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Extremes, extremal index estimation, records, moment problem for the Pseudo-Lindley distribution and applications

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Abstract. The pseudo-Lindley distribution which was introduced in Zeghdoudi and Nedjar (2016) is studied with regards to it upper tail. In that regard, and when the underlying distribution function follows the Pseudo-Lindley law, we investigate the behavior of its values, the asymptotic normality of the Hill estimator and the double-indexed generalized Hill statistic process (Ngom and Lo, 2016), the asymptotic normality of the records values and the moment problem.

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1. Introduction

1. General facts.

The following probability distribution function (*pdf*), named as the Pseudo-Lindley *pdf*,

$$f(x) = f(x, \theta, \beta) = \frac{\theta(\beta - 1 + \theta x)e^{-\theta x}}{\beta} 1_{(x \geq 0)} \tag{1}$$

with parameters $\theta > 0$ and $\beta > 1$, has been introduced by [15] as a generalization of the Lindley *pdf* :

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$$\ell(x) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta} 1_{(x \geq 0)}, \tag{2}$$

in the sense that for $\beta = 1+\theta$, $f(\circ)$ is identical to $\ell(\circ)$.

Actually, f derives from ℓ by a mixture of a Lindley distributed random variable and an independent $\Gamma(2,\theta)$ random variables with mixture coefficients $r_1 = (\beta-1)/\beta$ and $r_2 = 1/\beta$, where $1 < r_1,\ r_2 < 1$ and $r_1 + r_2 = 1$.

The cumulative distribution *cdf* function is given by

$$1 - F(x) = \left(\beta^{-1}(\beta + \theta x)e^{-\theta x}\right) 1_{(x \geq 0)}.$$

The Lindley original distribution is an important law that has been used and still is being used in Reliability, in Survival analysis and other important disciplines. Because of its original remarkable qualities, it kicked off a considerable number generalizations as pointed out by [15]. The current generalization (1) has been tested on real data and simulated. The results of those studies and simulations have shown a real interest of that model in survival analysis. In ([15]) for example, that model has been tested on Guinean Ebola. The paper of [6] focused on asymptotic tests of that law based on moments estimators of the new law. The interest that distribution demonstrated in real data modeling motivated us to give some asymptotic theories on it, in view of statistical tests. In this paper, we deal with the properties of the upper tail, the extreme value distribution and the record values. etc., each of them providing statistical tests.

Throughout the paper, X, X_1, X_2, $\cdots$ is a sequence of independent real-valued random (rv), defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, with common cumulative distribution function F, with the first asymptotic moment function and the generalized inverse function defined by

$$R(x,F) = \frac{1}{1-F(x)} \int_x^{+\infty} (1 - F(y))\ dy,\ x \in]0, +\infty[$$

and

$$F^{-1}(u) = \inf\{x \in \mathbb{R},\ F(x) \geq u\}\ for\ u \in]0,1[\ and\ F^{-1}(0) = F^{-1}(0+).$$

For each $n \geq 1$, we denote the ordered statistics of the sample $X_1, \cdots, X_n$ by

$$X_{1,n} \leq \cdots \leq X_{n,n}.$$

Usually, in extreme value theory, we focus on upper extreme and the hypothesis $X > 0$ and the log-transform $Y = \log X$ is instrumental in all major results in that field. We denote the *cdf* of Y by $G(x) = F(e^x)$, $x \in \mathbb{R}_+$. The Renyi representation is also of common use in the following form. The sequence is replaced as follows

$$\{\{X_{1,n} \leq \cdots \leq X_{n,n}\},\ n \geq 1\} =_d \{\{F^{-1}(1 - U_{n-j+1,n}), 1 \leq j \leq n\},\ n \geq 1\}, \tag{3}$$

where $=_d$ stands for the equality in distribution. Finally, the following Malmquist representation (see [14], also [9], page 127) is also used : for each $n \geq 1$, there exist a finite sequence of standard independent exponential random variables $E_{1,n}, \cdots, E_{n,n}$ such that

$$\left\{ \left(\frac{U_{i+1,n}}{U_{i,n}} \right)^i,\ 1 \leq i \leq n \right\} =_d \{E_{i,n}, 1 \leq i \leq n\}. \tag{4}$$

2. Extremes

We can directly see that F is the Gumbel distribution G_0 by three different arguments. First, by using the Von Mises' argument (see [2] or [7], Proposition 24, page 184)

$$\lim_{x \to +\infty} \frac{f'(x)(1 - F(x))}{f^2(x)} = -1. \tag{5}$$

A second argument comes from that $Y = \exp(X)$ has the distribution $G(x) = F(\log x) = \beta^{-1}(\beta + \theta \log x)x^{-\theta x}$. Since

$$\forall \lambda > 0,\ \lim_{x \to +\infty} \frac{1 - G(\lambda x)}{1 - G(x)} = \lambda^{-\theta}, \tag{6}$$

$G \in D(G_{1/\theta})$ and since $F(x) = G(e^x)$ for $x \geq 1$, by Theorem [4] (Lemmas 9 and 10), $F \in D(G_0)$.

