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On the Independent Neighborhood Polynomial of the Cartesian Product of Some Special Graphs

Normalah Abdulcarim[1,*], Susan Dagondon[2], Emmy Chacon[2]

[1] *Department of Mathematics, College of Natural Sciences and Mathematics, Mindanao State University Main Campus, 9700 Marawi City, Philippines*
[2] *Department of Mathematics and Statistics, College of Science and Mathematics, Center of Graph Theory, Algebra, and Analysis-Premier Research Institute of Science and Mathematics, Mindanao State University-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. Two vertices x, y of a graph G are adjacent, or neighbors, if xy is an edge of G. A set S of vertices in a graph G is a neighborhood set if $G = \bigcup_{v\in S} \langle N[v]\rangle$ where $\langle N[v]\rangle$ is the subgraph induced by v and all the vertices adjacent to v. If no two of the elements of S are adjacent, then S is called an independent neighborhood set. The independent neighborhood polynomial of G of order m is $N_i(G,x) = \sum_{j=\eta_i(G)}^{m} n_i(G,j)x^j$ where $n_i(G,j)$ is the number of independent neighborhood set of G of size j and $\eta_i(G)$ is the minimum cardinality of an independent neighborhood set of G. This paper investigates the independent neighborhood polynomial of the Cartesian product of some special graphs.

2020 Mathematics Subject Classifications: 05C31, 05C69, 05C76

Key Words and Phrases: Independent Neighborhood Set, Neighborhood Polynomial, Cartesian Product

1. Introduction

The history of graph theory may be specifically traced to 1735 when the Swiss Mathematician Leonhard Euler solve the königberg bridge problem. There are number of applications of graph theory that have been widely studied. A graph polynomial is one of the algebraic reperesentations for graph. In this paper, we study a new type of graph polynomial called the independent neighborhood polynomial [10]. Throughout this paper, we consider only a finite, simple, undirected graphs without loops and multiple edges.

*Corresponding author.
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Email addresses: n_abdulcarim@yahoo.com (N. Abdulcarim), susan.dagondon@g.msuiit.edu.ph (S. Dagondon), emmy.chacon@g.msuiit.edu.ph (E. Chacon)

A *graph* G is a pair $(V(G), E(G))$ consisting of a nonempty finite set of vertices $V(G)$ and a set of edges $E(G)$ of unordered pairs of elements of $V(G)$. The cardinalities of $V(G)$ and $E(G)$ are called the *order* and *size* of G, respectively. We write $x = uv$ and say that u and v are *adjacent* vertices; vertex u and edge x are *incident* with each other, so are v and x. The two vertices incident with an edge are its *endvertices* or *ends*, and an edge joins its ends. Two vertices of a graph G are said to be *neighbors* if they are adjacent in G.

The *neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{w : w \in V \text{ and } vw \in E(G)\}$. A vertex v is *pendant* if its neighborhood contains only one vertex; and edge $e = uv$ is pendant if one of its endvertices is a pendant vertex.

A graph H is called a *subgraph* of G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G. A subgraph F of a graph G is called an *induced subgraph* of G, denoted by $\langle F\rangle$, if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F as well.

A *path* is a nonempty graph $P = (V, E)$ of the form

$$V = \{v_1, \cdots, v_m\} \quad E = \{v_1v_2, v_2v_3, \cdots, v_{m-1}v_m\},$$

where the v_i are all distinct.

In this research, we focused to determine the independent neighborhood sets of the Cartesian product of some special graphs with path and represent them in a graph polynomial called independent neighborhood polynomial. The readers may also read on the following references: [1],[2],[3], [5],[9],[11] and [6].

2. Preliminaries

Definition 1. [7] A graph G is a *bipartite graph*, denoted by $K_{m,n}$, if $V(G)$ can be partitioned into two subsets V_m and V_n of order m and n, respectively, called partite sets such that every edge of G joins a vertex of V_n and a vertex of V_m. If G contains every edges joining V_n and V_m, then G is called *complete bipartite graph.* A *star* is complete bipartite $K_{1,n}$, the vertex in the singleton partition class is called the *apex vertex.* A star graph $K_{1,n-1}$ is also called an n-star graph.

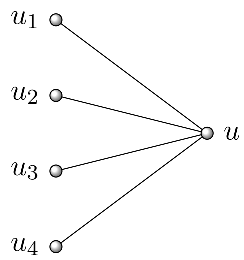

Figure 1: A star graph $K_{1,4}$ with apex vertex u and pendant vertices u_1, u_2, u_3, u_4

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Definition 2. [6] The *bistar graph* $B(m,n)$ is constructed by joining the apex vertices of two stars $K_{1,m}$ and $K_{1,n}$ for $m \geq 1$ and $n \geq 1$ with disjoint vertex sets.

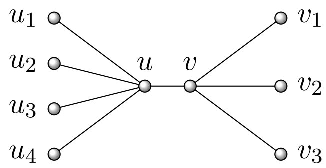


Figure 2: A bistar graph $B(4,3)$

Definition 3. [4] The *Banana tree graph* $B_{m,n}$ is the graph obtained by connecting one leaf of each m copies of an n-star graph with a single root vertex that is distinct for all the stars.

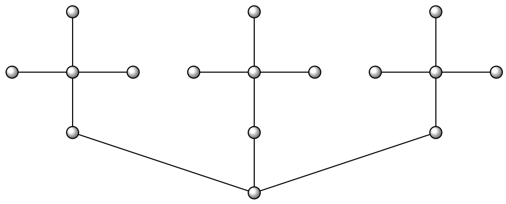


Figure 3: The Banana tree graph $B_{3,5}$

Definition 4. [4] The *Firecracker graph* $F_{m,n}$ is the graph obtained by the concatenation of mn-stars by linking one leaf from each.

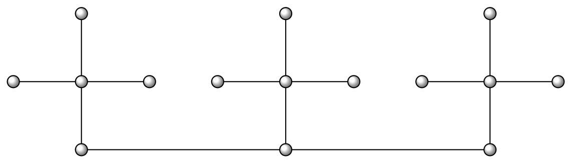


Figure 4: The Firecracker graph $F_{3,5}$

Definition 5. [8] The n-centipede graph or simply Cen_n is the tree on $2n$ vertices obtained by joining the bottoms of n copies of the path graph P_2 laid in a row with edges.

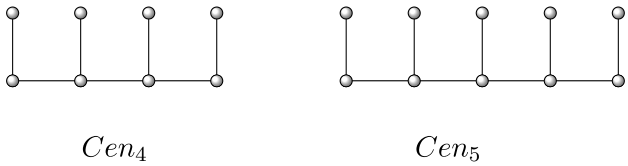


Figure 5: The Centipede graphs Cen_4 and Cen_5

Definition 6. [5] The *Cartesian product* of two graphs G and H, denoted $G\Box H$, is the graph where $V(G\Box H) = V(G)\times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G\Box H)$ if and only if either

(i.) $g_1 = g_2$ and $h_1h_2 \in E(H)$ or

(ii.) $h_1 = h_2$ and $g_1g_2 \in E(G)$.

