EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 4, 2020, 893-913
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Mathematical Modeling of Wastes Pile-Up in Kwadaso Municipality in Ashanti, Ghana

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Abstract. In this paper, a mathematical model is introduced to describe wastes pile-up in Kwadaso Municipality which categorize wastes on the streets X_1(t), wastes in gutters X_2(t), wastes in the dustbins X_3(t), wastes in households X_4(t), wastes in the market places X_5(t) and the wastes sent to dumpsites X_6(t). From the qualitative data, it was observed that wastes within the Municipal keeps on pilling up as time increases indefinitely. The increase is as a result of continuous enormous quantum generation of wastes which occur in the Municipality. It was also revealed that trucks were unable to carry out the expected task of carrying wastes to dumpsites regularly leading to daily overflow of wastes in Kwadaso Municipality.

2020 Mathematics Subject Classifications: 47B1, 47B2

Key Words and Phrases: Mathematical model, analytic solution, sensitivity analysis

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1. Introduction

Waste Generation (WG) is a world-wide problem which has various effects on human health if it is not properly managed. Karak et al., (2012) [8] defined municipal solid waste as a way of shortening waste generated through domestic activities, commercial processes, and construction activities collected and treated by persons within municipalities. In other words solid waste can be defined as the useless and unwanted products in solid state derived from the activities of human beings. On the average the developed countries generated $521.95-759.2$ kg per person per year (kpc) and the developing countries generated $109.5-525.6$ kpc [8].

Zhang et al., (2010) [7] observed that household waste as a major cause of Municipal solid waste (MSW). Wastes are basically generated in the homes of inhabitants. Improper

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DOI: https://doi.org/10.29020/nybg.ejpam.v13i4.3813

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wastes disposal can lead to soil contamination. Waste that ends up in landfills excrete hazardous chemicals that leak into the soil to contaminate the soil. Improper waste disposal can also lead to air contamination. Wastes burnt in landfills emit gas and chemicals that can deplete the ozone layer. The depletion of the ozone layer brings green house effect.

Chinchodkar and Jahdav (2017) [5] asserted that one important part of wastes management in Africa is how to transport waste materials from one place to another. This make it difficult to ensure effective wastes management especially in the developing countries as the transportation of wastes go with costs. Unfortunately, most African countries lack adequate funds to institute efficient systems to prevent spreading of wastes. Mathematical models have been established to serve as a tool to ensure overall management plan for an effective decision on MSW. But the model made mention of transportation costs which serve as a hindrance to most developing countries.

Ghana as a developing country has its share of the problem of wastes management. As observed by Boadi and Kuitunen (2003) [6] that, municipal solid waste management (MSWM) in Accra was carried out in an unsustainable manner. Large quantum of wastes are always generated daily in Ghana but only quantum part are transported daily to dumpsites and this among other key problems including urbanization continued to merge and injure the nation's rise in wastes generation. These lead to many harmful health effects on human beings and creating sanitary problems.

The Ashanti region of Ghana also experiences similar challenges in wastes management of which Asase et al. [1] tried to solve it and identified integrated management system (IMS) as operation that would help in implementing a sustainable waste management system (WMS) in the Kumasi Metropolitan Assembly. Solid waste generation and management (SWGM) is now a major problem confronting the residents of Kwadaso Municipal Assembly in the Ashanti Region of Ghana. Wastes are mostly found on the environment, specifically households, on the streets, gutters, markets areas, waste bins and at the dumpsite. Basically, waste generations start with direct dumping of waste into the environment by human beings. These solid wastes spread to other parts of the environment through certain human activities and some natural occurrences. Efforts are being made by the Assembly to place waste bins at some vantage points within the Municipality for residents to dispose off wastes but it is still not enough as wastes in the bins are not transported to dumpsites regularly which lead to overflow of wastes onto the streets and into gutters.

Differential equations have been used to model many social problems worldwide, for example see a paper by Manthey et al. (2008) [4] modeled the behavior of campus drinking. But the extensive review of literature revealed the dynamics of waste and its management applying mathematics with the use of optimization techniques with the exception of Barnes et al. (2019) [2] who as at now have modeled refuse build-up on the Kwame Nkrumah University of Science and Technology (KNUST) campus using a system of ODEs. But the pattern of refuse pile-up on KNUST is different from that of Kwadaso Municipal Assembly due to different modes of wastes collection. In this paper, the dynamics of wastes pile up in Kwadaso Municipality is modeled using a system of ODEs.

2. Preliminary Results

Theorem 1. *Consider the initial value problem* $\dot{\boldsymbol{x}} = \boldsymbol{f(x)}$, $\boldsymbol{x}(0) = \boldsymbol{x}_0$. *Suppose that* $\boldsymbol{f}$ *is continuous and that all its partial derivatives* $\partial f_i/\partial x_i$, $i, j = 1, ..., n$, *are continuous for* $\boldsymbol{x}$ *in some open connected set* $D \subset \mathbb{R}^n$. *Then for* $\boldsymbol{x}_0 \in D$, *the initial value problem has a solution* $\boldsymbol{x}(t)$ *on some time interval* $(-\tau, \tau)$ *about* $t = 0$, *and the solution is unique [3, p. 149].*

2.1. Main Results

This section contains the development of a mathematical model to describe the dynamics of wastes pile-up in Kwadaso Municipality and its analysis.

2.1.1. The assumptions of the model for describing wastes pile-up in Kwadaso Municipality in Ashanti, Ghana

(i) The model incorporates wastes on the streets X_1(t), wastes in gutters X_2(t), wastes in the dustbins X_3(t), wastes in households X_4(t), wastes in the market places X_5(t) and the wastes sent to dumpsites X_6(t) as they are the collection points where wastes can be located within the Municipality.

(ii) The rate at which waste flow from one collection point to the other is directly proportional to the volume of wastes at that collection point. Thus, there is no interaction of wastes flow between one collection point and another collection point.

