EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 13, No. 4, 2020, 794-806
ISSN 1307-5543 – www.ejpam.com
Published by New York Business Global

Robust Exponential Stability of Recurrent Neural Networks with Deviating Argument and Stochastic Disturbance

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Abstract. It is well known that deviating argument and stochastic disturbance may derail the stability of recurrent neural networks (RNNs). This paper discusses the robustness of global exponential stability (GES) of RNNs accompanied with deviating argument and stochastic disturbance. For a given global exponentially stable RNNs, it is interesting to know how much the length of the interval of piecewise function and the interference intensity so that the disturbed system may still be exponentially stable. The available upper boundary of the range of piecewise variables and the interference intensity in the disturbed RNNs to keep GES are the solutions of some transcendental equations. Finally, some examples are provided to demonstrate the efficacy of the inferential results.

2020 Mathematics Subject Classifications: 34A34, 34A50, 58A10, 93C10

Key Words and Phrases: Robustness, recurrent neural networks, deviating argument, stochastic disturbance

1. Introduction

As a kind of nonlinear dynamic system, the stability of recurrent neural networks (RNNs) mainly rely on its parameter allocation [5, 7, 18, 27]. It is known that arbitrary perturbations and various time delays in the process of neuron activation may led to instability or oscillation of RNNs [3, 4, 12, 14].

Piecewise argument, unifying the advanced and hysteretic arguments [13, 21, 26], is one of the nonlinear non-smooth actuators that play an important role in the operation of a nonlinear system [2, 22–25]. For example, in order to describe the stationary state of the wire length temperature, the nonlinear dynamic model with deviation parameters may be used for better fitting. Stochastic disturbance, which is hardly avoided in some real applications considered in some nonlinear systems, nontrivially generalizes the classical

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DOI: https://doi.org/10.29020/nybg.ejpam.v13i4.3848

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deterministic process [6, 9–11, 15]. In terms of stability analysis, there are a variety of unique stability theories and research results, which mainly include robust analysis, dissipative analysis and impulse control. In addition, discrete time delay and fuzzy systems have also attracted much attention [1, 8, 19, 21]. For a stable RNNs, it is significant to describe how much the length of the interval of piecewise argument and interference intensity of the perturbed RNNs can withstand without losing stability.

This paper characterizes the robustness of RNNs with piecewise argument and arbitrary disturbance. For globally exponential stable RNNs, the available upper boundary of the range of deflection arguments and the interference intensity in the perturbed RNNs to preserve globally exponential stability are the solutions of some transcendental equations. Roughly speaking, the contributions of this paper include: (i) The relationship between piecewise deviation variables and system solutions in recurrent neural networks is constructed; (ii) Some mathematical inequalities are used to enlarge the upper bound of the interval of piecewise arguments and the upper bound of random disturbances; (iii) Developing some approaches to analyze recurrent neural networks in the piecewise of deviating arguments.

The structure of the paper is outlined as follows. Section 2 gives preliminaries and model description. Main results are presented in Section 3. Several illustrative examples are given in Section 4. Some concluding remarks are presented in Section 5.

2. Preliminaries

Throughout this paper, let $\Re$ be the set of real numbers and $\Re^+$ be the set of positive real numbers. $\Re^n$ denotes the n-dimensional vectors space over $\Re$ and $\Re^{n\times m}$ be the set of $n \times m$ matrices over $\Re$, where n, m are included in the set $\mathbb{N}$ of nature numbers. The Euclidean norm in $\Re^n$ and $\Re^{n\times m}$ are expressed by the same symbol $\|\cdot\|$. Specifically, for an n-dimensional vector $v = (v_1, v_2, \ldots, v_n)^T \in \Re^n$ (T denotes the transpose of the vector), $\|v\| = (\sum_{i=1}^n v_i^2)^{1/2}$, and for a matrix $A \in \Re^{n\times n}$, $\|A\| = \sup\{\|Ay\| \mid \|y\| = 1,\ y \in \Re^n\}$. Moreover, $E(\cdot)$ is represented as mathematical expectation.

In this paper, we consider the following recurrent neural network with deviating argument and stochastic disturbance:

$$\begin{cases} dy(t) = [-Ay(t) + Bf(y(t)) + Df(y(\beta(t)))]dt + \sigma y(t)d\omega(t), \quad t \geq t_0 \geq 0, \\ y(t_0) = y_0 \in \Re^n. \end{cases} \tag{1}$$