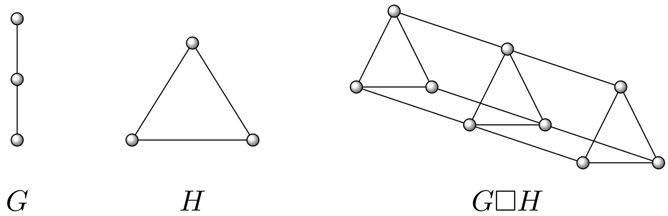


Figure 6: Cartesian product of G and H

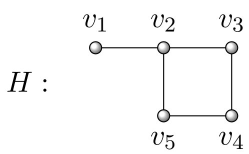
Definition 7. [11] A set $S \subseteq V(G)$ is an *independent neighborhood set* of G, if S is a neighborhood set and no two vertices in S are adjacent.

Definition 8. [11] Let $G = (V, E)$ be a graph with m vertices. Then the *independent neighborhood polynomial* of G of order m is

$$N_i(G, x) = \sum_{j=\eta_i(G)}^{m} n_i(G, j)x^j,$$

where $n_i(G, j)$ is the number of independent neighborhood set of G of size j and $\eta_i(G)$ is the minimum cardinality of an independent neighborhood set which is called the *independent neighborhood number* of G.

Example 1. Consider the graph H below



The only independent neighborhood sets of H are $\{v_2, v_4\}$ and $\{v_1, v_3, v_5\}$. Therefore, the independent neighborhood polynomial of H is $N_i(H, x) = x^2 + x^3$.

3. Independent Neighborhood Polynomial of the Cartesian product of Some Special Graphs with Path Graph

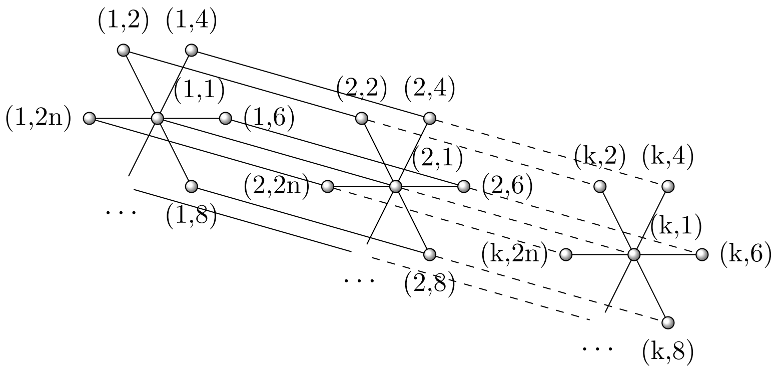
In this section, the independent neighborhood sets of the Cartesian product of some special graphs with path are determined and represented in an independent neighborhood polynomial.

Theorem 1. *For any path P_k and star $K_{1,n}$,*

$$N_i(P_k \square K_{1,n}, x) = \begin{cases} x^{\lfloor \frac{k}{2} \rfloor n + \lceil \frac{k}{2} \rceil} + x^{\lceil \frac{k}{2} \rceil n + \lfloor \frac{k}{2} \rfloor}, & k \text{ is odd} \\ 2x^{(\frac{k}{2})n + (\frac{k}{2})}, & k \text{ is even} \end{cases}$$

for any $k, n \in \mathbb{Z}^+$.

Proof: Label $V(P_k) = \{1, 2, \cdots, k\}$ and $V(K_{1,n}) = \begin{cases} 1, & \text{apex vertex} \\ 2n & \text{pendant vertices} \end{cases}$.



Then

$$V(P_k \square K_{1,n}) = \{(x, y) : x = 1, 2 \cdots, k, y = 1, 2, 4, 6, \cdots, 2n\}$$

and

$$E(P_k \square K_{1,n}) = \{(x, y)(w, z) : x = w \text{ and } yz \in E(K_{1,n}) \text{ or } xw \in E(P_k) \text{ and } y = z\}.$$

Observe that for any $(x, y)(w, z) \in E(P_k \square K_{1,n})$, we have the following cases:
case 1: $\{(x, y), (w, z) : x = w$ is odd while y is even and $z = 1\}$.
case 2: $\{(x, y), (w, z) : x = w$ is even while y is even and $z = 1\}$.
case 3: $\{(x, y), (w, z) : y = z = 1$ while x is even and w is odd$\}$.
case 4: $\{(x, y), (w, z) : y = z$ is even while x is even and w is odd$\}$.

Now, let $S = \{(p, q) \in V(P_k \square K_{1,n}) : p$ and q are both even or p and q are both odd$\}$ and $T = \{(r, s) \in V(P_k \square K_{1,n}) : r$ is odd and s is even or r is even and s is odd$\}$. We claim that S and T are the independent neighborhood sets of $P_k \square K_{1,n}$ that is, we show that

a. no two vertices in S are adjacent and $\bigcup_{u\in S} \langle N[u]\rangle = P_k\square K_{1,n}$; and

b. no two vertices in T are adjacent and $\bigcup_{v\in T} \langle N[v]\rangle = P_k\square K_{1,n}$.

a. Let $(x,y),(w,z)\in S$. We consider the following cases:

Case 1: If x,y,w,z are all odd, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. Thus, $(x,y),(w,z)$ are not adjacent.

Case 2: If x,y,w,z are all even, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. Thus, $(x,y),(w,z)$ are not adjacent.

Case 3: If x,y are even and w,z are odd, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. Thus, $(x,y),(w,z)$ are not adjacent.

Case 4: If x,y are odd and w,z are even, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. Thus, $(x,y),(w,z)$ are not adjacent.

Hence, in all above cases, none of the vertices of S are adjacent.

Next, we will show that $\bigcup_{u\in S} \langle N[u]\rangle = P_k\square K_{1,n}$. Assume to the contrary that $\bigcup_{u\in S} \langle N[u]\rangle \neq P_k\square K_{1,n}$. Then there exists $(x,y)(w,z)\in E(P_k\square K_{1,n})$ such that $(x,y)(w,z)\notin E\left(\bigcup_{u\in S} \langle N[u]\rangle\right)$, particularly, both (x,y) and (w,z) are not in S. It follows that both x and y are not odd or both x and y are not even. Similar case for w and z. Now, if x is odd, y is even, w is odd and z is even, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. This is a contradiction. If we consider x is odd, y is even, w is even and z is odd, then $(x,y)(w,z)\notin E(P_k\square K_{1,n})$. Similar case when x is even, y is odd, w is odd, z is even and for x is even, y is odd, w is even, z is odd. Hence, in either cases, $(x,y)(w,z)\notin E(P_k\square K_{1,n})$ which is a contradiction to the assumption. Therefore, $\bigcup_{u\in S} \langle N[u]\rangle = P_k\square K_{1,n}$. Consequently, S is an independent neighborhood set of $P_k\square K_{1,n}$. Following the same argument in (a) for (b), we can show that T is also an independent neighborhood set of $P_k\square K_{1,n}$.

Now, if we let $S_1=\{(x,y) : x \text{ and } y \text{ are odd}\}$, $S_2=\{(x,y) : x \text{ and } y \text{ are even}\}$, $T_1=\{(x,y) : x \text{ is odd and } y \text{ is even}\}$ and $T_2=\{(x,y) : x \text{ is even and } y \text{ is odd}\}$, then $S_1\cup S_2\cup T_1\cup T_2 = V(P_k\square K_{1,n})$ and that $S_1\cup S_2 = S$ and $T=T_1\cup T_2$. Notice that when k is odd,

$$|S_1| = \left\lceil \frac{k}{2}\right\rceil, |S_2| = \left\lfloor \frac{k}{2}\right\rfloor n, |T_1| = \left\lceil \frac{k}{2}\right\rceil n, |T_2| = \left\lfloor \frac{k}{2}\right\rfloor.$$

Hence, $|S| = \left\lfloor \frac{k}{2}\right\rfloor n + \left\lceil \frac{k}{2}\right\rceil$ and $|T| = \left\lceil \frac{k}{2}\right\rceil n + \left\lfloor \frac{k}{2}\right\rfloor$.

