set in $G\square H$. Thus, C is an independent kfd-set in $G\square H$. □

The next result immediately follows from Remark 3 and Corollary 6.

Corollary 16. *Let G and H be nontrivial connected graphs of orders m and n, respectively, and k a positive integer with $1 \leq k \leq min\{m,n\}$. If $G\square H$ admits an independent kfd-set, then*

$$\gamma^i_{kf}(G\square H) \leq min\{m \cdot \gamma_{kfd}(H), n \cdot \gamma_{kfd}(G)\}.$$

Remark 6. *The bound given in Corollary 16 is sharp. However, the strict inequality can be attained.*

To see this, consider the graphs shown in Figure 4. The shaded vertices in each graph form a γ^i_{kf}-set. Thus, $\gamma^i_{1f}(P_2\square P_3) = 2 = min\{2\cdot 1, 3\cdot 1\} = min\{|V(P_2)|\cdot\gamma_{1fd}(P_3), |V(P_3)|\cdot \gamma_{1fd}(P_2)\} = |V(P_2)| \cdot \gamma_{1fd}(P_3)$ and $\gamma^i_{3f}(P_3\square P_3) = 5 < min\{3\cdot 3, 3\cdot 3\} = min\{|V(P_3)| \cdot \gamma_{3fd}(P_3), |V(P_3)| \cdot \gamma_{3fd}(P_3)\}$.

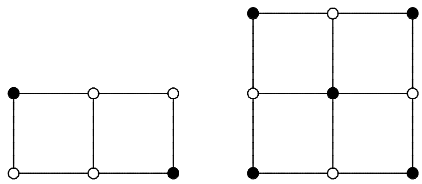

Figure 4: The graphs $P_2\square P_3$ and $P_3\square P_3$ with $\gamma^i_{1f}(P_2\square P_3) = 2$ and $\gamma^i_{3f}(P_3\square P_3) = 5$

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