A third argument is related to the development of the quantile function. In the appendix (page 753), we give a number of expansions of that quantile that could be used for different purposes. For example we have (see page 755),

$$\forall \lambda > 0,\ F^{-1}(1 - u) = \theta^{-1}(\log(1/u) - \log\log(1/u)) + \theta^{-1}K(u) \tag{7}$$

with $K(u) = O\Big((\log 1/u)^{-2}\Big)$. By using it, we get

$$\frac{F^{-1}(1-\lambda u)-F^{-1}(1-u)}{(1/\theta)} \rightarrow -\log \lambda \ as \ u \rightarrow 0.$$

By the π-variation criteria of [2] (See [9], Proposition 11, page 88), we have $F \in D(G_0)$ and $R(x,F) \rightarrow \gamma = 1/\theta$ as $x \rightarrow +\infty$. Formula (7) is actually a second-order condition for the quantile function (see [2]). We apply it right to get a rate of convergence of the maximum observations. Put $\gamma = 1/\theta$.

2. Expansion of the maximum values.

By the Renyi representation and by denoting $Z_n = -\log(nU_{1,n})$, we have that $\log(1 + Z_n/(\log n)) \rightarrow_{\mathbb{P}} 0$ and since $\log U_{1,n} = O_{\mathbb{P}}(\log n)^{-1}$

$$\begin{aligned} X_{n,n} - F^{-1}(1-1/n) &= \gamma Z_n + \gamma \log(1+Z_n/(\log n)) \\ &+ O((\log n)^{-2}) + O((\log U_{1,n})^{-2}) \end{aligned}$$

and hence

$$\frac{X_{n,n} - F^{-1}(1-1/n)}{\gamma} = Z_n + O_{\mathbb{P}}\left((\log n)^{-1}\right) = \Lambda + o_{\mathbb{P}}(1). \tag{8}$$

It is easy to see that Z_n converges to Gumbel law Λ with *cdf*

$$G_0(x) = \exp(-\exp(-x)), \ x \in \mathbb{R}.$$

So we have that $X_{n,n}$ converges to a Λ law. But we obtain the random rate of convergence $Z_n/\log n$, since

$$\frac{\log Z_n}{\log n}\left(\frac{X_{n,n} - F^{-1}(1-1/n)}{\gamma} - Z_n\right) = 1.$$

As well for $k = k(n) \rightarrow +\infty$ such that $k(n)/n \rightarrow 0$, and by taking $T_n = \log(nU_{k,n}/k)$ and $q_n = n/k(n)$ which goes to $+\infty$, we have

$$\frac{X_{n-k,n} - F^{-1}(1-k/n)}{\gamma} = T_n + \log(1+T_n/\log q_n)) + O_{\mathbb{P}}((\log q_n)^{-2}). \tag{9}$$

3. Estimating the extreme value index $\gamma = 1/\theta$.

The Hill's estimator ([3], 1975)

$$H_n = \frac{1}{k(n)} \sum_{j=1}^{k(n)} j \left(X_{n-j+1,n} - X_{n-j,n} \right), \tag{10}$$

is the most celebrated estimator of the extreme value index $\gamma = 1/\theta$ of $Z = \exp(X)$. Among a significant number of generalizations of the Hill's estimator, the Ngom-Lo generalization ([12], 2016), called the functional Double-indexed Hill estimator, is one the sharpest one. It is defined as

$$H_n(f,s) = \left(\sum_{j=1}^{k(n)} f(j) \left(X_{n-j+1,n} - X_{n-j,n} \right)^s / a_n(f,s) \right)^{1/s},$$

where $f : \mathbb{N} \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ is a measurable mapping and $s > 0$, and

$$a_n(f,s) = \Gamma(s+1) \sum_{j=1}^{k(n)} f(j) j^{-s}.$$

Let us define for $s > 0$ and $f : \mathbb{N} \setminus \{0\} \to \mathbb{R}_+ \setminus \{0\}$ measurable,

$$C^2(s) = \Gamma(2s+1) - \Gamma(s+1)^2, \ s_n^2(f,s) = C^2(s) \sum_{j=1}^{k(n)} f(j)^2 j^{-2s},$$

and

$$B_n(f,s) = \max\{f(j) j^{-s}/s_n(f,s), \ 1 \leq j \leq k(n)\}.$$

We simply notice that the classical Hill's estimator is $H_n(I_d, 1)$ where I_d is the identity function on $\mathbb{N} \setminus \{0\}$. Let us give asymptotic normality for the functional Double-indexed Hill estimator.

(a) Extreme Limit Theorem.

We begin with the simple Hill's estimator.