Thus, $N_i(P_k\square K_{1,n},x) = x^{\lfloor \frac{k}{2}\rfloor n+\lceil \frac{k}{2}\rceil} + x^{\lceil \frac{k}{2}\rceil n+\lfloor \frac{k}{2}\rfloor}$ when k is odd. For k is even, observe that $\left\lfloor \frac{k}{2}\right\rfloor = \left(\frac{k}{2}\right) = \left\lceil \frac{k}{2}\right\rceil$. It follows that $|S|=|T|$ and so, $N_i(P_k\square K_{1,n},x) = 2x^{\left(\frac{k}{2}\right)n+\left(\frac{k}{2}\right)}$.

Consequently,

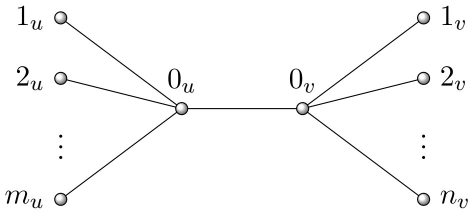
$$N_i(P_k\Box K_{1,n},x)=\begin{cases} x^{\lfloor\frac{k}{2}\rfloor n+\lceil\frac{k}{2}\rceil}+x^{\lceil\frac{k}{2}\rceil n+\lfloor\frac{k}{2}\rfloor}, & k \text{ is odd}\\ 2x^{(\frac{k}{2})n+(\frac{k}{2})}, & k \text{ is even.}\end{cases}$$

Theorem 2. *For any path P_k and Bistar graph $B(m,n)$,*

$$N_i(P_k\Box B_{m,n},x)=x^{\lceil\frac{k}{2}\rceil(m+1)+\lfloor\frac{k}{2}\rfloor(n+1)}+x^{\lfloor\frac{k}{2}\rfloor(m+1)+\lceil\frac{k}{2}\rceil(n+1)}$$

for any $k,m,n\in\mathbb{Z}^+$.

Proof: Label the vertices of $B(m,n)$ as $i_u,j_v,0_u,0_v,\ i=1,\cdots,m,\ j=1,\cdots,n$ where 0_u and 0_v are the apex vertices.



Then

$$V(P_k\Box B(m,n))=\{(r,i_a),(r,j_v),(r,0_u),(r,0_v):r=1,\cdots,k,\ i=1,\cdots,m,\ j=1,\cdots,n$$

and $E(P_k\Box B(m,n))=\{(w,x_a)(y,z_b): w=y, a=b$ and either $x=0$ or $z=0,\ w=y+1, a=b$ and $x=z,$ and $w=y, a=u, b=v$ and $x=0=z\}$.

Consider the following sets of vertices.

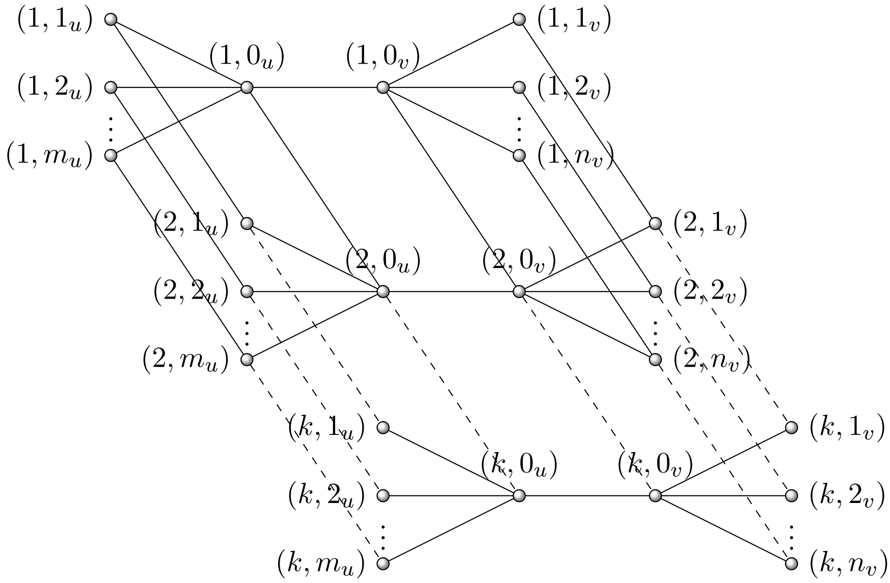
$$A_r=\{(r,i_u):r=1,\cdots,k,\ i=1\cdots,m\}\cup\{\{(r,0_v)\}$$
$$B_s=\{(s,j_v):s=1,\cdots,k,\ j=1\cdots,n\}\cup\{\{(s,0_u)\}.$$

Let

$$S=A_r\cup B_s \text{ such that } r \text{ is odd and } s \text{ is even and}$$
$$T=A_r\cup B_s \text{ such that } r \text{ is even and } s \text{ is odd.}$$

Then $S=\begin{cases}(r,i_u): & r \text{ is odd, } i=1,\cdots,m\\ (s,j_v): & s \text{ is even, } j=1,\cdots,n\\ (r,0_v): & r \text{ is odd}\\ (s,0_u): & s \text{ is even}\end{cases}$ and $T=\begin{cases}(r,i_u): & r \text{ is even, } i=1,\cdots,m\\ (s,j_v): & s \text{ is odd, } j=1,\cdots,n\\ (r,0_v): & r \text{ is even}\\ (s,0_u): & s \text{ is odd}\end{cases}$

We claim that S and T are the independent neighborhood sets of $P_k\Box B(m,n)$. First, we show that no two vertices in S are adjacent. Observe that for any $(r,i_u),(s,0_u)\in S$, $(r,i_u)(s,0_u)\notin E\left(P_k\Box B(m,n)\right)$ since r is odd in (r,ji_u) and s is even in $(s,0_u)$. Similarly,



$(s,j_v)(r,0_v) \notin E(P_k \square B(m,n))$ for any $(s,j_v),(r,0_v) \in S$. Hence, none of the vertices of S are adjacent.

Next, we show that $\bigcup\limits_{v\in S} \langle N[v] \rangle = P_k \square B(m,n)$. Assume to the contrary that $\bigcup\limits_{v\in S} \langle N[v] \rangle \neq P_k \square B(m,n)$. This implies there exists $(w,x_a)(y,z_b) \in E(P_k \square B(m,n))$ such that $(w,x_a)(y,z_b) \notin E\left(\bigcup\limits_{v\in S} \langle N[v]\rangle\right)$. Consider the following cases:

case I. $w=y, a=b$ and either $x=0$ or $y=0$
WLOG, let $z=0$.

i. If $(r_1,i_u)(r_2,0_u) \notin E\left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$, then both $(r_1,i_u),(r_2,0_u) \notin S$. It follows that r_2 is odd and so is r_1. But $(r_1,i_u) \in S$ for r_1 odd. This is a contradiction.

ii. If $(s_1,j_v)(s_2,0_v) \notin E\left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$, then both $(s_1,j_v),(s_2,0_v) \notin S$. It follows that s_2 is even and so is s_1 because $(s_1,j_v)(s_2,0_v) \in E(P_k \square B(m,n))$ when $s_1 = s_2$. But $(s_1,j_v) \in S$ for s_1 even which is a contradiction.

case II. $w=y+1, a=b$ and $x=z$
Assume that $x=z\neq 0$.

i. When $(r_1,i_{1u})(r_2,i_{2u}) \notin E\left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$, then both $(r_1,i_{1u}),(r_2,i_{2u}) \notin S$. This implies r_1 and r_2 are odd. But $(r_1,i_{1u})(r_2,i_{2u}) \notin E(P_k \square B(m,n))$ which is a contradiction.

ii. When $(s_1, j_{1u})(s_2, j_{2u}) \notin E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$, we will arrive contradiction similar to i.