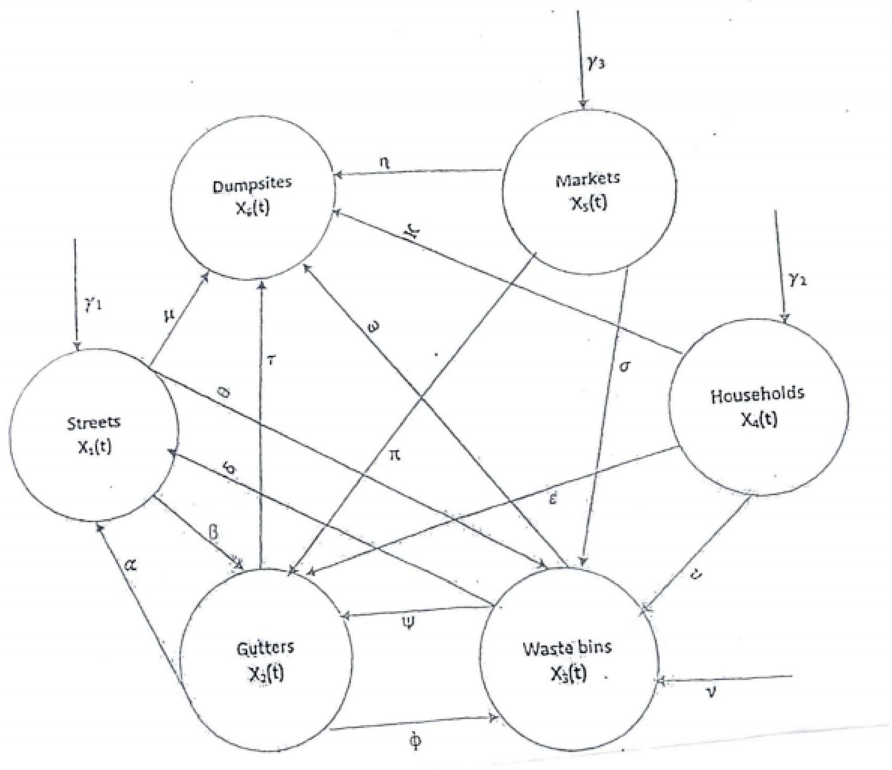

Figure 1: The Compartmental Diagram of the Model for Describing the Dynamics of Wastes Pile-up in Kwadaso Municipality

Based on the figure 1, we obtain the following system of ODEs:

$$\begin{aligned}
\frac{dX_1}{dt} &= \gamma_1 X_1 + \alpha X_2 + \delta X_3 - \beta X_1 - \mu X_1 - \theta X_1 \\
\frac{dX_2}{dt} &= \beta X_1 + \psi X_3 + \epsilon X_4 + \pi X_5 - \alpha X_2 - \phi X_2 - \tau X_2 \\
\frac{dX_3}{dt} &= \theta X_1 + \phi X_2 + \nu X_3 + \upsilon X_4 + \sigma X_5 - \psi X_3 - \delta X_3 - \omega X_3 \\
\frac{dX_4}{dt} &= \gamma_2 X_4 - \upsilon X_4 - \epsilon X_4 - \kappa X_4 \\
\frac{dX_5}{dt} &= \gamma_3 X_5 - \pi X_5 - \sigma X_5 - \eta X_5 \\
\frac{dX_6}{dt} &= \mu X_1 + \tau X_2 + \omega X_3 + \kappa X_4 + \eta X_5
\end{aligned} \tag{1}$$

Rewriting equation (1) in a matrix form

Thus, $\frac{dX}{dt} = AX$.

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \\ \frac{dx_6}{dt} \end{pmatrix} = \begin{bmatrix} (\gamma_1-\beta-\theta-\mu) & \alpha & \delta & 0 & 0 & 0 \\ \beta & -(\alpha+\phi+\tau) & \psi & \epsilon & \pi & 0 \\ \theta & \phi & (\nu-\psi-\delta-\omega) & \upsilon & \sigma & 0 \\ 0 & 0 & 0 & (\gamma_2-\upsilon-\epsilon-\kappa) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma_3-\pi-\sigma-\eta) & 0 \\ \mu & \tau & \omega & \kappa & \eta & 0 \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{pmatrix}$$

where,

$$A = \begin{bmatrix} (\gamma_1-\beta-\theta-\mu) & \alpha & \delta & 0 & 0 & 0 \\ \beta & -(\alpha+\phi+\tau) & \psi & \epsilon & \pi & 0 \\ \theta & \phi & (\nu-\psi-\delta-\omega) & \upsilon & \sigma & 0 \\ 0 & 0 & 0 & (\gamma_2-\upsilon-\epsilon-\kappa) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma_3-\pi-\sigma-\eta) & 0 \\ \mu & \tau & \omega & \kappa & \eta & 0 \end{bmatrix}$$

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \\ x_6(t) \end{pmatrix} \text{ and } \frac{dX}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \\ \frac{dx_5}{dt} \\ \frac{dx_6}{dt} \end{pmatrix}.$$

The eigenvalues of the coefficients matrix A are obtained by using the formula;

$$det(A-\lambda I) = 0. \tag{2}$$

$$\implies \begin{vmatrix} (\gamma_1-\beta-\theta-\mu)-\lambda & \alpha & \delta & 0 & 0 & 0 \\ \beta & -(\alpha+\phi+\tau)-\lambda & \psi & \epsilon & \pi & 0 \\ \theta & \phi & (\nu-\psi-\delta-\omega)-\lambda & \upsilon & \sigma & 0 \\ 0 & 0 & 0 & (\gamma_2-\upsilon-\epsilon-\kappa)-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & (\gamma_3-\pi-\sigma-\eta)-\lambda & 0 \\ \mu & \tau & \omega & \kappa & \eta & 0-\lambda \end{vmatrix} = 0.$$

The characteristic polynomial is obtained from the equation below:

$$\{-\lambda(D-\lambda)(E-\lambda)\}\left\{(A-\lambda)\begin{vmatrix} B-\lambda & \psi \\ \phi & C-\lambda \end{vmatrix} - \alpha\begin{vmatrix} \beta & \psi \\ \theta & C-\lambda \end{vmatrix} + \delta\begin{vmatrix} \beta & B-\lambda \\ \theta & \phi \end{vmatrix}\right\} = 0 \tag{3}$$

where;
$A = (\gamma_1-\beta-\theta-\mu)$, $B = -(\alpha+\phi+\tau)$,
$C = (\nu-\psi-\delta-\omega)$, $D = (\gamma_2-\upsilon-\epsilon-\kappa)$
and $E = (\gamma_3-\pi-\sigma-\eta)$.

The eigenvalues of the fixed points $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (0,0,0,0,0,0,0)$ are obtained as: $\lambda_1 = 0, \lambda_2 = D, \lambda_3 = E$ and $\lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0$

where; $a_1 = (A+B+C)$, $a_2 = (AB+AC+BC-\alpha\beta-\theta\delta-\phi\psi)$ and $a_3 = (ABC-\alpha\beta C+\alpha\theta\psi+\beta\phi\delta-\phi\psi A-\theta\delta B)$.

