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Cartan's Approach to Second Order Ordinary Differential Equations

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Abstract. In his work on projective connections, Cartan discusses his theory of second order differential equations. It is the aim here to look at how a normal projective connection can be constructed and how it relates to the geometry of a single second order differential equation. The calculations are presented in some detail in order to highlight the use of gauge conditions.

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1. Introduction

Two problems which have seen a resurgence of interest recently are two classical subjects, namely, the equivalence problem under different kinds of tranformations and study of the natural geometric structures induced on their solution spaces. In the early part of the twentieth century, Cartan proposed a theory of the second-order ordinary differential equation which came out of his theory of projective connections [3]. The point of interest here is that it is possible to construct certain kinds of geometric structures on the spaces of particular types of differential equations. In the case of the third order differential equation, it is possible to create from the equation a conformal class of Lorentzian metrics on the solution space, provided a function associated with the equation vanishes [6, 12, 13] This function is a relative invariant of the equation under contact transformations. These kinds of constructions turn out to be both useful and mathematically interesting. There are numerous applications of this kind of work, for example, in the study of general relativity [10]. The simplest one to begin with is the second order ordinary differential equation. A geometric structure can be constructed on the solution space of such an equation and a function which is a relative invariant of the equation can be identified. The concept of duality was also studied by Cartan. Considerable work has been done on the construction of a geometric structure on the solution space of a second order differential equation. The intention here is to discuss the theory of the second order ordinary differential equation

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which Cartan began and to study in particular Cartan's notion of duality between such types of equations [9].

The idea of what Cartan referred to as a manifold of elements with projective connection in the two-dimensional case is of great interest [2, 4, 5, 8]. By way of introduction, some definitions of basic concepts is introduced. An element is a pair consisting of a point of a differentiable manifold M and a one-dimensional subspace of the tangent space to M at that point. A manifold of elements is the projective tangent bundle PTM of a two-dimensional manifold M. Let P^2 denote real projective space with two-dimensions. Its projective tangent bundle PTP^2 can be expressed as the homogeneous space G/H where $G = SL(2, \mathbb{R})$ and H is the subgroup of G consisting of all upper triangular elements. A manifold of elements with projective connection is a Cartan geometry PTM modeled on PTP^2 in which certain conditions regarding the development of curves which arise out of the projective tangent structures of the underlying manifold and model geometry are satisfied [1, 14, 16]. It should also be stressed that there are many physical applications of this work especially in relativity [7, 11, 15].

2. Manifolds with Projective Connections

Local coordinates can be introduced on PTM by taking local coordinates (x, y) on M and noting that every equivalence class of tangent vectors

$$u\,\partial_x + v\partial_y$$

for which $u \neq 0$ has a unique representative of the form

$$\partial_x + y'\,\partial_y,$$

then (x, y, y') are local coordinates on PTM which we work with in noting they do not cover those equivalence classes of tangent vectors for which $u = 0$.

A *Klein geometry* is a homogeneous space G so that it is a manifold M with a transitive action of G. Take a point $x_0 \in M$ and let H be the stabilizer of x_0. Then M can be identified with the coset space G/H. Then G becomes a right principle H-bundle over M with projection $g \to gx_0$. The space (G, H) may be referred to as the Klein geometry.

Gauge changes may be made on a projective connection on such a manifold. If ω is a connection form that is a trace-free 3×3 matrix of local one forms on PTM, and h is an H-valued function, then the gauged connection form is

$$h^{-1}\,\omega\,h + h^{-1}\,dh. \tag{1}$$

If Ω is a curvature two-form corresponding to ω, the regauged curvature is $h^{-1}\Omega h$. Introducing

$$h = \begin{pmatrix} A & D & F \\ 0 & B & E \\ 0 & 0 & C \end{pmatrix} \tag{2}$$

where $ABC = 1$ and the inverse element is calculated to be

$$h^{-1} = \begin{pmatrix} A^{-1} & -CD & DE - BF \\ 0 & B^{-1} & -AE \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