Next, we assume $x = z = 0$.

iii. If $(r_1, 0_u)(r_2, 0_u) \notin E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$, then both $(r_1, 0_u), (r_2, 0_u) \notin S$. This implies r_1 is odd and r_2 is even. But $(r_2, 0_u) \in S$, a contradiction.

iv. If $(s_1, 0_v)(s_2, 0_v) \notin E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$, then both $(s_1, 0_v), (s_2, 0_v) \notin S$. Note that whenever s_1 is odd, s_2 is even. But $(s, 0_v) \in S$ for s even. This is a contradiction.

case III. $w = y, a = u, b = v$ and $x = 0 = z$

If $(r, 0_u)(s, 0_v) \notin E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$, then $(r, 0_u), (s, o_v) \notin S$. This implies r is odd. But $r = s$ and thus, s is also odd. This is a contradiction.

Hence, in either of the above cases, we arrived at a contradiction. Thus, $(w, x_a)(y, z_b) \in E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$. Consequently, $\bigcup_{v\in S}\langle N[v]\rangle = P_k\Box B(m, n)$. Hence, S is an independent neighborhood set of $P_k\Box B(m, n)$. Following the same argument in S, we can also show that T is an independent neighborhood set of $P_k\Box B(m, n)$

Now, observe that for each A_i and B_r, $|A_i| = m + 1$ and $|B_r| = |n + 1|$. Thus,

$$\begin{aligned}|S| &= \sum_{i \text{ is odd}} |A_i| + \sum_{r \text{ is even}} |B_r| \\ &= \left\lceil \frac{k}{2} \right\rceil (m+1) + \left\lfloor \frac{k}{2} \right\rfloor (n+1)\end{aligned}$$

and

$$\begin{aligned}|T| &= \sum_{i \text{ is even}} |A_i| + \sum_{r \text{ is odd}} |B_r| \\ &= \left\lfloor \frac{k}{2} \right\rfloor (m+1) + \left\lceil \frac{k}{2} \right\rceil (n+1).\end{aligned}$$

Therefore,

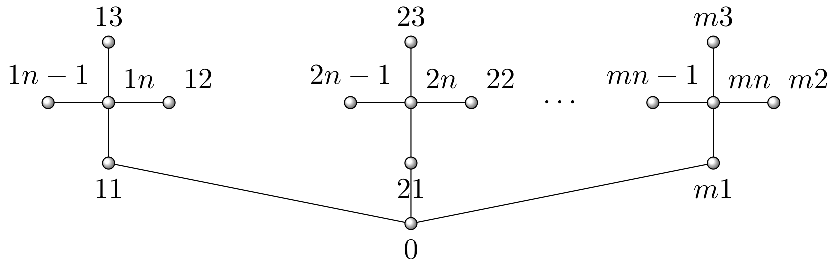
$$N_i(P_k\Box B(m, n), x) = x^{\lceil \frac{k}{2} \rceil (m+1) + \lfloor \frac{k}{2} \rfloor (n+1)} + x^{\lfloor \frac{k}{2} \rfloor (m+1) + \lceil \frac{k}{2} \rceil (n+1)}.$$

Theorem 3. *For any path* P_k *and Banana graph* $B_{m,n}$,

$$N_i(P_k \Box B_{m,n}, x) = x^{\lfloor \frac{k}{2} \rfloor m(n-1) + \lceil \frac{k}{2} \rceil (m+1)} + x^{\lceil \frac{k}{2} \rceil m(n-1) + \lfloor \frac{k}{2} \rfloor (m+1)}$$

for any $k, m, n \in \mathbb{Z}^+$.

Proof: Label the vertices of each star in $B_{m,n}$ by ij, $i = 1, \cdots, m$, $j = 1, \cdots, n$ and 0 as the root vertex in $B_{m,n}$.



Then $V(P_k \Box B_{m,n}) = \{(e, ij), (e, 0) : e = 1, \cdots, k, \ i = 1, \cdots, m, \ j = 1, \cdots, n\}$ as shown in the figure below:

Observe that

a. $(e_1, i_1j_1)(e_2, i_2j_2) \in E(P_k \Box B_{m,n})$ if

   i. $e_1 = e_2, i_1 = i_2$ and either $j_1 = n$ or $j_2 = n$; or

   ii. $e_1 = e_2 + 1, i_1 = i_2$ and $j_1 = j_2$.

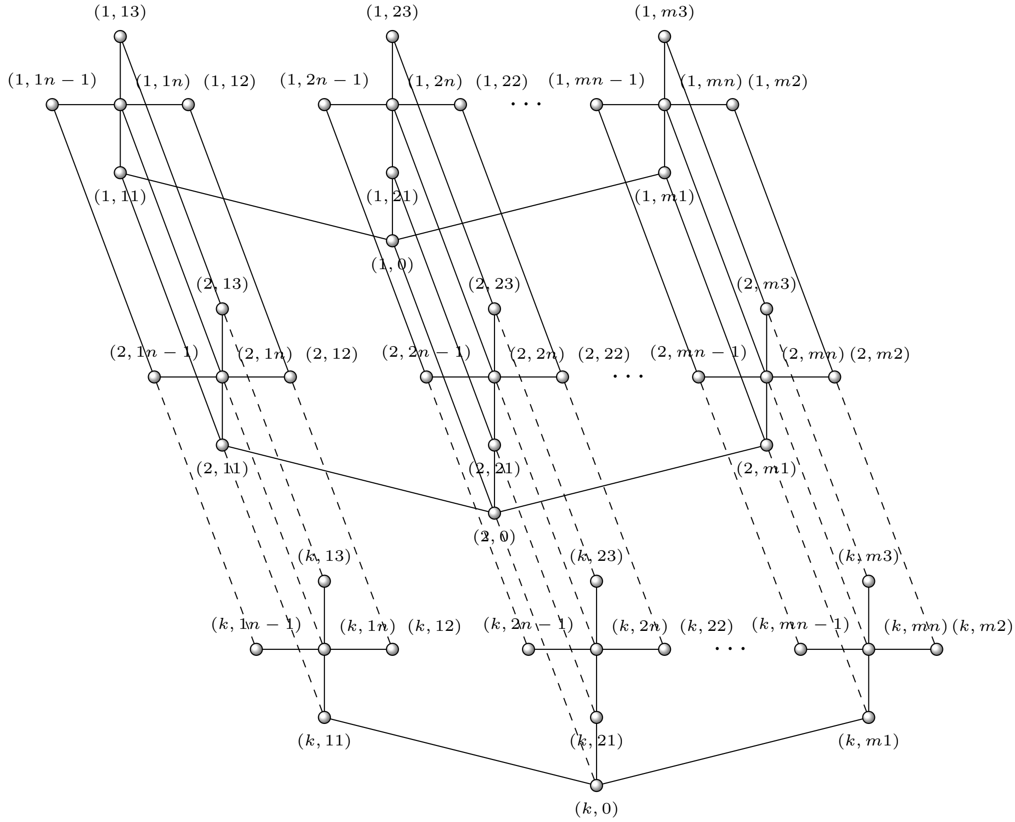
b. $(e_1, i1)(e_2, 0) \in E(P_k \Box B_{m,n})$ if $e_1 = e_2$, and

c. $(e_1, 0)(e_2, 0) \in E(P_k \Box B_{m,n})$ if $e_1 = e_2 + 1$.