The corresponding eigenvectors of the eigenvalues are obtained, using the formula:

$$(A-\lambda I)V = 0. \tag{4}$$

Substituting $\lambda_1 = 0$ into equation (4) yields,

$$\begin{bmatrix} A & \alpha & \delta & 0 & 0 & 0 \\ \beta & B & \psi & \epsilon & \pi & 0 \\ \theta & \phi & C & \upsilon & \sigma & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ \mu & \tau & \omega & \kappa & \eta & 0 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

.

Rewriting the above equation in an augmented matrix, we obtain

$$\left[\begin{array}{cccccc|c} A & \alpha & \delta & 0 & 0 & 0 & 0 \\ \beta & B & \psi & \epsilon & \pi & 0 & 0 \\ \theta & \phi & C & \upsilon & \sigma & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 \\ \mu & \tau & \omega & \kappa & \eta & 0 & 0 \end{array}\right]$$

.

Interchanging row one and row six, then row four and row six and followed by row operation $\frac{1}{\mu}R_1 \to R_1$ yields,

$$\left[\begin{array}{cccccc|c} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 & 0 \\ \beta & B & \psi & \epsilon & \pi & 0 & 0 \\ \theta & \phi & C & \upsilon & \sigma & 0 & 0 \\ A & \alpha & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 \end{array}\right]$$

.

Using the row operations: $\frac{1}{\beta}R_2 - R_1 \to R_2$, $\frac{1}{\theta}R_3 - R_1 \to R_3$, $\frac{1}{A}R_4 - R_1 \to R_4$ and followed by $\frac{1}{(\frac{\mu B-\beta\tau}{\beta\mu})}R_2 \to R_2$ yields,

$$\left[\begin{array}{cccccc|c} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 & 0 \\ 0 & 1 & (\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau}) & (\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau}) & (\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau}) & 0 & 0 \\ 0 & (\frac{\mu\phi-\theta\tau}{\mu\theta}) & (\frac{\mu C-\theta\omega}{\mu\theta}) & (\frac{\mu\upsilon-\theta\kappa}{\mu\theta}) & (\frac{\mu\sigma-\theta\eta}{\mu\theta}) & 0 & 0 \\ 0 & (\frac{\mu\alpha-\tau A}{\mu A}) & (\frac{\mu\delta-\omega A}{\mu A}) & (\frac{-\kappa}{\mu}) & (\frac{-\eta}{\mu}) & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 \end{array}\right]$$

.

Using the row operations: $\frac{1}{(\frac{\mu\phi-\theta\tau}{\mu\theta})}R_3 - R_2 \to R_3$, $\frac{1}{(\frac{\mu\alpha-\tau A}{\mu A})}R_4 - R_2 \to R_4$, and followed by $\frac{1}{\frac{(\mu C-\theta\omega)(\mu B-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)}{(\mu\phi-\theta\tau)(\mu B-\beta\tau)}}R_3 \to R_3$, yields,

$$\left[\begin{array}{cccccc|c} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 & 0 \\ 0 & 1 & (\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau}) & (\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau}) & (\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau}) & 0 & 0 \\ 0 & 0 & 1 & F & G & 0 & 0 \\ 0 & 0 & H & I & J & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 \end{array}\right]$$

,

where,
F = $\frac{(\mu\upsilon-\theta\kappa)(\mu B-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\phi-\theta\tau)}{(\mu C-\theta\omega)(\mu B-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)}$, G = $\frac{(\mu\sigma-\theta\eta)(\mu B-\beta\tau)-(\mu\pi-\beta\eta)(\mu\phi-\theta\tau)}{(\mu C-\theta\omega)(\mu B-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)}$
H = $\frac{(\mu\delta-\omega A)(\mu B-\beta\tau)-(\mu\psi-\beta\omega)(\mu\alpha-\tau A)}{(\mu\alpha-\tau A)(\mu B-\beta\tau)}$, I = $\frac{-\kappa A(\mu B-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\alpha-\tau A)}{(\mu\alpha-\tau A)(\mu B-\beta\tau)}$
and J = $\frac{-\eta A(\mu B-\beta\tau)-(\mu\pi-\beta\eta)(\mu\alpha-\tau A)}{(\mu\alpha-\tau A)(\mu B-\beta\tau)}$.

Using the row operation $\frac{1}{H}R_4 - R_3 \to R_4$ and followed by $\frac{1}{(\frac{I-HF}{H})}R_4 \to R_4$, yields,

$$\left[\begin{array}{cccccc|c} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 & 0 \\ 0 & 1 & (\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau}) & (\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau}) & (\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau}) & 0 & 0 \\ 0 & 0 & 1 & F & G & 0 & 0 \\ 0 & 0 & 0 & 1 & (\frac{I-HF}{J-HG}) & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 & 0 \end{array}\right]$$

Using the row operations $\frac{1}{D}R_6 - R_4 \to R_6$, $\frac{1}{E}R_5 \to R_5$, and followed by $\frac{-1}{(\frac{J-HG}{I-HF})}R_6 - R_5 \to R_6$ we obtain

$$\left[\begin{array}{cccccc|c} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 & 0 \\ 0 & 1 & (\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau}) & (\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau}) & (\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau}) & 0 & 0 \\ 0 & 0 & 1 & F & G & 0 & 0 \\ 0 & 0 & 0 & 1 & (\frac{J-HG}{I-HF}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rewriting the above augmented matrix in equation form, we obtain

$$\begin{bmatrix} 1 & \frac{\tau}{\mu} & \frac{\omega}{\mu} & \frac{\kappa}{\mu} & \frac{\eta}{\mu} & 0 \\ 0 & 1 & (\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau}) & (\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau}) & (\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau}) & 0 \\ 0 & 0 & 1 & F & G & 0 \\ 0 & 0 & 0 & 1 & (\frac{J-HG}{I-HF}) & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \\ k_5 \\ k_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus,
$k_1+\frac{\tau}{\mu}k_2+\frac{\omega}{\mu}k_3+\frac{\kappa}{\mu}k_4+\frac{\eta}{\mu}k_5=0,$
$k_2+(\frac{\mu\psi-\beta\omega}{\mu B-\beta\tau})k_3+(\frac{\mu\epsilon-\beta\kappa}{\mu B-\beta\tau})k_4+(\frac{\mu\pi-\beta\eta}{\mu B-\beta\tau})k_5=0,$
$k_3+Fk_4+Gk_5=0, \quad k_4+(\frac{J-HG}{I-HF})k_5=0,$ and $k_5=0,$
k_6 is arbitrary, choosing $k_6=1$, we obtain; $k_{1234}=0$.