where $\beta(t)$ is called deviating argument which is a piecewise function defined as $\beta(t) = \eta_k$, if $t \in [\alpha_k, \alpha_{k+1})$, $k \in \mathbb{N}$, the real-value sequences $\{\alpha_k\}$, $\{\eta_k\}$, $k \in \mathbb{N}$ satisfy $\alpha_k < \alpha_{k+1}$, $\alpha_k \leq \eta_k \leq \alpha_{k+1}$ and $\alpha_{k+1} - \alpha_k \leq \alpha$ for arbitrary $k \in \mathbb{N}$ with $\alpha_k \to +\infty$ as $k \to +\infty$ and $\alpha > 0$. $y(t) = (y_1(t), \ldots, y_n(t))^T \in \Re^n$ is the state vector of the system (1). $A = \mathrm{diag}\{a_1, \ldots, a_n\} \in \Re^{n\times n}$ is the self-feedback connection weight matrix, $B \in \Re^{n\times n}$ is the connection matrix, $f(y(t)) = (f_1(y_1(t)), \ldots, f_n(y_n(t)))^T \in \Re^n$ represents an activation function. $D \in \Re^{n\times n}$ is the connection matrix related with the deviating function $\beta(t)$. σ denotes the intensity of the stochastic disturbance and $\omega(t)$ represents Brownian motion on a complete probability space (Ω, F, P) with a natural filtration $\{F_t\}_{t\geq t_0\geq 0}$ generated by $\{\omega(s) : t_0 \leq s \leq t\}$. It can be referred to [15] for the basic knowledge of stochastic differential equations.

Without the deviating argument and stochastic disturbance, then the model (1) degrades into the following classical recurrent neural networks:

$$\begin{cases} \dot{x}(t) = -Ax(t) + Bf(x(t)), \quad t \geq t_0 \geq 0, \\ x(t_0) = x_0 \in \Re^n. \end{cases} \tag{2}$$

Remark 1. *The concept of deviating argument was first introduced in [17] to describe some alternately advanced and retarded systems. Actually, if the system (1) is defined on the interval $[\alpha_k, \alpha_{k+1})$, $k \in \mathbb{N}$, if $\alpha_k \leq t < \eta_k$, then (1) is an advanced system, and if $\eta_k < t < \alpha_{k+1}$, (1) is a retarded system.*

In order to state the main results, we need the following assumption:

(A1) There exist a nonnegative constant $k \in \Re$ such that for arbitrary $u, v \in \Re$

$$\|f(u) - f(v)\| \leq k\|u - v\|, \tag{3}$$

and $f(0) = 0$.

Under the assumption (A1), it is clear that $x = 0$ is a trivial state of system (2) and $y = 0$ is also a trivial state of system (1). Moreover, the system (2) has a unique state $x(t; t_0, x_0)$ for any t_0 and x_0, and (1) also has a unique solution $y(t; t_0, y_0)$ satisfying initial conditions [16].

The exponential stability of system (1) is defined as follows.

Definition 1. *([16]) The state vector $x(t; t_0, y_0)$ in (2) is globally exponentially stable, if for arbitrary $t_0 \in \Re^+$ and $x_0 \in \Re^+$, there exist $u, v \in \Re^+$, such that the inequality*

$$\|x(t; t_0, x_0)\| \leq v\|x_0\|\exp\{-u(t - t_0)\}$$

holds for all $t \geq t_0 \geq 0$.

For stochastic differential equations, various definitions of exponential stability are proposed, two of the most important ones are given in the following.

Definition 2. *([16]) The state vector $y(t; t_0, y_0)$ in (1) is almost surely exponentially stable if for any $t_0 \in \Re^+$ and $y_0 \in \Re^+$, there exist $u, v \in \Re^+$ such that*

$$\|y(t; t_0, y_0)\| \leq v\|y_0\|\exp\{-u(t - t_0)\}, t \geq t_0 \geq 0$$

holds almost surely for all $t \geq t_0 \geq 0$.

Definition 3. *([16]) The state vector $y(t; t_0, y_0)$ in (1) is mean square exponentially stable if for any $t_0 \in \Re^+$ and $y_0 \in \Re^+$, there exist $u, v \in \Re^+$ such that*

$$E\|y(t; t_0, y_0)\|^2 \leq v\|y_0\|^2\exp\{-u(t - t_0)\}, t \geq t_0 \geq 0$$

holds for all $t \geq t_0 \geq 0$.

It is shown that almost surely exponentially stable implies mean square exponentially stable, but the converse is not true [15]. However, under Assumption (A1), we have the following lemma.

Lemma 1. *([15]) If (A1) holds, then almost surely exponential stability could be derived from mean square exponential stability.*

Lemma 2. *([20]) For any $x_i > 0$, $0 < \lambda_i < 1$ and $\lambda_1 + \cdots + \lambda_n = 1$ we have*

$$(x_1 + \cdots + x_n)^2 \leq \frac{x_1^2}{\lambda_1} + \cdots + \frac{x_n^2}{\lambda_n}. \tag{4}$$

Next, to get the main result, we need another assumption:

$$\begin{aligned}(A2): \quad \bar{\gamma}(\alpha) = &\big(\frac{3\alpha^2}{\lambda_2}(\|A\|^2 + k^2\|B\|^2) + \frac{\alpha\sigma^2}{\lambda_3}\big)\big(\frac{1}{\lambda_1} + \frac{3\alpha}{\lambda_2}k^2\|D\|^2\big) \\ &\times \exp\{\frac{3\alpha^2}{\lambda_2}(\|A\|^2 + k^2\|B\|^2) + \frac{\alpha\sigma^2}{\lambda_3}\} + \frac{3}{\lambda_2}\alpha^2k^2\|D\|^2 < 1.\end{aligned}$$