Theorem 1. *For* $]0, \ n] \ni k(n) \to +\infty$ *such that*

$$k(n)^{3/4} / \log n \to 0. \tag{K1}$$

We have, as $n \to +\infty$,

$$\sqrt{k(n)} \left(H_n - \gamma \right) \rightsquigarrow \mathcal{N}(0, \gamma^2). \tag{11}$$

We want to establish the random rate of convergence associated with the convergence 11 in the part (a) of the following corollary. In the part (b), we want to share that we do not need any other condition on top of $k(n)/n \to 0$ to have the central limit theorem if F^{-1} is reduced to

$$F_*^{-1}(1-u) = \gamma \log u - C(\gamma) \log\log(1/u),\ \ u \in]0,1[,\ \ C(\gamma) \geq 1. \tag{12}$$

Corollary 1. *We have the following results.*

(a) Here again F is the cdf of the Pseudo-Lindley distribution with parameters $\theta > 0$ and $\beta > 0$ and the notation above. Let $k(n)/\log n \to 0$.

Let $W(1)$ is a standard Gaussian random variable. Then we have

$$\frac{\log n}{\gamma\sqrt{k(n)}}\left(\sqrt{k(n)}(H_n-\gamma) - \gamma W(1)\right) \to_{\mathbb{P}} 1,$$

(b) If F^{-1} were reduced as in Formula (12), we have the asymptotic normality

$$\sqrt{k(n)}(H_n - \gamma) \to \mathcal{N}(0, \gamma^2)$$

whenever $k(n)/n \to 0$ and

$$\log n\left(\sqrt{k(n)}(H_n-\gamma) - \gamma W(1)\right) = O_{\mathbb{P}}(1).\ \Diamond$$

Proof of Theorem 1. By the Malmquist representation (See [14] or [9], Proposition 32, page 135), by Formula (38), we have for any $1 \leq j \leq k$,

$$\begin{aligned} X_{n-j+1,n} - X_{n-j,n} &= F^{-1}(1-U_{j,n}) - F^{-1}(1-U_{j+1,n}) \\ &= \gamma j^{-1} E_{j,n} - \gamma \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)} + O_{\mathbb{P}}\left((logn)^{-2}\right) \end{aligned} \tag{13}$$

and next

$$j\left(X_{n-j+1,n} - X_{n-j,n}\right) = \gamma E_{j,n} - \gamma j \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)} + O_{\mathbb{P}}\left(k\,(logn)^{-2}\right).$$

So for $Z_n = \log nU_{1,n}$ (which converges in law to Λ) and

$$\left|\int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)}\right| \leq \frac{j^{-1}E_{j,n}}{|\log n - Z_n|}. \tag{14}$$

Hence

$$\frac{1}{k(n)}\left|\sum_{j=1}^{k(n)} j \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)}\right| \leq \frac{S^*_{k(n)}}{k} O_{\mathbb{P}}((\log n)^{-1}),$$

where $S^*_{k(n)} = E_{j,n} + \cdots + E_{k,n}$. We finally get

$$\sqrt{k(n)}\,(H_n - \gamma) = \gamma\frac{S^*_{k(n)} - k}{\sqrt{k(n)}} + O_{\mathbb{P}}\left(\frac{1}{\log n}, \frac{k^{3/2}}{(\log n)^2}\right)$$

We conclude that, whenever (K1) holds, we have

$$\sqrt{k(n)}\,(H_n - \gamma) = \gamma\frac{S^*_{k(n)} - k}{\sqrt{n}} + o_{\mathbb{P}}(1). \square$$

Proof of the Corollary 1. The proof of Part (b) is the conclusion of the proof of Theorem 1 up to the formula (14). If (12) holds, further steps are dismissed. And we need only $k(n)/n \to 0$ to conclude. Let us set

$$Z^*_n = \frac{1}{\sqrt{k(n)}}\sum_{j=1}^{k(n)} j \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)},\ n \geq 1.$$

From the first part, we already know that $Z^*_n = O_{\mathbb{P}}(1/\log n)$. We denoted by $W(1)$ a standard Gaussian random variable. By the classical Kómlos-Màjor-Tusnàdy (KMT) approximation, we have

$$\left|\frac{S^*_k(n) - k(n)}{\sqrt{k(n)}} - \gamma W(1)\right| = O_{\mathbb{P}}\left(\frac{\log k(n)}{\sqrt{k(n)}}\right).$$

Straightforward expansions using the different rates of convergence lead to

$$\frac{\sqrt{k(n)}(H_n - \gamma) - \gamma W(1)}{\gamma Z^*_n} \to_{\mathbb{P}} 1,$$

whenever $k(n)/n \to 0$. Now we apply Proposition in [9], page 22. Since the function $\log(1/u)$ is slowly varying and that $U_{1,n}/U_{k+1,n}$ and $U_{k+1,n}/U_{1,n}$ are both asymptotically bounded in probability, we have

$$t_n = \sup_{1 \leq j \leq k(n)} \sup_{s \in [U_{j,n}, U_{j+1,n}]} \left| \frac{\log(1/s)}{\log n} - 1 \right| \rightarrow_{\mathbb{P}} 0.$$

It comes that

$$Z_n^* = \frac{\sqrt{k(n)}}{\log n}(k_{(n)}^{-1} S_{k(n)}^*)(1 + O(t_n)) = \frac{\sqrt{k(n)}}{\log n}(1 + o(1)),$$

which gives the desired result. ■

We have the following convergence of the Double-indexed functional Hill statistics.