We observed that the eigenvector $v_1=\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$, corresponds to $\lambda_1=0$.

Similarly, substituting $\lambda_2=D$ and $\lambda_3=E$ into equation (4) the corresponding v_2 and v_3 are obtained as follows:

$$v_2=\begin{pmatrix}\frac{-\omega(OM-KQ)-D(O-NK)-\tau Y(Q-MN)-K(Q-MN)}{\mu(Q-MN)}\\ \frac{\beta D(O-NK)-(\mu\psi-\beta\omega)(OM-QK)-(\mu\epsilon-\beta\kappa)(Q-MN)}{(\mu(B-D)-\beta\tau)(Q-MN)}\\ (\frac{OM-KQ}{Q-MN})\\ 1\\ 0\\ (\frac{NK-O}{Q-MN})\end{pmatrix} \text{ and}$$

$$v_3=\begin{pmatrix}\frac{E(US-W)-\tau c_2(X-UT)-\omega c_3(X-UT)-\kappa c_4(X-UT)-\eta(X-UT)}{\mu(X-UT)}\\ \frac{c_3(\beta\omega-\mu\psi)(X-UT)-C_4(\mu\epsilon-\beta\kappa)(X-UT)-(\mu\pi-\beta\eta)(X-UT)-\beta E(US-W)}{(X-UT)(\mu(B-E)-\beta\tau)}\\ \frac{T(W-US)-S(X-UT)-Rc_4(X-UT)}{X-UT}\\ \frac{(W-US)(V-US)-(W-US)(V-UR)}{(V-UR)(V-US)}\\ 1\\ (\frac{US-W}{X-UT})\end{pmatrix},$$

where,
$\mathrm{K}=\frac{(\mu\upsilon-\theta\kappa)(\mu(B-D)-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\phi-\theta\tau)}{(\mu(C-B)-\theta\omega)(\mu(B-D)-\beta\tau)-(\mu\phi-\theta\tau)(\mu\psi-\beta\omega)},$

$\mathrm{L}=\frac{(\mu\sigma-\theta\eta)(\mu(B-D)-\beta\tau)-(\mu\phi-\theta\tau)(\mu\pi-\beta\eta)}{(\mu(C-B)-\theta\omega)(\mu(B-D)-\beta\tau)-(\mu\phi-\theta\tau)(\mu\psi-\beta\omega)},$

$\mathrm{M}=\frac{D\theta(\mu(B-D)-\beta\tau)-\beta D(\mu\phi-\theta\tau)}{(\mu(C-B)-\theta\omega)(\mu(B-D)-\beta\tau)-(\mu\phi-\theta\tau)(\mu\psi-\beta\omega)},$

$\mathrm{N}=\frac{(\mu\delta-\omega(A-D))(\mu(B-D)-\beta\tau)-(\mu\psi-\beta\omega)(\mu\alpha-\tau(A-D))}{(\mu\alpha-\tau(A-D))(\mu(B-D)-\beta\tau)},$

$\mathrm{O}=\frac{-\kappa(A-D)(\mu(B-D)-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\alpha-\tau(A-D))}{(\mu\alpha-\tau(A-D))(\mu(B-D)-\beta\tau)},$

$\mathrm{P}=\frac{-\eta(A-D)(\mu(B-D)-\beta\tau)-(\mu\pi-\beta\eta)(\mu\alpha-\tau(A-D))}{(\mu\alpha-\tau(A-D))(\mu(B-D)-\beta\tau)},$

$$Q = \frac{D(A-D)(\mu(B-D)-\beta\tau)(\mu\alpha-\tau(A-D))}{(\mu\alpha-\tau(A-D))(\mu(B-D)-\beta\tau)},$$

$$R = \frac{(\mu\upsilon-\theta\kappa)(\mu(B-E)-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\phi-\theta\tau)}{(\mu(C-E)-\theta\omega)(\mu(B-E)-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)},$$

$$S = \frac{(\mu\sigma-\theta\eta)(\mu(B-E)-\beta\tau)-(\mu\pi-\beta\eta)(\mu\phi-\theta\tau)}{(\mu(C-E)-\theta\omega)(\mu(B-E)-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)},$$

$$T = \frac{\theta E(\mu(B-E)-\beta\tau)-\beta E(\mu\phi-\theta\tau)}{(\mu(C-E)-\theta\omega)(\mu(B-E)-\beta\tau)-(\mu\psi-\beta\omega)(\mu\phi-\theta\tau)},$$

$$U = \frac{(\mu\delta-\omega(A-E))(\mu(B-E)-\beta\tau)-(\mu\psi-\beta\omega)(\mu\alpha-\tau(A-E))}{(\mu\alpha-\tau(A-E))(\mu(B-E)-\beta\tau)},$$

$$V = \frac{-\kappa(A-E)(\mu(B-E)-\beta\tau)-(\mu\epsilon-\beta\kappa)(\mu\alpha-\tau(A-E))}{(\mu\alpha-\tau(A-E))(\mu(B-E)-\beta\tau)},$$

$$W = \frac{-\eta(A-E)(\mu(B-E)-\beta\tau)-(\mu\pi-\beta\eta)(\mu\alpha-\tau(A-E))}{(\mu\alpha-\tau(A-E))(\mu(B-E)-\beta\tau)},$$

$$X = \frac{E(A-E)(\mu(B-E)-\beta\tau)-\beta E(\mu\alpha-\tau(A-E)}{(\mu\alpha-\tau(A-E))(\mu(B-E)-\beta\tau)},$$

$$c_2 = \frac{c_3(\beta\omega-\mu\psi)(X-UT)-C_4(\mu\epsilon-\beta\kappa)(X-UT)-(\mu\pi-\beta\eta)(X-UT)-\beta E(US-W)}{(X-UT)(\mu(B-E)-\beta\tau)},$$

$$c_3 = \frac{T(W-US)-S(X-UT)-Rc_4(X-UT)}{X-UT} \quad \text{and } c_4 = \frac{(W-US)(V-US)-(W-US)(V-UR)}{(V-UR)(V-US)}$$

The values of the parameters are summarized in table 1. The parameter values in the table below was estimated based on the data obtained from Kwadaso Municipal Assembly in the year 2019 to 2020.