Consider the following sets:

$$\begin{aligned} A_p &= \{(e, ij) : e \text{ is odd}, \ i = 1, \cdots, m, \ j = 1, \cdots, n-1\}, \\ A_q &= \{(e, ij) : e \text{ is even}, \ i = 1, \cdots, m, \ j = 1, \cdots, n-1\}, \\ B_p &= \{(e, 0) : e \text{ is odd}\} \cup \{(e, in) : e \text{ is odd}, \ i = 1, \cdots, m\}, \\ B_q &= \{(e, 0) : e \text{ is even}\} \cup \{(e, in) : e \text{ is even}, \ i = 1, \cdots, m\}. \end{aligned}$$

Let $S = A_p \cup B_q$ and $T = A_q \cup B_p$. We claim that S and T are the independent neighborhood sets of $P_k \Box B_{m,n}$. First, we show that no two vertices in S are adjacent. Observe that for any $(e_1, i_1j_1), (e_2, i_2j_2) \in S$ such that $e_1 = e_2$ and $i_1 = i_2$, we have $j_1 \neq n$ and $j_2 \neq n$. For the case when either $j_1 = n$ or $j_2 = n$, $e_1 \neq e_2$. Also, for $(e_1, i_1j_1), (e_2, i_2j_2) \in S$ such that $e_1 = e_2 + 1$ and $i_1 = i_2, j_1 \neq j_2$. This implies (e_1, i_1j_1) and (e_2, i_2j_2) are non-adjacents. Note that for any $(e_1, i1), (e_2, 0) \in S$, e_1 is odd while e_2 is even and so, $(e_1, i1), (e_2, 0)$ are non-adjacents. Hence, none of the vertices in S are adjacent.



Now, we will show that $\bigcup\limits_{v\in S}\langle N[v]\rangle = P_k\square B_{m,n}$. Assume to the contrary that $\bigcup\limits_{v\in S}\langle N[v]\rangle \neq P_k\square B_{m,n}$. Then there exists $xy \in E(P_k\square B_{m,n})$ such that $xy \notin E\left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$.

case I: $x = (e_1, i_1j_1), y = (e_2, i_2j_2)$

i. Consider $e_1 = e_2, i_1 = i_2$ and either $j_1 = n$ or $j_2 = n$. When $e_1 = e_2$ is even, $(e, in) \in B_q \subseteq S$ while when $e_1 = e_2$ is odd, $(e, ij) \in A_p \subseteq S$. This implies either $(e_1, i_1j_1) \in S$ or $(e_2, i_2j_2) \in S$ and follows that $(e_1, i_1j_1)(e_2, i_2j_2) \in E\left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$ which is a contradiction.

ii. Consider $e_1 = e_2 + 1, i_1 = i_2$ and $j_1 = j_2$. Then either e_1 is odd and e_2 is even or e_1 is even and e_2 is odd. But in either cases, $(e, ij) \in S$ when e is odd and consequently, $(e_1, i_1j_1)(e_2, i_2j_2) \in \left(\bigcup\limits_{v\in S}\langle N[v]\rangle\right)$. This is a contradiction.

case II: $x = (e_1, i_1), y = (e_2, 0)$

Since $(e_1, i_1)(e_2, 0) \in E\left(P_k \Box B_{m,n}\right), e_1 = e_2$. Then e_2 must be odd. But $(e_1, i_1) \in S$ when e_1 is odd and so, $(e_1, i_1)(e_2, 0) \in E\left(\bigcup_{v \in S} \langle N[v] \rangle\right)$ which is a contradiction.

case III: $x = (e_1, 0), y = (e_2, 0)$
Clearly, when e_1 is odd, e_2 is even and vice versa. But $(e, 0) \in S$ when e is even. This implies either $(e_1, 0) \in S$ or $(e_2, 0) \in S$. So, $(e_1, 0)(e_2, 0) \in E\left(\bigcup_{v \in S} \langle N[v] \rangle\right)$ which is a contradiction.

In either of the above cases, we arrived at a contradiction. Thus, $xy \in E\left(\bigcup_{v \in S} \langle N[v] \rangle\right)$. Consequently, $\bigcup_{v \in S} \langle N[v] \rangle = P_k \Box B_{m,n}$. Following same argument in S, we can easily show that T is also an independent neighborhood set in $P_k \Box B_{m,n}$.

Now, observe that

$$\begin{aligned}
|A_p| &= \{(e, ij) : e \text{ is odd}, i = 1, \cdots, m,\ j = 1, \cdots, n-1\} \\
&= \left\lceil \frac{k}{2} \right\rceil m(n-1), \\
|A_q| &= \{(e, ij) : e \text{ is even}, i = 1, \cdots, m,\ j = 1, \cdots, n-1\} \\
&= \left\lfloor \frac{k}{2} \right\rfloor m(n-1), \\
|B_p| &= \{(e, 0) : e \text{ is odd}\} \cup \{(e, in) : e \text{ is odd}, i = 1, \cdots, m\} \\
&= \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil m \\
&= \left\lceil \frac{k}{2} \right\rceil (m+1)
\end{aligned}$$

and

$$\begin{aligned}
|B_q| &= \{(e, 0) : e \text{ is even}\} \cup \{(e, in) : e \text{ is even}, i = 1, \cdots, m\} \\
&= \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor m \\
&= \left\lfloor \frac{k}{2} \right\rfloor (m+1).
\end{aligned}$$

Thus,

$$|S| = |A_p| + |B_q| = \left\lceil \frac{k}{2} \right\rceil m(n-1) + \left\lfloor \frac{k}{2} \right\rfloor (m+1)$$

and

$$|T| = |A_q| + |B_p| = \left\lfloor \frac{k}{2} \right\rfloor m(n-1) + \left\lceil \frac{k}{2} \right\rceil (m+1).$$

Therefore,

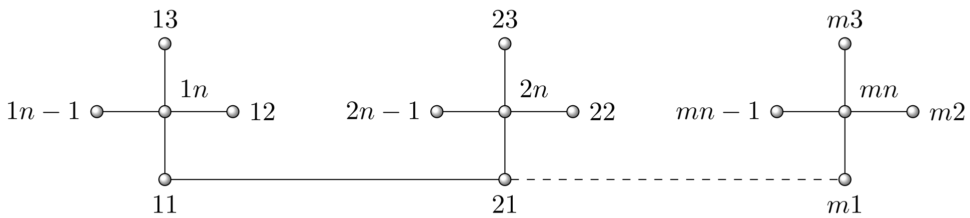
$$N_i(P_m \Box B_{m,n}, x) = x^{\lfloor \frac{k}{2} \rfloor m(n-1) + \lceil \frac{k}{2} \rceil (m+1)} + x^{\lceil \frac{k}{2} \rceil m(n-1) + \lfloor \frac{k}{2} \rfloor (m+1)}.$$

Theorem 4. *For any path P_k and Firecracker graph $F_{m,n}$,*

$$\begin{aligned} N_i(P_k \Box F_{m,n}, x) &= x^{\lceil \frac{k}{2} \rceil (\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor) + \lfloor \frac{k}{2} \rfloor (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor (n-1))} \\ &\quad + x^{\lfloor \frac{k}{2} \rfloor (\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor) + \lceil \frac{k}{2} \rceil (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor (n-1))} \end{aligned}$$

for any $k, m, n \in \mathbb{Z}^n$.