The effect of a change of gauge on the lower triangular terms is of most interest. Since $h^{-1}\, dh$ is upper triangular, it has no effect on these terms. In any event, the entire matrix can be computed. Suppose ω is given as

$$\omega = \begin{pmatrix} 0 & \omega_2^1 & \omega_3^1 \\ -\omega_2^1 & 0 & \omega_3^2 \\ -\omega_3^1 & -\omega_3^2 & 0 \end{pmatrix}, \qquad \omega^T = -\omega. \tag{4}$$

Using (2) and (3), we calculate

$$h^{-1}\,\omega\,h =$$
$$\begin{pmatrix} ACD\omega_2^1 - A(DE-BF)\omega_3^1 & A^{-1}B\omega_2^1 + CD^2\omega_2^1 & A^{-1}(E\omega_2^1 + C\omega_3^1) + CD(E\omega_2^1 - C\omega_3^2) \\ & -(DE-BF)(D\omega_3^1 - B\omega_3^2) & -(DE-BF)(F\omega_3^1 + E\omega_3^2) \\ -AB^{-1}\omega_2^1 + A^2E\omega_3^1 & -B^{-1}D\omega_2^1 + AE(D\omega_3^1 + B\omega_3^2) & B^{-1}(-F\omega_2^1 + C\omega_3^2) \\ & & +AE(F\omega_3^1 + E\omega_3^2) \\ -C^{-1}A\omega_3^1 & -C^{-1}(D\omega_3^1 + B\omega_3^2) & -C^{-1}(F\omega_3^1 + E\omega_3^2) \end{pmatrix} \tag{5}$$

Let Ω be the curvature form given by

$$\Omega = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ 0 & 0 & \Omega_{23} \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

The effect of a change of gauge on (6) using (3) is found to be

$$h^{-1}\,\Omega\,h = \begin{pmatrix} 0 & A^{-1}B\Omega_{12} & A^{-1}(E+C)\Omega_{13} - C^2D\Omega_{23} \\ 0 & 0 & B^{-1}C\Omega_{23} \\ 0 & 0 & 0 \end{pmatrix} \tag{7}$$

The equation for the development of a curve will be required so to this end, the following mapping is proposed

$$(\xi, \eta, \eta') \to \begin{pmatrix} 1 & 0 & 0 \\ \xi & 1 & 0 \\ \eta & \eta' & 1 \end{pmatrix} \tag{8}$$

This is a local section of $SL(3, \mathbb{R}) \to PTP^2$. The corresponding Maurer-Cartan form is obtained by working out the exterior derivative of (8), At this point, the connection form needed to calculate the development equations later is given,

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_2^0 \\ \omega_0^1 & \omega_1^1 & \omega_2^1 \\ \omega_0^2 & \omega_1^2 & \omega_2^2 \end{pmatrix} \tag{9}$$

The development equation for a curve γ in PTM gives

$$a\dot{\xi} - b(\dot{\eta} - \eta'\dot{\xi}) = \langle \dot{\gamma}, \omega_0^1 \rangle, \qquad c(\dot{\eta} - \eta'\dot{\xi}) = \langle \dot{\gamma}, \omega_0^2 \rangle, \tag{10}$$

for some functions $a(t), b(t), c(t)$.

Knowing these formulas, the conditions to be imposed on the development of a curve should be considered further. There are two kinds of curve on PTM. There are vertical curves and there are natural lifts. With respect to the coordinates (x, y, y') introduced above, a curve in PTM is a natural lift if its tangent vector is annihilated by the contact form $dy - y'dx$. The conditions are then (i) the development into PTP^2 of a natural curve in PTM is vertical (ii) the development into PTP^2 of a natural lift in PTM is a natural lift. These conditions require that if γ is vertical then $\langle \gamma, \omega_0^1 \rangle = \langle \gamma, \omega_0^2 \rangle = 0$, while if γ is a natural lift, then $\langle \gamma, \omega_0^2 \rangle = 0$. It follows that

$$\omega_0^1 = \lambda \, dx + \mu \, dy, \qquad \omega_0^2 = \nu(dy - y' \, dx). \tag{11}$$

for some functions λ, μ, ν on PTM.