Lemma 3. *If Assumptions (A1) and (A2) hold, then for any $t \in \Re^+$, the solution $y(t)$ of (1) satisfies*

$$E\|y(\beta(t))\|^2 \leq \lambda E\|y(t)\|^2 \tag{5}$$

where the coefficient

$$\lambda = \lambda_1^{-1}(1 - \bar{\gamma}(\alpha))^{-1}. \tag{6}$$

Proof. For any $t \geq t_0$, by the definition of $\beta(t)$, there exist sequences $\{\alpha_k\}$ and $\{\eta_k\}$ $(k \in \mathbb{N})$ such that

$$\beta(t) = \eta_k \in [\alpha_k, \alpha_{k+1}), \;\; t \in [\alpha_k, \alpha_{k+1}).$$

For $t \geq \eta_k$, from (1) we have

$$y(t) = y(\eta_k) + \int_{\eta_k}^{t} \Big[- Ay(s) + Bf(y(s)) + Df(y(\eta_k))\Big] ds + \int_{\eta_k}^{t} \sigma y(s) d\omega(s). \tag{7}$$

By applying Lemma 2 in the case $n = 3$, Assumption (A1), the properties of mathematical expectation $E(\cdot)$ and Cauchy-Schwarz inequality we have

$$\begin{aligned}E\|y(t)\|^2 &= E\|y(\eta_k) + \int_{\eta_k}^{t} \big[- Ay(s) + Bf(y(s)) + Df(y(\eta_k))\big] ds + \int_{\eta_k}^{t} \sigma y(t) d\omega(t)\|^2 \\ &\leq \frac{1}{\lambda_1}E\|y(\eta_k)\|^2 + \frac{1}{\lambda_2}E\|\int_{\eta_k}^{t} \big[- Ay(s) + Bf(y(s)) + Df(y(\eta_k))\big] ds\|^2 \\ &+ \frac{1}{\lambda_3}E\|\int_{\eta_k}^{t} \sigma y(t) d\omega(t)\|^2 \\ &\leq \frac{1}{\lambda_1}E\|y(\eta_k)\|^2 + \frac{1}{\lambda_2}E\|\int_{\eta_k}^{t} 1 \times \big[- Ay(s) + Bf(y(s)) + Df(y(\eta_k))\big] ds\|^2\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_3}E\|\int_{\eta_k}^{t}\sigma y(t)d\omega(t)\|^2\\
&\le \frac{1}{\lambda_1}E\|y(\eta_k)\|^2+\frac{1}{\lambda_2}E\|\int_{\eta_k}^{t}1ds\times\int_{\eta_k}^{t}\big[-Ay(s)+Bf(y(s))+Df(y(\eta_k))\big]ds\|^2\\
&+\frac{1}{\lambda_3}E\|\int_{\eta_k}^{t}\sigma y(t)d\omega(t)\|^2\\
&\le \frac{1}{\lambda_1}E\|y(\eta_k)\|^2+\frac{3\alpha}{\lambda_2}\big[E\int_{\eta_k}^{t}\big(\|A\|^2+k^2\|B\|^2\big)\|y(s)\|^2ds\\
&+E\int_{\eta_k}^{t}k^2\|D\|^2\|y(\eta_k)\|^2ds\big]+\frac{1}{\lambda_3}E\int_{\eta_k}^{t}\sigma^2\|y(s)\|^2ds\\
&\le (\frac{1}{\lambda_1}+\frac{3\alpha}{\lambda_2}k^2\|D\|^2)E\|y(\eta_k)\|^2+[\frac{3\alpha}{\lambda_2}(\|A\|^2+k^2\|B\|^2)+\frac{\sigma^2}{\lambda_3}]\int_{\eta_k}^{t}E\|y(s)\|^2ds.
\end{aligned}
\tag{8}
$$

Utilizing the Gronwall-Bellman inequality to (8), we have

$$
E\|y(t)\|^2\le(\frac{1}{\lambda_1}+\frac{3\alpha}{\lambda_2}k^2\|D\|^2)E\|y(\eta_k)\|^2\exp\{\alpha[\frac{3\alpha}{\lambda_2}(\|A\|^2+k^2\|B\|^2)+\frac{\sigma^2}{\lambda_3}]\}. \tag{9}
$$

By exchanging the position of $y(t)$ and $y(\eta_k)$ in (7) and similar deduction as above, we have

$$
\begin{aligned}
E\|y(\eta_k)\|^2 &\le \frac{1}{\lambda_1}E\|y(t)\|^2+\frac{3\alpha}{\lambda_2}k^2\|D\|^2E\|y(\eta_k)\|^2\\
&+[\frac{3\alpha}{\lambda_2}(\|A\|^2+k^2\|B\|^2)+\frac{\sigma^2}{\lambda_3}]\int_{\eta_k}^{t}E\|y(s)\|^2ds.
\end{aligned}
\tag{10}
$$