Table 1: Shows the estimation of model parameters

Parameter	Description	Typical Value/unit
γ_1	rate of individuals depositing wastes on the street	0.65/day
γ_2	rate of waste generated in the various households	0.86/day
γ_3	rate of waste generated in the market	0.75/day
α	rate at which wastes flow from gutters to the streets	0.15/day
β	rate of wastes on the streets entering gutters	0.36/day
δ	rate of waste in waste bins overflowed onto the streets	0.22/day
ϵ	rate of waste in households ending in gutters	0.10/day
η	rate of waste transported from markets to dumpsites	0.24/day
θ	rate of waste on the streets transfered into waste bins	0.45/day
κ	rate of flow of wastes from households to dumpsites	0.28/day
μ	rate of waste on the streets transported to dumpsites	0.16/day
ν	rate at which individuals deposit waste into waste bins	0.78/day
π	rate of waste in the markets transfered into gutters	0.25/day
σ	rate of waste transfered from markets to waste bins	0.44/day
τ	rate of transfer of waste from gutters to dumpsites	0.16/day
υ	rate of waste transfered from households to waste bins	0.65/day
ϕ	rate of transferring waste from gutters to waste bins	0.20/day
ψ	rate of overflowed waste from waste bins into gutters	0.31/day
ω	rate of transporting waste in waste bins to dumpsites	0.82/day

Substituting the parameter values in table 1, into equation (2) yields, $\lambda_1 = 0$, $\lambda_2 = 0.0825$, $\lambda_3 = -0.6648$, $\lambda_4 = -0.8177$, $\lambda_5 = -0.17$ and $\lambda_6 = -0.18$. Since $\lambda_2 = 0.0825 > 0$ and $\lambda_{1,3,4,5,6} \leq 0$, then $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (0, 0, 0, 0, 0, 0, 0)$ is an unstable fixed point. This implies that the refuse is being pilling up in Kwadaso Municipality. Thus, the refuse or garbage generated in houses (households), in the markets, on the streets and refuse in gutters are always placed in collection bins/containers provided by the Municipal Assembly at some vantage points within the Municipality but tracks are either unable to transport or do not transport wastes in the collection containers regularly to the dumpsites, thereby resulting in daily overflow of wastes which create unpleasant conditions for the inhabitants.

Also, substituting the values of the parameters in table 1, into equation (3) the corresponding eigenvectors are obtained as follows:

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} -0.0682 \\ -0.0787 \\ -0.0712 \\ 0 \\ 0 \\ 0.9920 \end{pmatrix}, \; v_3 = \begin{pmatrix} 0.4053 \\ -0.9025 \\ -0.0199 \\ 0 \\ 0 \\ 0.1443 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} 0.1416 \\ 0.5016 \\ -0.6623 \\ 0 \\ 0 \\ 0.5383 \end{pmatrix}, v_5 = \begin{pmatrix} -0.5711 \\ -0.4979 \\ -0.0499 \\ 0.5179 \\ 0 \\ 0.3939 \end{pmatrix}, \text{ and } v_6 = \begin{pmatrix} -0.5094 \\ -0.2797 \\ -0.1335 \\ 0 \\ 0.5299 \\ 0.6030 \end{pmatrix}$$

At $X_1(0) = 10$, $X_2(0) = 8$, $X_3(0) = 14$, $X_4(0) = 12$, $X_5(0) = 10$ and $X_6(0) = 15$, we obtain the particular solution below:

$$\begin{pmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \\ X_6(t) \end{pmatrix} = 13.4678 \begin{pmatrix} 0.1416 \\ 0.5016 \\ -0.6623 \\ 0 \\ 0 \\ 0.5383 \end{pmatrix} e^{-0.8177t} + 12.8759 \begin{pmatrix} 0.4053 \\ -0.9025 \\ -0.0199 \\ 0 \\ 0 \\ 0.1443 \end{pmatrix} e^{-0.6648t}$$
$$+ 18.8715 \begin{pmatrix} -0.5094 \\ -0.2797 \\ -0.1335 \\ 0 \\ 0.5299 \\ 0.6030 \end{pmatrix} e^{-0.18t} + 23.1705 \begin{pmatrix} -0.5711 \\ -0.4979 \\ -0.0499 \\ 0.5179 \\ 0 \\ 0.3939 \end{pmatrix} e^{-0.17t} \tag{5}$$
$$+ 359.4971 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - 377.1283 \begin{pmatrix} -0.0682 \\ -0.0787 \\ -0.0712 \\ 0 \\ 0 \\ 0.9920 \end{pmatrix} e^{0.0825t}.$$

We can see that polynomial functions are on the right hand side of system of equation (1) which are $C'(\mathbb{R})$ then the function that appears in equation is the solution of a system of equation (1). Also, using the initial conditions for $x_1(0) = 10, ..., x_6(0) = 15$, the solution in equation (5) is unique. Thus, the differential operator $\frac{d}{dt}$ is a bijective operator on $\mathbb{R}^6$.

2.2. Numerical Simulation

In this section, the performance of the sensitivity analysis on the model parameters are carried out by varying each parameter value and maintaining the rest of the values of parameters in table 1 at equilibrium. The sensitivity analysis performed on the model parameters revealed γ_1, μ and τ as the most sensitive parameters and γ_2, γ_3 and α as the less sensitive parameters.

Varying γ_1 from 0.65 to 0.98 and to 0.32 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

Table 2: Interpolations for $\gamma_1 = 0.98$ and $\gamma_1 = 0.32$

Parameter	Description	Value	Eigenvalues	Decision
γ_1	rate of wastes on the streets	0.98	$\lambda_1 = 0$, $\lambda_2 = 0.2759$, $\lambda_3 = -0.5438$, $\lambda_4 = -0.8020$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Unstable
γ_1	rate of wastes on the streets	0.32	$\lambda_1 = 0$, $\lambda_2 = -0.0435$, $\lambda_3 = -0.7614$, $\lambda_4 = -0.9252$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Stable