The connection matrix ω can now be simplified by introducing a change of gauge, so setting

$$A = ((\lambda + \mu y')\nu)^{-1/3}, \qquad B = (\lambda + \mu y')A, \qquad C = \nu A, \qquad E = \mu A, \tag{12}$$

it can be ensured that $\omega_0^1 = dx$ and $\omega_0^2 = dy - y' \, dx = \vartheta$. It is possible to write the form ω_1^2 as $\omega_1^2 = k(dy' - tdx) + md\vartheta$ for some functions f, k and m on PTM. The function k must be nonzero because the forms ω_0^1, ω_0^2 and ω_1^2 have to be linearly independent. Upon setting $dy' - f \, dx = \varphi$, the set of forms dx, ϑ and φ constitute a local basis of one-forms. It can be summarized by saying that a gauge has been chosen such that

$$\omega = \begin{pmatrix} \omega_0^2 & \omega_1^0 & \omega_2^0 \\ dx & \omega_1^1 & \omega_2^1 \\ \vartheta & k\varphi + m\vartheta & \omega_2^2 \end{pmatrix} \tag{13}$$

The remaining gauge freedom involves the functions D and F. Consider a further gauge change which is specified by taking

$$h = \begin{pmatrix} 1 & D & F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h^{-1} = \begin{pmatrix} 1 & -D & -F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

By using (13) and (14), we find that

$$h^{-1} \, \omega \, h + h^{-1} \, dh$$

$$= \begin{pmatrix} \omega_0^0 - D \, dx - F\vartheta & \omega_1^0 - D\omega_1^1 - F\omega_1^2 + dD & \omega_2^0 - D\omega_2^1 - F\omega_2^2 + dF \\ dx & \omega_1^1 & \omega_2^1 \\ \vartheta & \omega_1^2 & \omega_2^2 \end{pmatrix} \tag{15}$$

Therefore, a gauge can be chosen for any projective connectionof a manifold of elements such that

$$\omega_0^1 = dx, \quad \omega_0^2 = \vartheta, \quad \omega_0^0 = \kappa\, \varphi, \tag{16}$$

for some function κ. This is referred to as the standard gauge for the projective connection.

A geodesic of this projective connection is a curve whose development satisfies the equations $\dot{\eta} - \eta' \xi = 0$ and $\dot{\eta}' = 0$. Put another way, a geodesic is a curve whose tangents are annihilated by ϑ and φ, and therefore is a solution of the second-order equation

$$\frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx}). \tag{17}$$

Geodesics are the base integral curves of the following vector field on PTM,

$$\Gamma = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f \frac{\partial}{\partial y'}. \tag{18}$$

This can be referred to as the second-order differential equation field corresponding to the projective connection. Note that Γ is determined by the conditions

$$\langle \Gamma, dx \rangle = 1, \qquad \langle \Gamma, \vartheta \rangle = \langle \Gamma, \varphi \rangle = 0. \tag{19}$$

Under a change of coordinates on the base manifold M, with induced change on PTM, the field Γ will acquire an overall factor which depends on the coordinate transformation functions. It may be said that we are really working not with a vector field Γ, but with a line element field.

Having fixed the gauge, the next step is to impose gauge-invariant conditions on the curvature in order to single out a particular connection form out of the class of connections under consideration. In fact, Cartan showed in effect that there is a unique choice of the remaining connection forms so that the curvature Ω is upper triangular, with Ω_1^0 a multiple of $dx \wedge \vartheta$. The unique connection obtained this way is usually referred to as the normal projective connection on the manifold of elements associated with the second order differential equation.

3. Model Geometry and Duality of Points and Lines in Projective Geometry

Let M and $\bar{M}$ be two-dimensional manifolds, and S a co-dimension one submanifold of $M \times \bar{M}$, which is fibered over both M and $\bar{M}$. For any $\bar{p} \in \bar{M}$ the set $\{p \in M | (p, \bar{p}) \in S\}$ is a path in M, call it $\sigma_{\bar{p}}$. For $p \in M$, $\{\bar{p} \in \bar{M} | (p, \bar{p}) \in S\}$ determines a one-parameter family of paths $\sigma_{\bar{p}} \subset M$ such that $p \in \sigma_{\bar{p}}$ for all such $\bar{p}$. Require that this construction define a path space on M. For every $p \in M$ and $[u] \in PT_p M$, there is a unique $\bar{p} \in \bar{M}$ with $(p, \bar{p}) \in S$ the direction of the tangent to $\sigma_{\bar{p}}$ at x is $[u]$.