Proof: Label the vertices of $F_{m,n}$ by $ij,\ i = 1, \cdots, m,\ j = 1, \cdots, n$ as shown in the figure below:



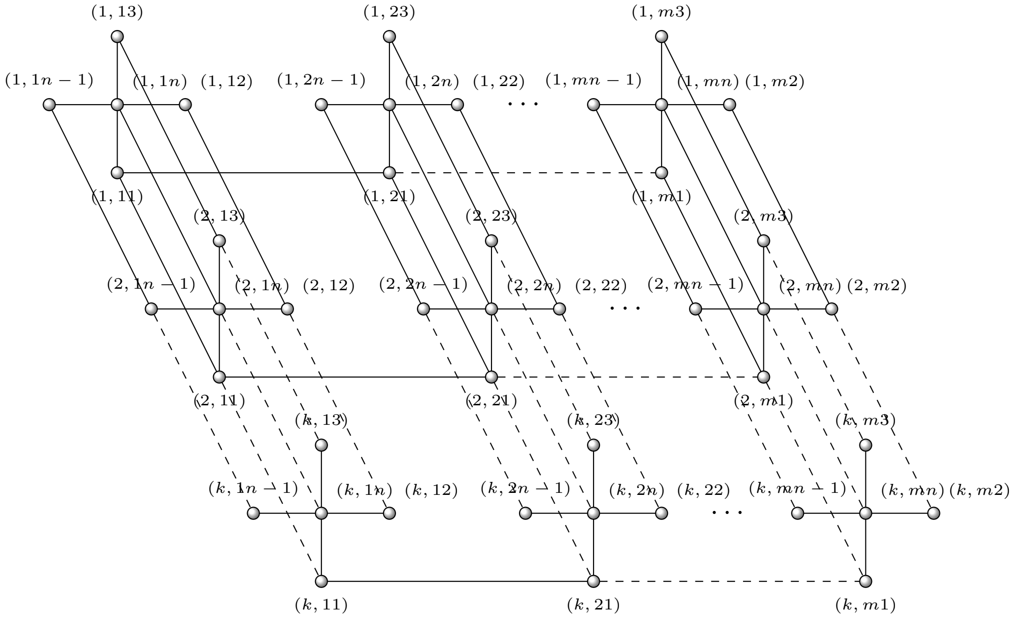
Then $V(P_k \Box F_{m,n}) = \{(e, ij) : e = 1, \cdots, k,\ i = 1, \cdots, m,\ j = 1, \cdots, n\}$ and $E(P_k \Box F_{m,n}) = \{(e_1, i_1 j_1)(e_2, i_2 j_2) : e_1 = e_2$ and $i_1 j_1 i_2 j_2 \in E(F_{m,n})$ or $e_1 e_2 \in E(P_k)$ and $i_1 = i_2, j_1 = j_2\}$ as shown in the figure below:

Observe that for any $(e_1, i_1 j_1)(e_2, i_2 j_2) \in E(P_k \Box F_{m,n})$, either

i.) $e_1 = e_2,\ i_1 = i_2,\ j_1 = n$ or $j_2 = n$;

ii.) $e_1 = e_2,\ i_1 = i_2 + 1,\ j_1 = 1 = j_2$; or

iii.) $e_1 = e_2 + 1,\ i_1 = i_2,\ j_1 = j_2$.



Consider the following sets:

$$A_p = \{(p, ij) : p \text{ is odd, } i \text{ is odd, } j = 1, \cdots, n-1\}, \qquad B_p = \{(p, in) : p \text{ is odd, } i \text{ is odd}\}$$
$$A_q = \{(q, ij) : q \text{ is even, } i \text{ is odd, } j = 1, \cdots, n-1\}, \qquad B_q = \{(q, in) : q \text{ is even, } i \text{ is odd}\}$$
$$C_p = \{(p, ij) : p \text{ is odd, } i \text{ is even, } j = 1, \cdots, n-1\}, \qquad D_p = \{(p, in) : p \text{ is odd, } i \text{ is even}\}$$
$$C_q = \{(q, ij) : q \text{ is even, } i \text{ is even, } j = 1, \cdots, n-1\}, \qquad D_q = \{(q, in) : q \text{ is even, } i \text{ is even}\}$$

Let $S = A_p \cup D_p \cup B_q \cup C_q$ and $T = A_q \cup D_q \cup B_p \cup C_p$. We claim that S and T are the independent neighborhood sets of $P_k \square F_{m,n}$. First, we show that no two vertices in S are adjacent. Since each A_p and C_q consist of the pendant vertices in each star, A_p and C_q are independent sets. Also, since each B_q and D_p consist of apex vertices in each star, B_q and D_p are independent sets. We note that elements of A_p and D_p are not adjacent since i is odd in A_p and i is even in D_p. Similarly, elements of B_q and C_q are non-adjacent. Hence, the set $A_p \cup D_p \cup B_q \cup C_q$ have non-adjacent vertices.

Next, we show that $\bigcup_{v\in S} \langle N[v] \rangle = P_k \square F_{m,n}$. Assume to the contrary that $\bigcup_{v\in S} \langle N[v] \rangle \neq P_k \square F_{m,n}$. Then there exists $(e_1, i_1j_1)(e_2, i_2j_2) \in E\left(P_k \square F_{m,n}\right)$ such that $(e_1, i_1j_1)(e_2, i_2j_2) \notin E\left(\bigcup_{v\in S} \langle N[v] \rangle\right)$. This implies $(e_1, i_1j_1), (e_2, i_2j_2) \notin S$.

case 1: $e_1 = e_2,\ i_1 = i_2,$ either $j_1 = n$ or $j_2 = n$.
WLOG, we may assume $j_1 = n$. Since $(e_1, i_1j_1) \notin S$, either either e_1 and i_1 are odd or e_1 and i_1 are even. Consider e_1 and i_1 are odd. Since $e_1 = e_2$ and $i_1 = i_2$, e_2

and i_2 are odd. But $(e_2, i_2j_2) \in A_p \subseteq S$. This is a contradiction. Similarly, if we consider e_1 and i_1 to be even, then $(e_2, i_2j_2) \in C_q \subseteq S$

case 2: $e_1 = e_2,\ i_1 = i_2 + 1$ and $j_1 = 1 = j_2$.
Since $(e_1, i_1j_1) \notin S$, either e_1 is odd and i_1 is even or e_1 is even and i_1 is odd. When e_1 is odd and i_1 is even, e_2 and i_2 are odd. But $(e_2, i_2j_2) \in A_p \subseteq S$. This is a contradiction. Similarly, a contradiction will arrive when e_1 is even and i_1 is odd.

case 3: $e_1 = e_2 + 1,\ i_1 = i_2$ and $j_1 = j_2$.
Since $(e_1, i_1j_1) \notin S$, we consider the following cases: For $j = 1, \cdots, n-1$, if e_1 is even, i_1 is odd, then it follows that e_2 and i_2 are odd. But $(e_2, i_2j_2) \in S$. This is a contradiction. Similarly, when e_1 is odd and i_1 is even, then e_2 and i_2 are even for which $(e_2i_2j_2) \in S$, a contradiction. For $j = n$, if e_1 and i_1 are even, then e_2 is odd and i_1 is even. So, $(e_2, i_2j_2) \in S$, a contradiction. Also, for if e_1 and i_1 are odd, e_2 is even and i_2 is odd and that $(e_2, i_2j_2) \in S$ which is a contradiction.