Substituting (9) into (10) we have

$$
\begin{aligned}
E\|y(\eta_k)\|^2 &\le \frac{1}{\lambda_1}E\|y(t)\|^2+\frac{3\alpha}{\lambda_2}k^2\|D\|^2E\|y(\eta_k)\|^2\\
&+\alpha[\frac{3\alpha}{\lambda_2}(\|A\|^2+k^2\|B\|^2)+\frac{\sigma^2}{\lambda_3}](\frac{1}{\lambda_1}+\frac{3\alpha}{\lambda_2}k^2\|D\|^2)\\
&\times\exp\{\alpha[\frac{3\alpha}{\lambda_2}(\|A\|^2+k^2\|B\|^2)+\frac{\sigma^2}{\lambda_3}]\}E\|y(\eta_k)\|^2\\
&=\frac{1}{\lambda_1}E\|y(t)\|^2+\bar{\gamma}(\alpha)E\|y(\eta_k)\|^2.
\end{aligned}
\tag{11}
$$

By using Assumption (A2), it follows that

$$
E\|y(\eta_k)\|^2\le\frac{1}{\lambda_1}(1-\bar{\gamma}(\alpha))^{-1}E\|y(t)\|^2=\lambda E\|y(t)\|^2.
$$

For $t<\eta_k$, we can get the same result as above inequality. So, (5) holds for any $t\in[\alpha_k,\alpha_{k+1})$ and $k\in\mathbb{N}$. The proof is finished.

Remark 2. *The inequality (12) in Lemma 3 tells the relationship between the norm $\|y(\xi_i)\|$ and $\|y(t)\|$. It brings convenience for the study of the recurrent neural network system (1).*

3. Main Results

In this section, we will quantitatively analyze the influence of the deviation function and random disturbance on the global exponential stability of recurrent neural network system (1).

Theorem 1. *Let Assumptions (A1) and (A2) hold, and assume that (2) is globally exponentially stable. Then (1) is mean square globally exponentially stable, which implies (1) is almost surely exponentially stable, if $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$ and $\alpha < \min(\frac{\rho}{2}, \bar{\alpha})$, where $\bar{\sigma}$ is the unique solution $\hat{\sigma}$ in the following transcendental equation*

$$2v\exp\{-u\rho\} + \frac{8\rho v}{u}\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\frac{1}{\lambda_1}) + \frac{\hat{\sigma}^2}{\bar{\lambda}_4}\Big)\exp\{2\rho c_0\} = 1 \tag{12}$$

and the interval length $\bar{\alpha}$ is a unique solution $\hat{\alpha}$ respect to the equation (13)

$$2v\exp\Big\{-u(\rho-\hat{\alpha})\Big\} + \frac{8\rho v}{u}\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\hat{\sigma}^2}{\bar{\lambda}_4}\Big)\exp\{2\rho c_1\} = 1 \tag{13}$$

where

$$c_0 = 2\rho\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}\Big) + 2\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\frac{1}{\lambda_1}) + \frac{\hat{\sigma}^2}{\bar{\lambda}_4}\Big)$$
$$c_1 = 2\rho\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}\Big) + 2\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\hat{\sigma}^2}{\bar{\lambda}_4}\Big)$$

and $\lambda = \lambda_1^{-1}(1-\hat{\gamma})^{-1}$, $\hat{\gamma} = \bar{\gamma}(\hat{\alpha})$, $\rho > \frac{\ln(v)}{u} > 0$.

Proof. Denote by $x(t;t_0,x_0) \equiv x(t)$ and $y(t;t_0,x_0) \equiv y(t)$ the state of (2) and (1) respectively. For $t_0 \le t \le t_0 + 2\rho$, from (1), we have

$$\begin{aligned} E\|y(t)-x(t)\|^2 = E\|\int_{t_0}^{t} & -A(y(s)-x(s)) + B(f(y(s)) - f(x(s))) \\ & + D[f(y(\beta(t))) - f(x(s))]ds + \int_{t_0}^{t}\sigma y(s)d\omega(s)\|^2. \end{aligned} \tag{14}$$

Let $n = 4$ in Lemma 2, we get

$$(x_1+x_2+x_3+x_4)^2 \le \frac{x_1^2}{\bar{\lambda}_1} + \frac{x_2^2}{\bar{\lambda}_2} + \frac{x_3^2}{\bar{\lambda}_3} + \frac{x_4^2}{\bar{\lambda}_4},$$

where $\bar{\lambda}_i \in (0,1)$ and $\sum_{i=1}^{4} \bar{\lambda}_i = 1$. Applying the above inequality, Cauchy-Schwarz inequality and Assumption (A1) on (14) and together with the properties of $E(\cdot)$, we have