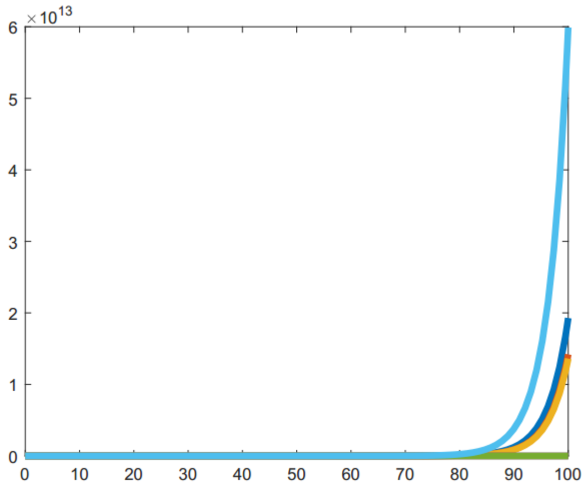

Figure 2: Interpolation for $\gamma_1 = 0.98$

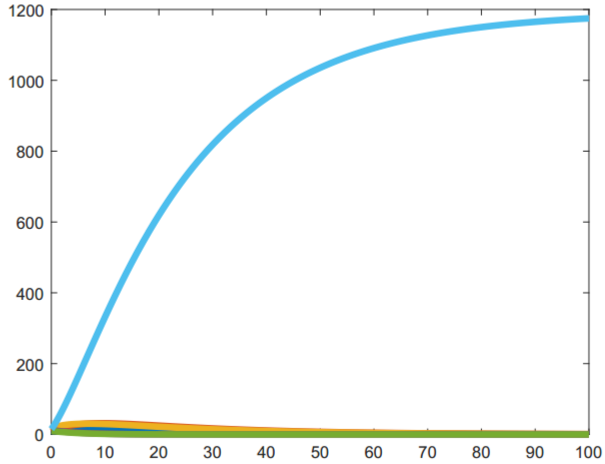

Figure 3: Interpolation for $\gamma_1 = 0.32$

The results in table 2 and the corresponding graphs are as follows;
Increasing γ_1 makes the system remains unstable with the graphs exhibiting an exponential growth and all trajectories moving in opposite direction. Decreasing γ_1, most of the trajectories remain at equilibrium with uniform acceleration and a trajectory showing an exponential decay making it a stable node. Wastes could therefore be controlled in the Kwadaso Municipality if the rate at which individuals deposit wastes on the streets is reduced to its barest minimum.

Varying μ from 0.16 to 0.86 and to 0.04 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

Table 3: Interpolations for $\mu = 0.86$ and $\mu = 0.04$

Parameter	Description	Value	Eigenvalues	Decision
μ	rate of wastes on streets to dump-sites	0.86	$\lambda_1 = 0$, $\lambda_2 = -0.1235$, $\lambda_3 = -0.7799$, $\lambda_4 = -1.1966$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Stable
μ	rate of wastes on streets to dump-sites	0.04	$\lambda_1 = 0$, $\lambda_2 = 0.1451$, $\lambda_3 = -0.6162$, $\lambda_4 = -0.8089$, $\lambda_5 = 0.17$, $\lambda_6 = -0.18$.	Unstable

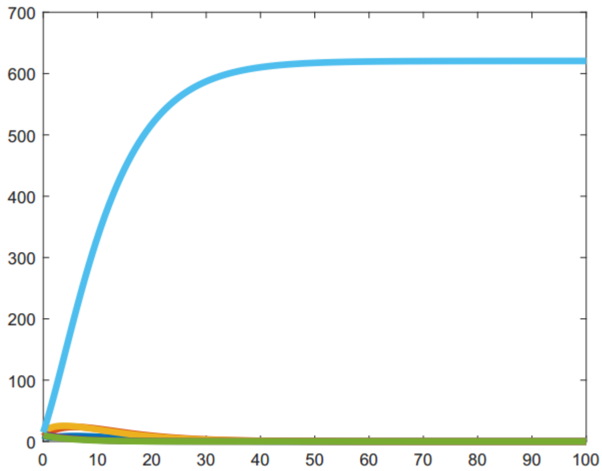

Figure 4: Interpolation for $\mu = 0.86$

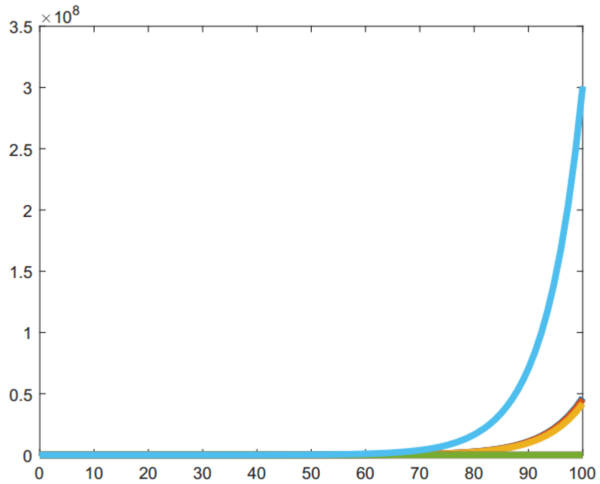

Figure 5: Interpolation for $\mu = 0.04$

The results in table 3 and the corresponding graphs are as follows;
Increasing μ, most of the trajectories remain at equilibrium with little effect on those collection points and a decrease of wastes on the streets showing an exponential decay from the graph.This in addition to the eigenvalues in the table indicates a stable equilibrium point. Decreasing μ makes the system unstable with the graphs exhibiting an exponential growth and all trajectories moving in opposite direction . Wastes could therefore be controlled in the Kwadaso Municipality if the rate of wastes on the streets are cleaned and transported regularly to dumpsites.

Varying τ from 0.16 to 0.94 and to 0.05 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

Table 4: Interpolations for $\tau = 0.94$ and $\tau = 0.05$

Parameter	Description	Value	Eigenvalues	Decision
τ	rate of wastes in gutters to dump-sites	0.94	$\lambda_1 = 0$, $\lambda_2 = -1.3853$, $\lambda_3 = -0.0188$, $\lambda_4 = -0.7759$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Stable
τ	rate of wastes in gutters to dump-sites	0.05	$\lambda_1 = 0$, $\lambda_2 = 0.1123$, $\lambda_3 = -0.5977$, $\lambda_4 = -0.8048$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Unstable

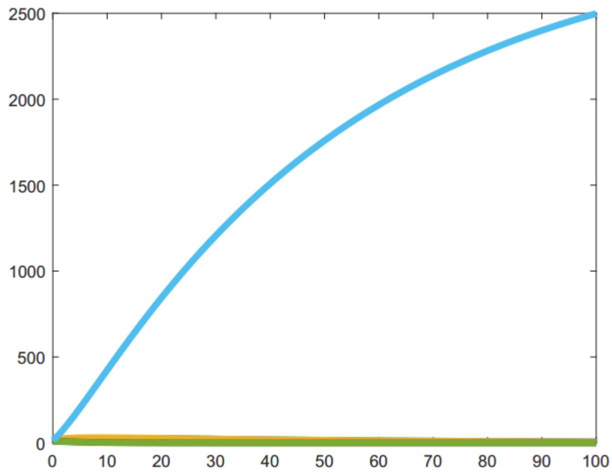

Figure 6: Interpolation for $\tau = 0.94$

The results in table 4 and the corresponding graphs are as follows;
Increasing τ, most of the trajectories remain at equilibrium with little effect on those collection points and a decrease of wastes in the gutters which indicates an exponential decay from the graph. This in addition to the eigenvalues in the table indicates a stable equilibrium point. Decreasing μ makes the system unstable with the graphs exhibiting an exponential growth and all trajectories moving in opposite direction. Wastes could therefore be controlled in the Kwadaso Municipality if the rate of wastes in gutters are distilled and transported regularly to dumpsites.