Theorem 2. *We have the following two results.*

(a) If the following conditions hold, as $n \rightarrow +\infty$

$$s_n(f,1)/(s_n(f,s)\log n) \rightarrow 0 \text{ and } B_n(f,s) \rightarrow 0,$$

then

$$\frac{T_n(f,s) - \gamma^s a_n(f,s)}{s_n(f,s)} \rightsquigarrow \mathcal{N}\left(0, \gamma^{2s}\right).$$

(b) Furthermore, if $a_n(f,s)/s_n(f,s) \rightarrow +\infty$, then

$$\frac{a_n(f,s)}{s_n(f,s)}\left(\left(\frac{T_n(f,s)}{a_n(f,s)}\right)^{1/s} - \gamma\right) \rightsquigarrow \mathcal{N}(0, s^{-2}\gamma^2).$$

Proof. Let us exploit the proof of Theorem 1. We have for $j \in \{1, \cdots, k(n)\}$, $s \geq 1$,

$$\begin{aligned} A_{i,n} &= f(j)\left(X_{n-j+1,n} - X_{n-j,n}\right)^s \\ &= f(j)\left(\gamma j^{-1} E_{j,n} - \gamma \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)} + O_{\mathbb{P}}\left(F_{k(n)}\,(logn)^{-2}\right)\right)^s \\ &=: f(j)\left(\gamma j^{-1} E_{j,n} - R_{j,n} + C_{j,n}\right)^s, \end{aligned}$$

with

$$C_{j,n} = O_{\mathbb{P}}\left((logn)^{-2}\right)(uniformly \ in \ j), \left|\gamma \int_{U_{j,n}}^{U_{j+1,n}} \frac{du}{u\log(1/u)}\right| \leq \frac{\gamma j^{-1} E_{j,n} b(n)}{|\log n - Z_n|}.$$

We get, by the mean value theorem, $j \in \{1, \cdots, k(n)\}$, $s \geq 1$,

$$\begin{aligned}
&A_{i,n} - \gamma^s f(j) j^{-s} E_{j,n}^s \\
&\leq s f(j) \left| R_{j,n} + C_{j,n} \right| \left(\gamma j^{-1} E_{j,n} + |R_{j,n}| + |C_{j,n}| \right)^{s-1} \\
&\leq \left(\frac{s\gamma f(j) j^{-1} E_{j,n}}{|\log n - Z_n|} \right) \left(\gamma j^{-1} E_{j,n} + |R_{j,n}| + |C_{j,n}| \right)^{s-1}.
\end{aligned}$$

In the lines below, we will bound the term with the power $s-1$. If $s=1$, there will is nothing to bound. So formulas regarding that term are dismissed for $s=1$ and are used only for $s > 2$. For $s \geq 1$, we will use the C_{s-1} inequality (for $s \leq 2$, with $|a+b|^{s-1} \leq 2^{s-2}|a|^{s-1} + |b|^{s-1}$ $C_{s-1} = 2^{s-2}$). For $0 < r < 1$, it can be easily checked that, for $u > 0$ fixed, the function $g(v) = (u+v)^r - u^r - v^r$ of $v \geq 0$ takes the value $g(v) = 0$ and has a non-positive derivative function, so that $g(v) \leq g(0) = 0$ for any $v \geq 0$, which is equivalent to $(u+v)^r \leq u^r + v^r$. We finally have that $|a+b|^{s-1} \leq D_s |a|^{s-1} + |b|^{s-1}$ with $D_s = 1$ for $1 < s < 2$ and $D_s = C_{s-1}$ for $s \geq 2$. Applying that inequality leads, $j \in \{1, \cdots, k(n)\}$, $s \geq 1$, to

$$\begin{aligned}
&A_{i,n} - \gamma^s f(j) j^{-s} E_{j,n}^s \qquad (A)\\
&\leq \left(\frac{s\gamma f(j) j^{-1} E_{j,n}}{|\log n - Z_n|} \right) \left(D_s \gamma^{s-1} j^{s-1} E_{j,n}^{s-1} + \frac{D_s^2 \gamma^{s-1} j^{s-1} E_{j,n}^{s-1}}{(|\log n - X_n|^{s-1})} + O_{\mathbb{P}}\left(\frac{D_s^2}{(\log n)^{2(s-1)}} \right) \right).
\end{aligned}$$