Thus, in either of the above cases, we arrived a contradiction. Hence, $(e_1, i_1j_1)(e_2, i_2j_2) \in E\left(\bigcup_{v\in S}\langle N[v]\rangle\right)$. Consequently, $\bigcup_{v\in S}\langle N[v]\rangle = P_k \Box F_{m,n}$. Following the same argument in S, we can verify that T is also an independent neighborhood set of $P_k \Box F_{m,n}$.

Finally,

$$\begin{aligned}
|S| &= |A_p| + |D_p| + |B_q| + |C_q| \\
&= \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil (n-1) + \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor (n-1) \\
&= \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{m}{2} \right\rceil (n-1) + \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor (n-1) \right)
\end{aligned}$$

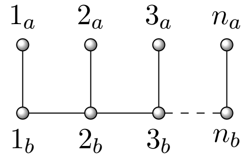
and

$$\begin{aligned}
|T| &= |A_q| + |D_q| + |B_p| + |C_p| \\
&= \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{m}{2} \right\rceil (n-1) + \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{m}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{m}{2} \right\rfloor (n-1) \\
&= \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lceil \frac{m}{2} \right\rceil (n-1) + \left\lfloor \frac{m}{2} \right\rfloor \right) + \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{m}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor (n-1) \right).
\end{aligned}$$

Therefore,

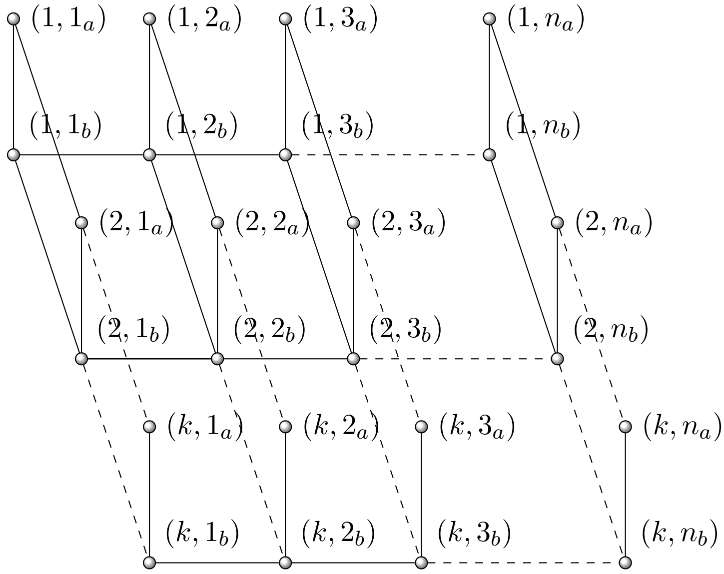
$$\begin{aligned}
N_i(P_k \Box F_{m,n}, x) &= x^{\lceil \frac{k}{2} \rceil (\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor) + \lfloor \frac{k}{2} \rfloor (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor (n-1))} \\
&\quad + x^{\lfloor \frac{k}{2} \rfloor (\lceil \frac{m}{2} \rceil (n-1) + \lfloor \frac{m}{2} \rfloor) + \lceil \frac{k}{2} \rceil (\lceil \frac{m}{2} \rceil + \lfloor \frac{m}{2} \rfloor (n-1))}.
\end{aligned}$$

Theorem 5. *For any path P_k and Centipede graph Cen_n, $N_i(P_k \Box Cen_n, x) = 2x^{nk}$ for any $k, n \in \mathbb{Z}^+$.*



Proof: Label the vertices of Cen_n by $j_a, j_b : j = 1, \cdots, n$ and define its edges by $E(Cen_n) = \{j_a j_b : j = 1, \cdots, n\} \cup \{j_b(j+1)_b : j = 1 \cdots, n-1\}$ as shown in the figure below:

Then $V(P_k \square Cen_n) = \{(i, j_a), (i, j_b) : i = 1, \cdots, k,\ j = 1, \cdots, n\}$.



Observe that

- $(i_1, j_{1a})(i_2, j_{2b}) \in E(P_k \Box Cen_n)$ if $i_1 = i_2$ and $j_1 = j_2$,
- $(i_1, j_{1a})(i_2, j_{2a}) \in E(P_k \Box Cen_n)$ if $i_1 = i_2 + 1$ and $j_1 = j_2$ and
- $(i_1, j_{1b})(i_2, j_{2b}) \in E(P_k \Box Cen_n)$ if either
  i.) $i_1 = i_2, j_1 = j_2 + 1$ or
  ii.) $i_1 = i_2 + 1, j_1 = j_2$.

Consider the following sets:

$$\begin{aligned}
A_p &= \{(i, j_a) : i \text{ and } j \text{ are odd}\}, & B_p &= \{(i, j_a) : i \text{ is odd and } j \text{ is even}\}\\
A_q &= \{(i, j_a) : i \text{ is even and } j \text{ is odd}\}, & B_q &= \{(i, j_a) : i \text{ and } j \text{ are even}\}\\
c_p &= \{(i, j_b) : i \text{ and } j \text{ are odd}\}, & D_p &= \{(i, j_b) : i \text{ is odd and } j \text{ is even}\}\\
C_q &= \{(i, j_b) : i \text{ is even and } j \text{ is odd}\}, & D_q &= \{(i, j_b) : i \text{ and } j \text{ are even}\}.
\end{aligned}$$

Let $S = A_p \cup D_p \cup B_q \cup C_q$ and $T = A_q \cup D_q \cup B_p \cup C_p$, that is,

$$S = \begin{cases} (i, j_a) : & i \text{ and } j \text{ are odd} \\ (i, j_a) : & i \text{ and } j \text{ are even} \\ (i, j_b) : & i \text{ is even and } j \text{ is odd} \\ (i, j_b) : & i \text{ is odd and } j \text{ is even} \end{cases} \text{ and } T = \begin{cases} (i, j_a) : & i \text{ is even and } j \text{ is odd} \\ (i, j_a) : & i \text{ is odd and } j \text{ is even} \\ (i, j_b) : & i \text{ and } j \text{ are odd} \\ (i, j_b) : & i \text{ and } j \text{ are even .} \end{cases}$$

We claim that S and T are the independent neighborhood sets of $P_k \Box Cen_n$. First, we show that no two vertices in S are adjacent. Observe that for any $(i_1, j_{1a}), (i_2, j_{2a}) \in S$, $(i_1, j_{1a})(i_2, j_{2a}) \notin E(P_k \Box Cen_n)$ since $j_1 \neq j_2$. Also, for any $(i_1, j_{1a}), (i_2, j_{2b}) \in S$, $(i_1, j_{1a})(i_2, j_{2b}) \notin E(P_k \Box Cen_n)$ since when $i_1 = i_2, j_1 \neq j_2$ and when $j_1 = j_2, i_1 \neq i_2$. Furthermore, any $(i_1, j_{1b}), (i_2, j_{2b}) \in S$, $(i_1, j_{1b})(i_2, j_{2b}) \notin E(P_k \Box Cen_n)$ since both $i_1 \neq i_2$ and $j_1 \neq j_2$. Hence, no two vertices in S are adjacent.