Varying γ_2 from 0.86 to 0.98 and to 0.32 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

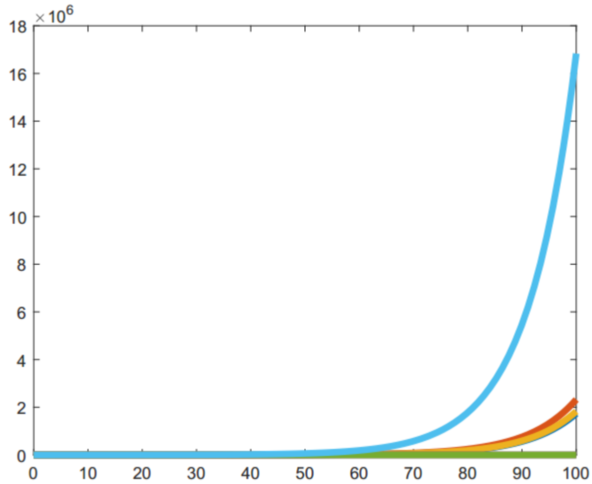

Figure 7: Interpolation for $\tau = 0.05$

Table 5: Interpolations for $\gamma_2 = 0.98$ and $\gamma_2 = 0.32$

Parameter	Description	Value	Eigenvalues	Decision
γ_2	rate of wastes in the households	0.98	$\lambda_1 = 0$, $\lambda_2 = 0.0825$, $\lambda_3 = -0.6648$,$\lambda_4 = -0.8177$, $\lambda_5 = -0.05$,$\lambda_6 = -0.18$.	Unstable
γ_2	rate of wastes in the households	0.32	$\lambda_1 = 0$, $\lambda_2 = 0.0825$, $\lambda_3 = -0.6648$,$\lambda_4 = -0.8177$, $\lambda_5 = -0.71$,$\lambda_6 = -0.18$.	Unstable

The results in table 5 and the corresponding graphs are as follows;
Increasing the value of γ_2 leads to the trajectories remaining constantly in line for a while and then dispersed later by moving exponentially in different dimensions with a single linear path. Decreasing the value of γ_2 exhibit similar characteristics like increasing it. They are both unstable because all the paths of the graphs have declined and also as there exist at least a positive eigenvalue determined in the table 5. Wastes are continuously being generated in the houses of the inhabitants of Kwadaso Municipal Assembly. Appropriate measures should be put in place to control household wastes.

Varying γ_3 from 0.75 to 0.96 and to 0.25 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

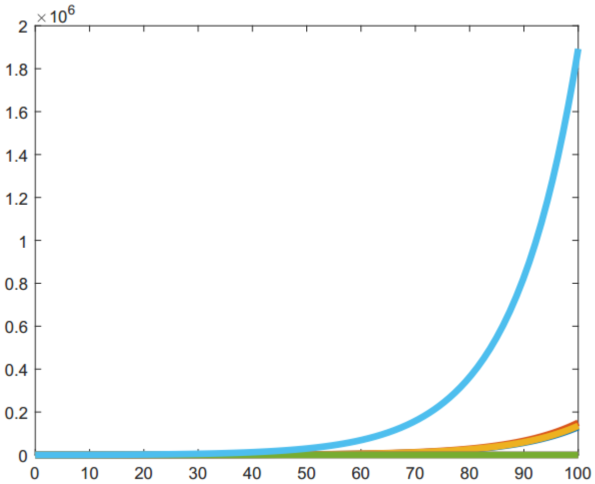

Figure 8: Interpolation for $\gamma_2 = 0.98$

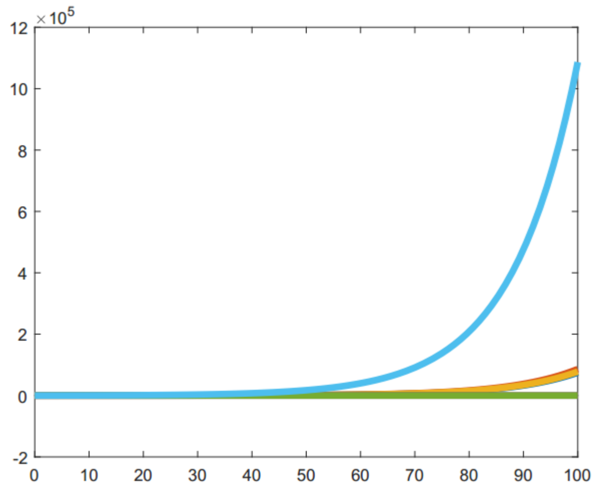

Figure 9: Interpolation for $\gamma_2 = 0.32$

The results in table 6 and the corresponding graphs are as follows;
Increasing the value of γ_3 leads to the trajectories remaining constantly in line for a while and then dispersed later by moving exponentially in different dimensions with a single linear path . Decreasing the value of γ_3 exhibit similar characteristics like increasing it. They are both unstable because none of the path of the graphs showed a declined and also as there exist at least a positive eigenvalue in the eigenvalues determined in the table 6. Wastes are continuously being generated in the markets of Kwadaso Municipal Assembly which requires effective and safety means of controlling it.