$$
\begin{aligned}
E\|y(t)-x(t)\|^2 \le{} & \frac{1}{\bar{\lambda}_1} E\Big\| \int_{t_0}^{t} -A(y(s)-x(s))ds\Big\|^2 + \frac{1}{\bar{\lambda}_2} E\Big\| \int_{t_0}^{t} B(f(y(s))-f(x(s)))ds\Big\|^2 \\
& + \frac{1}{\bar{\lambda}_3} E\Big\| \int_{t_0}^{t} D[f(y(\beta(s)))-f(x(s))]ds\Big\|^2 + \frac{1}{\bar{\lambda}_4} E\Big\| \int_{t_0}^{t} \sigma y(s) d\omega(s)\Big\|^2 \\
\le{} & 2\rho\bigg[\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2}\Big) \int_{t_0}^{t} E\|y(s)-x(s)\|^2 ds \\
& + \frac{k^2\|D\|^2}{\bar{\lambda}_3} \int_{t_0}^{t} E\|y(\beta(s))-x(s)\|^2 ds\bigg] + \frac{\sigma^2}{\bar{\lambda}_4} \int_{t_0}^{t} E\|y(s)\|^2 ds \\
\le{} & 2\rho\bigg[\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}\Big) \int_{t_0}^{t} E\|y(s)-x(s)\|^2 ds \\
& + \frac{4k^2\|D\|^2}{\bar{\lambda}_3} \int_{t_0}^{t} E\|y(\beta(s))\|^2 ds\bigg] + \Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3} + \frac{\sigma^2}{\bar{\lambda}_4}\Big) \int_{t_0}^{t} E\|y(s)\|^2 ds
\end{aligned}
\tag{15}
$$

From the inequality (5) in Lemma 3, we have

$$E\|y(\beta(t))\|^2 \le \lambda E\|y(t)\|^2,$$

where λ is given by (6). Substituting above inequality into (15), we have

$$
\begin{aligned}
E\|y(t)-x(t)\|^2 \le{} & 2\rho\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}\Big) \int_{t_0}^{t} E\|y(s)-x(s)\|^2 ds \\
& + \Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\sigma^2}{\bar{\lambda}_4}\Big) \int_{t_0}^{t} E\|y(s)\|^2 ds \\
\le{} & \bigg[2\rho(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}) \\
& + 2\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\sigma^2}{\bar{\lambda}_4}\Big)\bigg] \int_{t_0}^{t} E\|y(s)-x(s)\|^2 ds \\
& + \frac{4\rho v}{u}\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\sigma^2}{\bar{\lambda}_4}\Big)\|y_0\|^2
\end{aligned}
\tag{16}
$$

When $t_0 + \alpha \le t \le t_0 + 2\rho$, it follows from (16) that

$$E\|y(t)-x(t)\|^2 \le c_2 \int_{t_0}^{t} E\|y(s)-x(s)\|^2 ds + c_3\|y_0\|^2 \tag{17}$$

where

$$c_2 = 2\rho\Big(\frac{\|A\|^2}{\bar{\lambda}_1} + \frac{k^2\|B\|^2}{\bar{\lambda}_2} + \frac{2k^2\|D\|^2}{\bar{\lambda}_3}\Big) + 2\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\sigma^2}{\bar{\lambda}_4}\Big)$$

$$c_3 = \frac{4\rho v}{u}\Big(\frac{8\rho k^2\|D\|^2}{\bar{\lambda}_3}(1+\lambda) + \frac{\sigma^2}{\bar{\lambda}_4}\Big)$$

Utilized the well-known Gronwall Inequality to (17), for $t_0 + \alpha \leq t \leq t_0 + 2\rho$,

$$E\|y(t) - x(t)\|^2 \leq c_3 \exp\{2\rho c_2\}\|y_0\|^2. \tag{18}$$

Consequently, for $t_0 + \alpha \leq t \leq t_0 + 2\rho$,

$$\begin{aligned} E\|y(t)\|^2 &\leq 2E\|y(t) - x(t)\|^2 + 2E\|x(t)\|^2 \\ &\leq 2c_3\exp\{2\rho c_2\}\|y_0\|^2 + 2u\|y_0\|^2\exp\{-v(t-t_0)\}. \end{aligned} \tag{19}$$

Hence, for $t_0 + \rho - \alpha \leq t \leq t_0 + 2\rho - \alpha$,

$$\begin{aligned} E\|y(t)\|^2 &\leq \big[2c_3\exp\{2\rho c_2\} + 2u\exp\{-v(\rho-\alpha)\}\big]\|y_0\|^2 \\ &= \hat{c}\|y_0\|^2 \end{aligned} \tag{20}$$

where $\hat{c} = 2c_3\exp\{2\rho c_2\} + 2u\exp\{-v(\rho - \alpha)\}$.