Let us denote

$$S_n(f, s) = \sum_{j=1}^{k(n)} f(j) j^{-s} E_{j,n}^s$$

and

$$T_n(f, s) = \sum_{j=1}^{k(n)} f(j) \left(X_{n-j+1,n} - X_{n-j,n} \right)^s.$$

By combining the results above, we arrive at

$$\begin{aligned}
&\left| T_n(f, s) - \gamma^s S_n(f, s) \right| \qquad (B)\\
&\leq \left(\frac{s\gamma S_n(f, 1)}{|\log n - Z_n|} \right) \left(D_s \gamma^{s-1} S_n(Id, s-1) + \frac{D_s^2 \gamma^{s-1} S_n(Id, s-1)}{(|\log n - Z_n|^{s-1})} + O_{\mathbb{P}}\left(\frac{D_s^2}{(\log n)^{2(s-1)}} \right) \right).
\end{aligned}$$

Let us study $S_n(f,s)$. As a sequence of partial sums of real-value independent random variables indexed by $j \in \{1, \cdots, k(n)\}$ with first and second moments

$$\Gamma(s+1)f(j)j^{-s} \ and \ (\Gamma(2s+1) - \Gamma(s+1)^2)f(j)^2 j^{-2s},$$

the asymptotic normality is given by the the theorem of Levy-Feller-Linderberg (See Theorem 20 in [5]) we apply to the centered *rrv*'s $\xi_j = f(j)j^{-s}(E^s_{j,n} - \Gamma(s+1))$, after remarking that

$$\left\{ \frac{\mathbb{V}ar(\xi_j)}{\sum_{j=1}^{k(n)} \mathbb{V}ar(\xi_j)}, \ 1 \leq j \leq k(n) \right\} = C(s)B_n(f,s).$$

So, as $n \to +\infty$,

$$\left(\frac{1}{s_n(f,s)} \sum_{j=1}^{k(n)} \left(f(j)j^{-s}(E^s_{j,n} - \Gamma(s+1)) \right) \rightsquigarrow \mathcal{N}(0,1) \right) \ and \ B_n(f,s) \to 0$$

and the Lynderberg condition holds, that is, for any $\varepsilon > 0$,

$$g(n,\varepsilon) = \frac{1}{s_n(f,s)} \sum_{j=1}^{k(n)} \int_{(|\xi_j| > \varepsilon s_n(f,s))} \xi_j^2 \ d\mathbb{P} \to 0.$$

But, for $K^2(s) = \Gamma(4s+1) - 4\Gamma(3s+1)\Gamma(s+1) + \Gamma(2s+1)\Gamma(s+1)2 - 3\Gamma(3s+1)^4$,

$$\mathbb{E}\xi^4 = K(s)f(s)^4 j^{-4s}$$

and, by the Cauchy-Schwarz inequality

$$\begin{aligned}
&\int_{(|\xi_j| > \varepsilon s_n(f,s))} \xi_j^2 \ d\mathbb{P} \\
&\leq \left(\int \xi_j^4 \ d\mathbb{P} \right)^{1/2} \left(\int 1_{(|\xi_j| > \varepsilon s_n(f,s))} \ d\mathbb{P} \right)^{1/2} \\
&= Kf(j)^2 j^{2s} \left(\int 1_{(|\xi_j| > \varepsilon s_n(f,s))} \ d\mathbb{P} \right)^{1/2} \\
&= Kf(j)^2 j^{2s} \mathbb{P}\left(|\xi_j| > \varepsilon s_n(f,s) \right))^{1/2} \\
&\leq Kf(j)^2 j^{2s} \left(\frac{K(s)^2 f(j)^4 j^{-4s}}{\varepsilon^4 s_n^4(f,s)} \right)^{1/2} \\
&= K(s)^2 \left(f(j)^2 j^{2s} \right)^2 (s_n^{-2}(f,s)^2
\end{aligned}$$

$$= \frac{C(s)}{K(s)} B_n(f,s) \frac{\mathbb{V}ar(\xi_j)}{s^2\,(f,s)}$$

So

$$g(n,\varepsilon) = \left(\frac{K(s)}{C(s)}\right)^2 B_n(f,s) \to 0.$$

Our hypothesis $B_n(f,s) \to 0$ makes the Lynderberg hold and the central limit theorem holds for $S_n(f,s)$, that is

$$\frac{S_n(f,s) - \gamma^s a_n(f,s)}{s_n(f,s)} \rightsquigarrow \mathcal{N}(0,1).$$

Now, let us return to the approximation (B) at page 747. We have that for $s = 1$, the expression denoted as C_n between the pair of big parentheses should be equal to one as explained before. If $s > 1$, we have $\sigma^2(s) = \sum_{j\geq 1} j^{-2(s-1)} < +\infty$, we apply a theorem of Kolmogorov (see [5], Proposition 25, page 233), $S_n(Id, s-1)$ weakly converges to the random variable $W(s)$ with variance $\sigma^2(s)$. Hence $C_n = O_{\mathbb{P}}(1)$. We arrive at

$$\left| \frac{T_n(f,s) - a_n(f,s)}{s_n(f,s)} - \frac{\gamma^s(S_n(f,s) - a_n(f,s))}{s_n(f,s)} \right| \leq O_{\mathbb{P}}\left(\frac{S_n(f,1)}{s_n(f,s)\log n}\right). \tag{15}$$