Varying α from 0.15 to 0.95 and to 0.05 respectively and maintaining the rest of parameter values in table 1 at equilibrium and substituting them into equation (2) yields,

Table 6: Interpolations for $\gamma_3 = 0.96$ and $\gamma_3 = 0.25$

Parameter	Description	Value	Eigenvalues	Decision
γ_3	rate of wastes in the markets	0.96	$\lambda_1 = 0$, $\lambda_2 = 0.0825$, $\lambda_3 = -0.6648$, $\lambda_4 = -0.8177$, $\lambda_5 = -17$, $\lambda_6 = 03$.	Unstable
γ_3	rate of wastes in the markets	0.25	$\lambda_1 = 0$, $\lambda_2 = 0.0825$, $\lambda_3 = -0.6648$, $\lambda_4 = -0.8177$, $\lambda_5 = -0.17$, $\lambda_6 = -0.68$.	Unstable

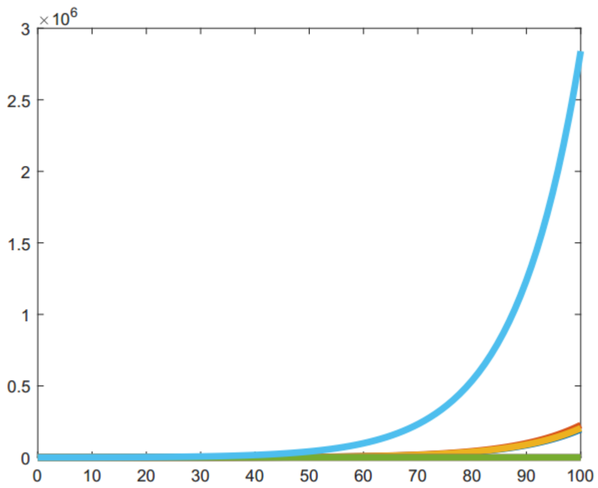

Figure 10: Interpolation for $\gamma_3 = 0.96$

Table 7: Interpolations for $\alpha = 0.95$ and $\alpha = 0.05$

Parameter	Description	Value	Eigenvalues	Decision
α	rate of wastes in gutters to streets	0.95	$\lambda_1 = 0$, $\lambda_2 = 0.1939$, $\lambda_3 = -1.5480$, $\lambda_4 = -0.8459$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Unstable
α	rate of wastes in gutters to streets	0.05	$\lambda_1 = 0$, $\lambda_2 = 0.0438$, $\lambda_3 = -0.5161$, $\lambda_4 = -0.8277$, $\lambda_5 = -0.17$, $\lambda_6 = -0.18$.	Unstable

The results in table 7 and the corresponding graphs are as follows;
Increasing the value of α, its associated eigenvalues show an unstable node. From the figure 12 all the trajectories are moving upwards and growing exponentially which describes the instability nature of that fixed point and growth of wastes on the streets. This

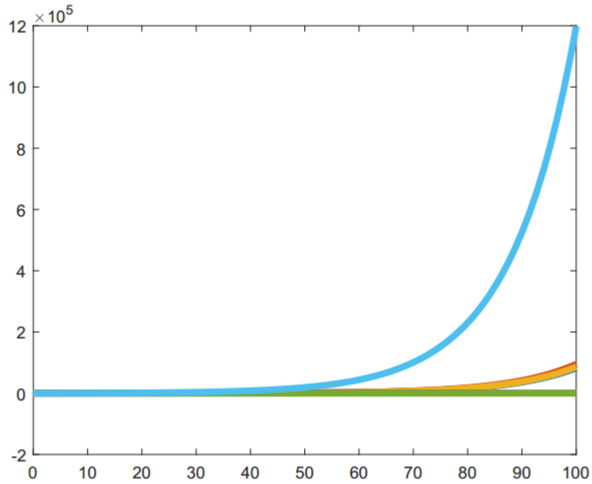

Figure 11: Interpolation for $\gamma_3 = 0.25$

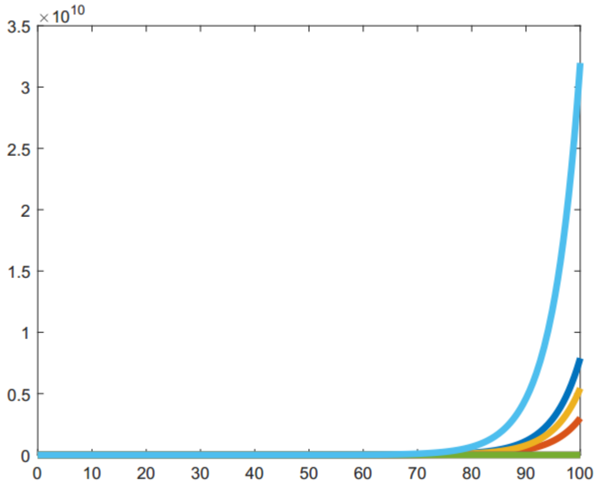

Figure 12: Interpolation for $\alpha = 0.95$

indicates that wastes in gutters within the Municipality should regularly be emptied and sent to dumpsites. Decreasing the value of α, we observe from figure 13 that most of the trajectories move constantly in line with a slight rate of dispersion, except one trajectory showing an exponential growth rate. This confirms that when wastes are transferred from gutters to streets the problem improper wastes management in Kwadaso Municipality will still continue to exist.

3. Conclusions

The $X_1X_2X_3X_4X_5X_6$ model has been introduced for describing the flow of wastes generated in the six collection points within Kwadaso Municipality, Ashanti Region, Ghana.

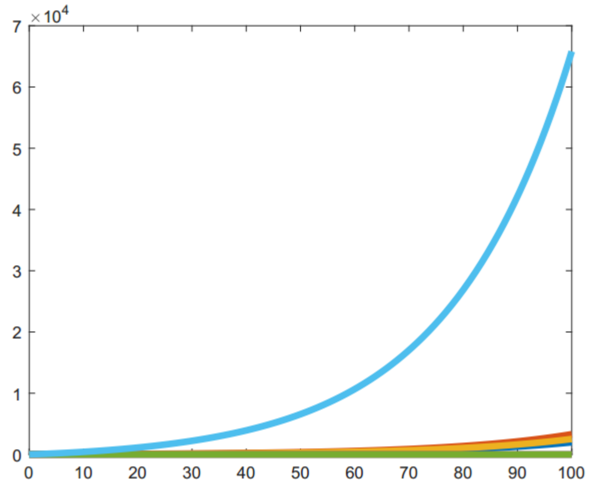

Figure 13: Interpolation for $\alpha = 0.05$

Based on the data on the various compartments of wastes generation in the Municipality, it is revealed that wastes keep on pilling up and hence $\lambda_2 = 0.0825 > 0$, which implies $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*) = (0, 0, 0, 0, 0, 0, 0)$ is an unstable fixed point. Thus, despite the efforts being made by the Assembly and the inhabitants by distilling gutters, sweeping the streets, market places and the houses of the residents and keeping these wastes in waste bins, only a very small amount of the wastes generated daily are transported to dumpsites by the trucks. Consequently, these overflows of wastes in the containers/bins, provided by the Assembly, in market places, on streets and in front of homes of residents create unpleasant sanitary conditions and outbreaks of communicable diseases for the inhabitants and the Assembly as a whole. It is also revealed that the most sensitive parameters are γ_1, μ and τ and the less sensitive parameters are γ_2, γ_3 and α.

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