From (13), combining the monotonicity of the function, we know that when $\alpha < \bar{\alpha}$, we have $\hat{c} < 1$. Therefore, when $t_0 - \alpha + \rho \leq t \leq t_0 - \alpha + 2\rho$, we discuss the existence of parameter $\bar{\lambda}_1$ as follows

$$\frac{\partial \ln \bar{c}}{\partial \bar{\lambda}_1} = \frac{\partial \hat{c}}{\partial \bar{\lambda}_1} = 0$$

where $\bar{c} = \hat{c} - 2u\exp\{-v(\rho - \alpha)\}$

$$\frac{\partial \ln \bar{c}}{\partial \bar{\lambda}_1} = \frac{\partial \ln c_3}{\partial \bar{\lambda}_1} + 2\rho\frac{\partial c_2}{\partial \bar{\lambda}_1} = \frac{\partial \ln c_3}{c_3 \partial \bar{\lambda}_1} + 2\rho\frac{\partial c_2}{\partial \bar{\lambda}_1} \tag{21}$$

Further more, from (21) we obtain

$$\begin{aligned} &(a_2\sigma^2 + a_1 - a_3)\bar{\lambda}_1^3 + (2\sigma^4\bar{\lambda}_3 - k_2\bar{\lambda}_3\sigma^2 - k_1a_2\sigma^2 - k_1a_3 - 3k_1a_1)\bar{\lambda}_1^2 \\ &\quad + (2k_2k_1\sigma^2 + 3k_1^2)\bar{\lambda}_1 - (k_2k_1^2\sigma^2 + a_1k_1^3) = 0 \end{aligned} \tag{22}$$

where $a_1 = 16\rho^2k^2\|A\|^2\|D\|^2$, $a_2 = -\frac{4\rho\sigma^2 v\bar{\lambda}_3}{u}$, $a_3 = \frac{64\rho^3k^2|D\|^2(1+\lambda)v\bar{\lambda}_3}{u}$, $k_1 = (1 - \bar{\lambda}_2 - \bar{\lambda}_3)$, $k_2 = 2\rho\bar{\lambda}_3\|A\|^2$

Therefore, according to (22), there exists a real solution $\bar{\lambda}_1$. For $\bar{\lambda}_2$, $\bar{\lambda}_3$ and $\bar{\lambda}_4$ can be similarly discussed as (22), substituting $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\lambda}_3$ and $\bar{\lambda}_4$ into $\hat{c}$, we know that $\hat{c}$ is a strictly monotone function. So, equation (13) exists a unique $\hat{\alpha}$ such that $\hat{\alpha} = \bar{\alpha}$, for $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3 \in (0,1)$.

On the basis of (12) and (13), we observe that $\hat{c} < 1$, when $|\sigma| < \bar{\sigma}$, $\alpha < \min(\frac{\rho}{2}, \bar{\alpha})$. Setting $\tau = -\frac{\ln(\hat{c})}{\rho}$, we have

$$E\|y(t)\|^2 \leq \exp\{-\rho\tau\}\|y_0\|^2 \tag{23}$$

Combined with the flow and the uniqueness of solution in RNN (1), for any positive integer m.

$$y(t;t_0,y_0) = y(t;t_0+(m-1)\rho, y(t_0+(m-1)\rho;t_0,y_0)) \tag{24}$$

Thus, considering (23) and (24), for $t \geq t_0 - \alpha + m\rho$,

$$\begin{aligned}\|y(t;t_0,y_0)\| &= \|y(t;t_0+(m-1)\rho, y(t_0+(m-1)\rho;t_0,y_0))\| \\ &\leq \exp(-\rho\tau)\|y(t_0+(m-1)\rho;t_0,y_0)\| \\ &\leq \exp(-m\rho\tau)\|y_0\|\end{aligned}$$

Consequently, when $t > t_0-\alpha+\rho$, the positive integer m that satisfies $t_0-\alpha+(m-1)\rho \leq t \leq t_0-\alpha+m\rho$,

$$\|y(t;t_0,y_0)\|^2 \leq \exp(-\tau(t-t_0))\exp(\tau(\rho-\alpha))\|y_0\|^2 \tag{25}$$

Obviously, (25) also holds for $t_0 \leq t \leq t_0-\alpha+\rho$. Therefore, RNN (1) is mean square exponentially stable. From Lemma 2.3, the almost surely exponential stability of system (1) can be fully proved.

4. Illustrative Numerical Examples

In this section, an example is provided to illustrate the results.

Example 1. *Given the two-dimensional original system*

$$\begin{cases}\dot{x_1}(t) = -x_1(t) - 2f(x_1(t)) + 2f(x_2(t)), \\ \dot{x_2}(t) = -x_2(t) + 2f(x_1(t)) - 2f(x_2(t)).\end{cases} \tag{26}$$

Under the vector form in (1), the matrix can be written as

$$A = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad B = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}, \quad D = \begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}.$$

If we choose $f(x_j) = \sin^2(x_j)$, $j = 1,2$. Then, by Theorem 1 in [26], the recurrent neural network (26) is globally exponentially stable. The system (26) with deviating argument and stochastic disturbance is modeled by

$$\begin{cases}\dot{y_1}(t) = -y_1(t) - f(y_1(t)) + f(y_2(t)) - f(y_1(\beta(t))) + f(y_2(\beta(t))) + \sigma y_1(t)d\omega(t), \\ \dot{y_2}(t) = -y_2(t) + f(y_1(t)) - f(y_2(t)) + f(y_1(\beta(t))) - f(y_2(\beta(t))) + \sigma y_1(t)d\omega(t),\end{cases} \tag{27}$$

where $\{\alpha_k\} = \{\frac{k}{4}\}$, $\{\eta_k\} = \{\frac{2k+1}{8}\}$, $k \in N$, $t \in [\alpha_k, \alpha_{k+1})$, $t \in \Re^+$. The deviating function $\beta(t) = \eta_k$, σ is the interference intensity, and $\omega(t)$ is a Brownian motion.