Let (x, y) be coordinates on M and $(\bar{x}, \bar{y})$ those on $\bar{M}$, and suppose the submanifold S of $M \times \bar{M}$ is given by $\Phi(x, y, \bar{x}, \bar{y}) = 0$. Then $\sigma(\bar{x}_0, \bar{y}_0)$ is $\Phi(x, y, \bar{x}_0, \bar{y}_0) = 0$ and vector

$$(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y})_{(\bar{x}_0, \bar{y}_0)} \tag{20}$$

is a representative of some tangent vector with $u \neq 0$ and is tangent to this path provided

$$\Phi_x(x_0, y_0, \bar{x}_0, \bar{y}_0) + y'\Phi_y(x_0, y_0, \bar{x}_0, \bar{y}_0) = 0, \tag{21}$$

where $\Phi(x_0, y_0, \bar{x}_0, \bar{y}_0) = 0$. Therefore, the map $S \to PTM$ is given by $(x, y, \bar{x}, \bar{y}) \to (x, y, y')$ such that

$$\Phi(x, y, \bar{x}, \bar{y}) = 0, \qquad \Phi_x(x, y, \bar{x}, \bar{y}) + y'\Phi_y(x, y, \bar{x}, \bar{y}) = 0. \tag{22}$$

It is required that Φ_x and Φ_y do not vanish simultaneously. In a similar way, the map $S \to PT\bar{M}$ is given by $(x, y, \bar{x}, \bar{y}) \to (\bar{x}, \bar{y}, \bar{y}')$ where

$$\Phi(x, y, \bar{x}, \bar{y}) = 0, \qquad \Phi_{\bar{x}}(x, y, \bar{x}, \bar{y}) + \bar{y}'\Phi_{\bar{y}}(x, y, \bar{x}, \bar{y}) = 0,$$

and $\Phi_{\bar{x}}$, $\Phi_{\bar{y}}$ must not vanish at the same time.

The condition that the first pair of equations determines $\bar{x}$ and $\bar{y}$ in terms of x, y and y' is that the following matrix be nonsingular,

$$\begin{pmatrix} 0 & \Phi_x & \Phi_y \\ \Phi_{\bar{x}} & \Phi_{x\bar{x}} & \Phi_{y\bar{x}} \\ \Phi_{\bar{y}} & \Phi_{x\bar{y}} & \Phi_{y\bar{y}} \end{pmatrix} \tag{23}$$

The same condition ensures that the second pair of equations can be solved for x and y in terms of bared variables with bars. This assumes that Φ_x, Φ_y do not both vanish simultaneously and the same for $\Phi_{\bar{x}}$ $\Phi_{\bar{y}}$ as well.

Assume that this condition holds on S. It can be stated in the form $\Delta \neq 0$, where Δ is the determinant

$$\Delta = -\Phi_y\Phi_{\bar{y}}(\Phi_{x\bar{x}} + y'\Phi_{y\bar{x}} + \bar{y}\Phi_{x\bar{y}} + y'\bar{y}'\Phi_{y\bar{y}}), \qquad \Phi_y\Phi_{y'} \neq 0. \tag{24}$$

Take some fixed point $(\bar{x}, \bar{y}) \in \bar{M}$ such that the path it takes in M can be parametrized by x. Then y' and y'' satisfy

$$\Phi_{xx} + 2\frac{dy}{dx}\Phi_{xy} + (\frac{dy}{dx})^2\Phi_{yy} + (\frac{d^2y}{dx^2})\Phi_y = 0. \tag{25}$$

The path then is a solution to the second order equation

$$\frac{d^2y}{dx^2} = f(x, y, y'), \tag{26}$$

where $f(x, y, y')$ is obtained by eliminating $\bar{x}$ and $\bar{y}$ between the equations

$$\Phi = 0, \qquad \Phi_x + y'\Phi_y = 0, \qquad \Phi_{xx} + 2y'\Phi_{xy} + (y')^2\Phi_{yy} + f\,\Phi_y = 0. \tag{27}$$