Next, we will show that $\bigcup_{v \in S} \langle N[v] \rangle = P_k \Box Cen_n$. Assume to the contrary that $\bigcup_{v \in S} \langle N[v] \rangle \neq P_k \Box Cen_n$. Then there exists $xy \in E(P_k \Box C_n)$ such that $xy \notin E\left(\bigcup_{v \in S} \langle N[v] \rangle\right)$. This implies both $x, y \notin S$.

case 1: $x = (i_1, j_{1a})$ and $y = (i_2, j_{2b})$
Since $(i_1, j_{1a}) \notin S$, either i_1 is even and j_1 is odd or i_1 is odd and j_1 is even. If i_1 is even, j_1 is odd, then i_2 is even and j_2 is odd. But $(i_2, j_{2b}) \in S$. This is a contradiction. For i_1 odd and j_1 even, i_2 is odd and j_2 is even. Similarly, $(i_2, j_{2b}) \in S$ which is a contradiction.

case 2: $x = (i_1, j_{1a})$ and $y = (i_2, j_{2a})$
When $(i_1, j_{1a}) \notin S$, it follows that either i_1 is even and j_1 is odd or i_1 is odd and j_1 is even. Consider i_1 to be even and j_1 to be odd. Since $(i_1, j_{1a})(i_2, j_{2a}) \in E(P_k \Box Cen_n)$

if $i_1 = i_2 + 1$ and $j_1 = j_2$, it follows that i_2 and j_2 are odd and this is a contradiction for $(i_2, j_{2a}) \in S$. Similarly, when i_1 is odd and j_1 is even, i_2 and j_2 are even and this is a contradiction since $(i_2, j_{2a}) \in S$.

case 3: $x = (i_1, j_{1b})$ and $y = (i_2, j_{2b})$
Since $(i_1, j_{1b}) \notin S$, either i_1 and j_1 are odd or even. Similarly, $(i_2, j_{2b}) \notin S$ implies i_2 and j_2 are odd or even. But if i_1, j_1, i_2, j_2 are all odd, $(i_1, j_{1b})(i_2, j_{2b}) \notin E(P_k \square Cen_n)$. Also, when i_1, i_2, j_1, j_2 are all even, $(i_1, j_{1b})(i_2, j_{2b}) \notin E(P_k \square Cen_n)$. If we consider i_1, j_1 to be odd and i_2, j_2 to be even, clearly, $(i_1, j_{1b})(i_2, j_{2b}) \notin E(P_k \square Cen_n)$. Hence, all possibilities yield a contradiction.

Thus, $xy \in E\left(\bigcup_{v\in S} \langle N[v]\rangle\right)$. Consequently, $\bigcup_{v\in S} \langle N[v]\rangle = P_k \square Cen_n$. Thus, S is an independent neighborhood set of $P_k \square Cen_n$. We can also verify that T is an independent neighborhood set of $P_k \square Cen_n$ by following the same argument in S.

Lastly, observe that $|A_p| = \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil, |D_p| = \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor, |B_q| = \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, |C_q| = \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, |A_q| = \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil, |D_q| = \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor, |B_p| = \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor$ and $|C_p| = \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$. Hence,

$$\begin{aligned}
|S| &= |A_p| + |D_p| + |B_q| + |C_q| \\
&= \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil \\
&= \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right) + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right) \\
&= \left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right) \left(\left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right) \\
&= nk
\end{aligned}$$

and

$$\begin{aligned}
|T| &= |A_q| + |D_q| + |B_p| + |C_p| \\
&= \left\lfloor \frac{k}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil \\
&= \left(\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \right) \left(\left\lceil \frac{k}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor \right) \\
&= nk.
\end{aligned}$$

Therefore, $N_i(P_k \square C_n, x) = 2x^{nk}$.

Remark 1. *When* $m = 2$ *in Theorem* 3.4,

$$\begin{aligned}
N_i(P_k \square F_{2,n}, x) &= x^{\left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{2}{2} \right\rceil (n-1) + \left\lfloor \frac{2}{2} \right\rfloor \right) + \left\lfloor \frac{k}{2} \right\rfloor \left(\left\lceil \frac{2}{2} \right\rceil + \left\lfloor \frac{2}{2} \right\rfloor (n-1) \right)} + x^{\left\lfloor \frac{k}{2} \right\rfloor \left(\left\lceil \frac{2}{2} \right\rceil (n-1) + \left\lfloor \frac{2}{2} \right\rfloor \right) + \left\lceil \frac{k}{2} \right\rceil \left(\left\lceil \frac{2}{2} \right\rceil + \left\lfloor \frac{2}{2} \right\rfloor (n-1) \right)} \\
&= x^{\left\lceil \frac{k}{2} \right\rceil n + \left\lfloor \frac{k}{2} \right\rfloor n} + x^{\left\lfloor \frac{k}{2} \right\rfloor n + \left\lceil \frac{k}{2} \right\rceil n}
\end{aligned}$$

$$= 2x^{n(\lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor)}$$
$$= 2x^{nk}$$
$$= N_i(P_k \Box Cen_n, x).$$

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References

[1] A. Alwardi and P.M. Shivaswamy. On the neighbourhood polynomial of graphs. *Bulletin of the Society of Mathematicians Banja Luka*, 6:13–24.

[2] F. Buckley and F. Harary. *Distance in Graphs.* Addison-Wesley Series in Mathematics, 1990.

[3] R. Diestel. *Graph Theory.* Springer Nature, 2017.

[4] M.S. Sardar et.al. Computing topological indices of the line graphs of banana tree graph and firecracker graph. *Applied Mathematics and Nonlinear Sciences*, 2:83–92, 2017.

[5] R. Hammack et.al. *Handbook of Product of Graphs.* CRC Press Taylor and Francis Group, London and New York, 2011.

[6] S. Ganesh and R. Revathi. Analysis of some bistar related mmd graphs. *International Journal of Pure and Applied Mathematics*, 118(10):407–413, 2018.

[7] F. Harary. *Graph Theory.* Addison-Wesley Publishing Company, 1969.

[8] M. Khasif. Chromatic polynomials and chromaticity of some linear h-hypergraphs. 1969.

[9] V.R. Kulli. The neighborhood graph of a graph. *International Journal of Fuzzy Mathematical Archive*, 8(2):93–99, 2015.

[10] V. E. Levit and E. Mandrescui. The independence polynomial of a graph-a survey. *In Proceedings of the* 1^{st} *International Conference on Algebraic Informatics*, pages 233–254, 2005.

[11] K.B. Murthy and Puttaswamy. On the independent neighbourhood polynomial of graphs. *Indian Streams Research Journal*, 5(12):1–7, 2015.