Let $\rho = 1 \geq \frac{\ln(1.2)}{0.9} = 0.2026$, $k = 0.01$, $\lambda_1 = \frac{1}{3}$, $\bar{\lambda}_i = \frac{1}{4}$, substituting them into (12) and (13), then we get $\bar{\sigma} = 0.0468$. From $|\sigma| < \frac{\bar{\sigma}}{\sqrt{2}}$, we know that $|\sigma| < 0.0325$. In reference

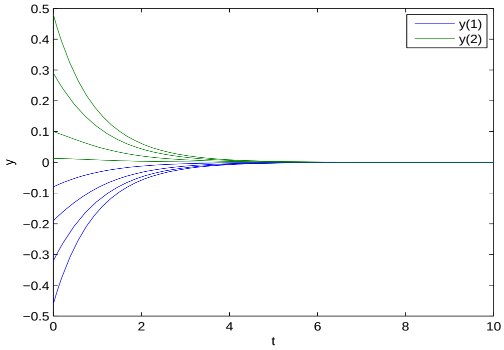

Figure 1: Stability behavior of system (26)

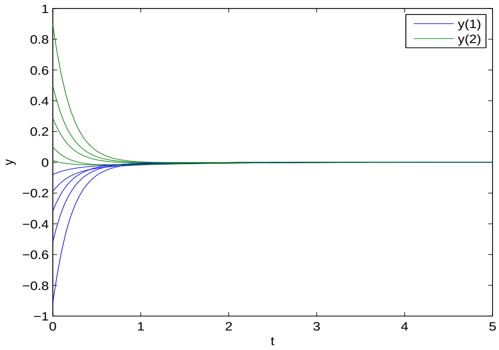

Figure 2: Stability behavior of RNN (27)

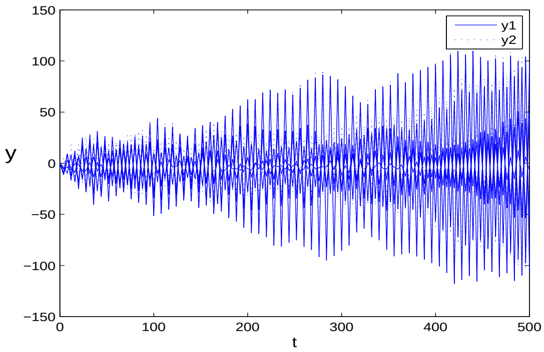

Figure 3: Instability behavior of RNN (27)

[26], the upper bound of the $|\sigma|$ is 0.0265, which is smaller than 0.0325. This means that the system (27) can withstand higher intensity random disturbance than the system in [26]. Moreover, by substituting those parameters in (13), we have $\bar{\alpha} = 25.3565$. Note that $\alpha < \min(\frac{\varrho}{2}, \bar{\alpha})$, then $\alpha < 0.1013$. In the Example 2 of [26], $\alpha < 0.0159$, which is smaller than 0.1013. This shows that the system (27) has wider range of piecewise argument and implies that the system (27) can withstand higher intensity of impact from time delay or advance.

Figure 2 describes the stability performance of (27) with $\sigma = 0.04$, $\{\alpha_k\} = \{\frac{k}{100}\}$, $\{\eta_k\} = \{\frac{2k+1}{200}\}$, $k \in \mathbb{N}$. Figure 3 illustrates a degenerative performance of RNN (27) for $\sigma = 1$, $\{\alpha_k\} = \{\frac{k}{100}\}$, $\{\eta_k\} = \{\frac{2k+1}{200}\}$, $k \in \mathbb{N}$. Certainly, in this respect, these parameters of the conditions in Theorem (1) were no longer effective, the system will involve into unstable.

5. Conclusion

This paper investigates global robust exponential stability of recurrent neural networks with piecewise argument and arbitrary disturbance. The upper bound of perturbation intensity is estimated by using inequalities and transcendental equations with restricted parameters. The theoretical results of this paper provide a reliable basis for the application and design of RNNs. The authors would like to search for larger upper bounds of deviating variables and improve the interference intensity such that the system remains stable. The treatment methods in this paper can provide references for more flexible control systems.

Future work will extend the results to the multi-stability of bidirectional associative memory neural networks or fuzzy neural networks in the presence of deviating arguments and random disturbances. The main problem is the construction of the relationship between piecewise deviating variables and system solution vectors in neural networks.