The later bound goes to zero in probability if and only if $S_n(f,1)/(s_n(f,s)\log n) \to 0$. Now, we have

$$\frac{a_n(f,s)}{s_n(f,s)}\left(\frac{T_n(f,s)}{a_n(f,s)} - \gamma^s\right) = Z_n + o_{\mathbb{P}}(1).$$

If $a_n(f,s)/s_n(f,s) \to +\infty$, we can use the δ-method applied to $g(t) = t^{1/s}$ to get

$$\frac{a_n(f,s)}{s_n(f,s)}\left(\left(\frac{T_n(f,s)}{a_n(f,s)}\right)^{1/s} - \gamma\right) \rightsquigarrow \mathcal{N}(0, s^{-2}\gamma^2).\blacksquare.$$

Remark. In [12], we gave a direct proof of the asymptotic normality of $S_n(f,s)$ by using the two hypotheses $B_n(f,s) \to 0$ and $s_n(f,s) \to +\infty$. Here, it seems that we only used the first one. But that one could not hold if $S_n(f,s)$ contains a sub-sequence converging to a finite and positive number. That remark should be recalled in interpreting the results in [12].

3. Upper records values

The main result is :

Theorem 3. *If, for each $n \geq 1$, $X^{(n)}$ stands for n-th record value, we have as $n \to +\infty$,*

$$\frac{X^{(n)} - \gamma n}{\gamma\sqrt{n}} \rightsquigarrow \mathcal{N}(0,1).$$

Remark. We refer the reader to [8] for a simple introduction to records theory.

Proof. We already noticed that $Z = exp(X)$ is the extremal domain of attraction of $G_\gamma(x) = \exp(-(1+\gamma x))$, for $\gamma x > -1$. From Part (b) of Theorem 1 in [8], the n-th record $Z^{(n)} = exp(X^{(n)})$ have the representation

$$\left(\frac{\exp(X^{(n)})}{H^{-1}(1-e^{-n}}\right)^{1/\sqrt{n}} = \exp(\gamma S_n^*) + o_{\mathbb{P}}(1) \tag{16}$$

where S_n^* has the same law as $\gamma^{-1}(T_n - n)/\sqrt{n}$ with T_n denoting a γ law with parameters n and 1. Since $H^{-1}(1-u) = \exp(F^{-1}(1-u))$, we have

$$\frac{X^{(n)} - F^{-1}(1-e^{-n})}{\gamma\sqrt{n}} = S_n^* + o_{\mathbb{P}}(1) \tag{17}$$

By the central limit theorem, it comes that

$$\frac{X^{(n)} - F^{-1}(1-e^{-n})}{\gamma\sqrt{n}} = \mathcal{N}(0,1) + o_{\mathbb{P}}(1). \tag{18}$$

By using Formula (7), we get

$$\frac{X^{(n)} - \gamma n}{\gamma\sqrt{n}} = S_n^* + o_{\mathbb{P}}(1) \tag{19}$$

$$\frac{X^{(n)} - \gamma n}{\gamma\sqrt{n}} = \mathcal{N}(0,1) + o_{\mathbb{P}}(1). \tag{20}$$

The proof is over. $\square$

4. The moment problem

Typically, the moment problem on $\mathbb{R}$(see [13] and more recently in [10]) is the following. Given a sequences real numbers $(m_n)_{n\geq 1}$, can we find a distribution (not necessarily a *cdf*) F on $\mathbb{R}$ as the unique solution of the moments equations.

$$\forall n \geq 1,\ m_n = \int x^n\ dF(x).$$

This is a nice but a difficult mathematical question treated in [13] and more recently [10]. But in the context of probability theory on $\mathbb{R}$, we may have a fixed *cdf* F of random variable X having moments

$$\forall n \geq 1,\ \mathbb{E}X^n = m_n\ finite.$$

The moment problem becomes : Is the sequence of moments $(m_n)_{n\geq 1}$ characterize the probability law of X. In that regard, we have

Theorem 4. *The moments of the pseudo-Lindely probability law are the following*

$$\forall n \geq 1,\ m_n = \frac{n!(\beta+n)}{\theta^n \beta}.$$

Any real-valued random variable have the moments $(m_n)_{n\geq 1}$ follows the pseudo-Limdley law.

Proof. At the place of a simple proof, we proceed to slight round-up of the moment problem and explain how to find a simple criteria based on Analysis. A possible tool is the characteristic function which characterize its associated probability law. We have the following expansion of any characteristic function of X (see [11] or [5], Lemma 5, page 255), we have

$$\mathbb{E}e^{iuX} = 1 + \sum_{k=1}^{n} \frac{(iu)^k mk}{k!} + \theta 2^{1-\delta} \mu^{n+\delta} \frac{|u|^{n+\delta}}{(n+1)!}. \tag{21}$$

By usual analysis tools, the series in Formula (21) converges in the $]-R,R[$ where R is found according the Cauchy rule

$$\limsup_{n\to+\infty} (m_n)^{1/n} = R > 0.$$

The conclusion is that two random variables have the same moments of all orders have characteristic functions coinciding on $]-R,R[$. Finally, (see [11], page 225, Part B.; see also [1]) two characteristic functions coinciding on an interval $]-R,R[$ coincide everywhere and thus, are associated to the same probability law.

Let us apply to the pseudo-Lindley law. In [15], the moments are given by

$$\forall n \geq 1,\ m_n = \frac{n!(\beta + n)}{\theta^n \beta}.$$

Straightforward computation based on the Stirling formula leads to $R = 1/\theta$. This is enough to prove the claim of the theorem. (As remarked by the anonymous referee, We might have use the Carleman criteria).$\square$

Appendix . Let $R = \beta/\theta$. In the computations below, $u \in (0,1)$ and $x \geq 0$ are linked by $u = 1 - F(x)$. So $u \to 0$ if and only if $x \to +\infty$. Also, below, functions of x are functions of u actually. We denote $A(u) = \log(1 + R/x)$. We have $A(u) \to 0$ as $u \to 0$. By writing

$$\log(\beta + \theta x) = \log(\beta + \theta x) - \log \theta x + \log \theta x = \log \theta x + A(u),$$

we see that $u = 1 - F(x)$ gives

$$\theta x = \log(1/u) + \log R + \log x + A(u). \tag{22}$$

So, we have

$$F^{-1}(1 - u) = \theta^{-1} \log(1/u)(1 + o(1)). \tag{23}$$

and

$$\log x = \log \log(1/u)(1 + o(1)). \tag{24}$$

Now, we wish to develop that asymptotic equivalence with rates of convergence. Let $B(u) = \log R + \log x + A(u)$. From Formula 22, we have

$$\frac{x}{\theta^{-1} \log(1/u)} - 1 = \frac{B(u)}{\log(1/u)}. \tag{25}$$

By Formula (25), we notice that

$$B(u) = \log R + \log x + (R/x) - (R/x)^2/2 + O(\log(1/u)^{-3}) = O(\log x) = (\log \log u)(1 + o(1)), \tag{26}$$

and hence, for $D(u) = \log R + A(u)$,

$$\frac{\log(1/u)}{\log x} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) = 1 + \frac{D(u)}{\log x}. \tag{27}$$

Also

$$\frac{D(u)}{\log x} = \frac{\log R + (R/x) - (R/x)^2/2 + O(x^{-3}}{\log x}$$

Next, we have

$$\frac{\log x}{-\log R} \left(\frac{\log(1/u)}{\log x} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) - 1 \right) \tag{28}$$

$$= 1 + \frac{R}{x \log R} - \frac{R^2}{2x^2 \log R} + O(x^{-3})$$

and finally

$$\begin{aligned} &\frac{x \log R}{R} \left(\frac{\log x}{\log R} \left(\frac{\log(1/u)}{\log x} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) - 1 \right) - 1 \right) \\ &= 1 - \frac{R}{2x} + O(x^{-2}). \end{aligned} \tag{29}$$

Now we want to do the same for $\log x$. Hence, we get.

$$\log(\theta x) = \log\log(1/u) + \log(1 + B(u)/\log(1/u)) \tag{30}$$

from which we get

$$\log x - \log\log(1/u) = -\log\theta + (B(u)/\log(1/u)) + O\left((B(u)/\log(1/u)^2\right). \tag{31}$$

From Formula (27), we have

$$\begin{aligned} &\frac{\log(1/u)}{\log x} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) - \frac{\log(1/u)}{\log\log 1/u} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) \\ &= \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) \frac{-(\log(1/u))(\log x - \log\log 1/u)}{(\log x)(\log\log 1/u)} \\ &= (1 + D(u)/\log x) \left(\frac{1}{(\log x)(\log\log 1/u)} \left(-\log\theta + (B(u)/\log(1/u)) + O(B(u)/\log(1/u)^2) \right) \right) \\ &= O((\log\log 1/u)^2) \end{aligned}$$

Formula (27) becomes

$$\frac{\log(1/u)}{\log\log 1/u} \left(\frac{x}{\theta^{-1} \log(1/u)} - 1 \right) = 1 + \frac{D(u)}{\log x} + O((\log\log 1/u)^2). \tag{32}$$

That formula will be used with Formula 31 and

$$\begin{aligned} \frac{B(u)}{\log 1/u} &= \frac{\log R}{\log 1/u} + \frac{\log\log 1/u}{\log 1/u}(1 + o(1)) \\ &+ \frac{(R/x) - (R/x)^2/2}{\log 1/u} + O((\log 1/u)^{-4}). \end{aligned} \tag{33}$$

From 22, and from the following formula we can check by using differentiation methods to establish monotonicity

$$x - x^2/2 \leq \log(1+x) \leq x$$

we have

$$(R/x) - R^2/(2x^2) + \log R + \log x \leq \theta x - \log(1/u) \leq (R/x) + \log R + \log x. \tag{34}$$