References

[1] M Syed Ali, G Narayanan, V Shekher, H Alsulami, and T Saeed. Dynamic stability analysis of stochastic fractional-order memristor fuzzy bam neural networks with delay and leakage terms. *Applied Mathematics and Computation*, 369:124896, 2020.

[2] C Aouiti and E Abed Assali. Stability analysis for a class of impulsive bidirectional associative memory (bam) neural networks with distributed delays and leakage time-varying delays. *Neural Processing Letters*, 50(1):851–885, 2019.

[3] J Cao and J Wang. Global asymptotic stability of a general class of recurrent neural networks with time-varying delays. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 50(1):34–44, 2003.

[4] T Chen. Global exponential stability of delayed hopfield neural networks. *Neural Networks*, 14(8):977–980, 2001.

[5] M A Cohen and S Grossberg. Absolute stability of global pattern formation and parallel memory storage by competitive neural networks. *IEEE transactions on systems, man, and cybernetics*, (5):815–826, 1983.

[6] B Guo, Z Wu, and H Zhou. Active disturbance rejection control approach to output-feedback stabilization of a class of uncertain nonlinear systems subject to stochastic disturbance. *IEEE Transactions on Automatic Control*, 61(6):1613–1618, 2015.

[7] C Huang and J Cao. Impact of leakage delay on bifurcation in high-order fractional bam neural networks. *Neural Networks*, 98:223–235, 2018.

[8] T Huang, C Li, S Duan, and Janusz A Starzyk. Robust exponential stability of uncertain delayed neural networks with stochastic perturbation and impulse effects. *IEEE Transactions on Neural Networks and Learning Systems*, 23(6):866–875, 2012.

[9] V Hien Le and L Dao Hai. Exponential stability of positive neural networks in bidirectional associative memory model with delays. *Mathematical methods in the applied sciences*, pages 1–19, 2019.

[10] Z Li, T Li, and G Feng. Adaptive neural control for a class of stochastic nonlinear time-delay systems with unknown dead zone using dynamic surface technique. *International Journal of Robust and Nonlinear Control*, 26(4):759–781, 2016.

[11] X Liao. *Theory and application of stability for dynamical systems.* National Defense Industry Press, Beijingm, China, 2000.

[12] C Liu, X Wang, and Y Xue. Global exponential stability analysis of discrete-time genetic regulatory networks with time-varying discrete delays and unbounded distributed delays. *Neurocomputing*, 372:100–108, 2020.

[13] L Liu, A Wu, Z Zeng, and T Huang. Global mean square exponential stability of stochastic neural networks with retarded and advanced argument. *Neurocomputing*, 247:156–164, 2017.

[14] L Liu, Z Yu, J Yu, and Q Zhou. Global output feedback stabilisation for a class of stochastic feedforward non-linear systems with state time delay. *IET Control Theory & Applications*, 9(6):963–971, 2014.

[15] X Mao. Stability and stabilisation of stochastic differential delay equations. *IET Control Theory and Applications*, 1(6):1551–1566, 2007.

[16] X Mao. *Stochastic differential equations and applications.* Elsevier, 2007.

[17] E. Yılmaz M.U. Akhmet, D. Aruğaslan. Stability analysis of recurrent neural networks with piecewise constant argument of generalized type. *Neural Networks*, 23(7):805–811, 2010.

[18] Y Shen and J Wang. Robustness analysis of global exponential stability of non-linear systems with time delays and neutral terms. *IET Control Theory & Applications*, 7(9):1227–1232, 2013.

[19] Y Shen and J Wang. Robustness of global exponential stability of nonlinear systems with random disturbances and time delays. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 46(9):1157–1166, 2015.

[20] J Michael Steele. *The Cauchy-Schwarz master class: an introduction to the art of mathematical inequalities.* Cambridge University Press, 2004.

[21] X Wei, Z Wu, and H Karimi. Disturbance observer-based disturbance attenuation control for a class of stochastic systems. *Automatica*, 63:21–25, 2016.

[22] A Wu, L Liu, T Huang, and Z Zeng. Mittag-leffler stability of fractional-order neural networks in the presence of generalized piecewise constant arguments. *Neural Networks*, 85:118–127, 2017.

[23] A Wu and Z Zeng. Output convergence of fuzzy neurodynamic system with piecewise constant argument of generalized type and time-varying input. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 46(12):1689–1702, 2016.

[24] G Xu and H Bao. Further results on mean-square exponential input-to-state stability of time-varying delayed bam neural networks with markovian switching. *Neurocomputing*, 376:191–201, 2020.

[25] R Yan, X He, and D Zhou. Detecting intermittent sensor faults for linear stochastic systems subject to unknown disturbance. *Journal of the Franklin Institute*, 353(17):4734–4753, 2016.

[26] J Zhang. Robustness analysis of global exponential stability of nonlinear systems with deviating argument and stochastic disturbance. *IEEE Access*, 5:13446–13454, 2017.

[27] J Zhang, K Yang, R Qi, S Zhao, and Y Li. Robustness analysis method for orbit control. *Acta Astronautica*, 137:15–24, 2017.