But we also have

$$x = \log(1/u)\left(1 + \frac{\log \beta^{-1} + \log x + A(u)}{\log(1/u)}\right)$$

which implies

$$\log x = \log\log(1/u) + \log\left(1 + \frac{\log \beta^{-1} + \log x + A(u)}{\log(1/u)}\right)$$

By putting

$$H(u) = \frac{\log \beta^{-1} + \log x + A(u)}{\log(1/u)},$$

we finally get

$$H(u) - H(u)^2/2 \leq \log x - \log\log(1/u) \leq H(u). \tag{35}$$

By combining Formulas (34) and (35), we get

$$|\theta x - \log(1/u) - \log(1/u)| \leq \frac{1}{2}\left(\frac{R^2}{x^2} + H(u)^2\right). \tag{36}$$

Since (R/x^2) and $H(u)^2$ are both $O(\log 1/u)^{-2})$, we have

$$F^{-1}(1-u) = \theta^{-1}(\log(1/u) - \log\log(1/u)) + O(\log 1/u)^{-2}). \tag{37}$$

But since the derivative $\log\log(1/u)$ is $(-u\log(1/u))^{-1}$, we have for $d = -\log\log 2$,

$$\forall u \in]0,1[,\ \log\log(1/u) - = \int_u^{1/2} \frac{1}{u\log(1/u} \ du,$$

and finally

$$F^{-1}(1-u) = d + \theta^{-1}(\log(1/u) - \int_u^{1/2} \frac{1}{u\log(1/u)}\, du + O\left((\log 1/u)^{-2}\right). \tag{38}$$

References

[1] P. Billingsley. *Probability and Measure.* Wiley, Third Edition, 1995.

[2] L. de Haan. *On regular variation and its application to the weak convergence of sample extremes.* Mathematical Center Tracts,Amsterdam. (MR0286156), 1970.

[3] B. Hill. A simple general approach to the inference about the tail index of a distribution. *Ann. Statist.*, 3:1163–1174, 1975.

[4] G.S. Lo. *Sur quelques estimateurs de l'Index d'une loi de Pareto : Estimateur de Hill, de S.Csörgő-Deheuvels-Mason, de de Haan-Resnick et loi limites de sommes de valeurs extrêmes pour une variable aléatoire dans le domaine d'attraction de Gumbel.* Université Pierre et Marie Curie, Sorbonne. Thèse de Doctorat, 1986.

[5] G.S. Lo. *Mathematical foundation of probability theory.* arXiv:1808.01713, doi : 10.16929/sbs/2016.0008, 2018.

[6] G.S. Lo, T. A. Kpanzou, and C.M. Haidara. Statistical tests for the pseudo-lindley distribution and applications. *Afrik. Statist., Doi :10.16929/as/2019.2127.151*, 14 (4):2127–2139, 2019.

[7] G.S. Lo, T. A. Kpanzou, M. Ngom, and M. Diallo. *Weak Convergence (IIA) - Functional and Random Aspects of the Univariate Extreme Value Theory.* Arxiv : 1810.01625, 2018.

[8] G.S. Lo and Ahnsanullah M. Asymptotic laws of the strong upper records values in the extreme domain of attraction and beyond. *Arxiv :1905.03380*, 2019.

[9] G.S. Lo, M. Ngom, and T. A. Kpanzou. *Weak Convergence (IA) - Sequences of random vectors.* arXiv:1810.01625, doi :10.16929/sbs/2016.0001, 2016.

[10] G.S. Lo, M. Touré, and A.B. Niang. *Recent and Modern Approach to The Moment Problem on R. In : Theory and Practice of Mathematics and Computer Science Vol. 1. [TPMCS-V1] (Ed. B. Chaouchi). ISBN: 978-93-90149-72-8 (print), 978-93-90149-22-3 (ebook). Doi : DOI: 10.9734/bpi/tpmcs/v1.* Book Publishing International, Hooghly and London, 2020.

[11] M. Loève. *Probability Theory I.* Springer-Verlag, 4th Edition, 1997.

[12] M. Ngom and G.S. Lo. A double-indexed functional hill process and applications. *Journal of Mathematical Research (e-ISSN 1916-9809), Doi : 105539/jmr/v8n4p144*, 8 (4):144–165, 2016.

[13] J.A. Shohat and J.D. Tamarkin. *The problem of moment, Mathematical Surveys and monographs.* American Society of Mathematics (Re-edited in 1950, 1963 and 1970), 1943.

[14] G.R. Shorack and J.A. Wellner. *Empirical Processes with Applications to Statistics.* Wiley-Interscience, New-York, 1986.

[15] H. Zeghdoudi and S. Nedjar. A pseudo lindley-distribution and its application. *Afrika Statistika, DOI: http://dx.doi.org/10.16929/as/2016.923.83*, 11 (1):923–932, 2016.