The right-hand side $\bar{f}(\bar{x}, \bar{y}, \bar{y}')$ of the equation giving the dual path is obtained by eliminating x and y between the equations

$$\Phi = 0, \qquad \Phi_{\bar{x}} + \bar{y}'\Phi_{\bar{y}} = 0, \qquad \Phi_{\bar{x}\bar{x}} + 2\bar{y}'\Phi_{\bar{x}\bar{y}} + (\bar{y}')^2\Phi_{\bar{y}\bar{y}} + \bar{f}\Phi_{\bar{y}} = 0. \tag{28}$$

There are two Cartan normal projective connection forms associated with this structure. There is one corresponding to $d^2y/dx^2 = f$ and the other to $d^2\bar{y}/d\bar{x}^2 = \bar{f}$. Each can be represented by a connection form on S. The connection form associated with the bar equation takes a lower triangular form

$$\bar{\omega} = \begin{pmatrix} d\bar{x} & \\ \bar{\vartheta} & \bar{\varphi} - \frac{1}{3}\bar{f}_{\bar{y}}\bar{\vartheta} \end{pmatrix}, \quad \bar{\vartheta} = d\bar{y} - \bar{y}'\,d\bar{x}, \quad \bar{\varphi} = d\bar{y}' - \bar{f}\,d\bar{x}. \tag{29}$$

Now (x, y, y') and $(\bar{x}, \bar{y}, \bar{y}')$ can be regarded as alternative coordinates on S with transformation given by

$$\Phi(x, y, \bar{x}, \bar{y}) = 0, \qquad \Phi_x(x, y, \bar{x}, \bar{y}) + y'\Phi_y(x, y, \bar{x}, \bar{y}) = 0, \qquad \Phi_{\bar{x}}(x, y, \bar{x}, \bar{y}) + \bar{y}'\Phi_{\bar{y}}(x, y, \bar{x}, \bar{y}) = 0. \tag{30}$$

It can be shown that the Jacobian matrix of $(\bar{x}, \bar{y}, \bar{y}')$ with respect to (x, y, y') is nonsingular since it is assumed that $\Delta \neq 0$ and moreover $\Phi_{xx} + 2y'\Phi_{xy} + (y')^2\Phi_{yy} + f\Phi_y = 0$. There is a similar equation in $\bar{f}$ of course.

Consider now everything expressed in terms of the unbarred coordinates. By taking the exterior derivative of the equation $\Phi = 0$ and expressing dy and $d\bar{y}$ in terms of ϑ and $\bar{\vartheta}$, dx and $d\bar{x}$, we obtain that

$$(\Phi_x + y'\Phi_y)\,dx + \Phi_y\vartheta + (\Phi_{\bar{x}} + \bar{y}'\Phi_{\bar{y}})\,d\bar{x} + \Phi_{\bar{y}}\bar{\vartheta} = 0, \tag{31}$$

so there is the relation

$$\bar{\vartheta} = -\frac{\Phi_y}{\Phi_{\bar{y}}}\,\vartheta. \tag{32}$$

In a similar way, the exterior derivative of $\Phi_x + y'\Phi = 0$ is worked out and the differentials dy, $d\bar{y}$ are replaced yielding an expression in terms of $dx, d\bar{x}, \vartheta, \bar{\vartheta}$ and φ,

$$\begin{aligned} d(\Phi_x + y'\Phi_y) &= \Phi_{xx}\,dx + \Phi_{xy}\,dy + \Phi_{\bar{x}x}\,d\bar{x} + \Phi_{\bar{y}x}\,d\bar{y} + dy'\,\Phi_y + y'(\Phi_{xy}\,dx + \Phi_{yy}\,dy + \Phi_{\bar{x}y}\,d\bar{x} + \Phi_{\bar{y}y}) \\ &= (\Phi_{xx} + 2y'\Phi_{xy} + (y')^2\Phi_{yy} + f\Phi_y)\,dx + (\Phi_{x\bar{x}} + y'\Phi_{y\bar{x}} + y'\Phi_{x\bar{y}} + y'\bar{y}'\Phi_{y\bar{y}})\,d\bar{x} \\ &\quad + (\Phi_{xy} + y'\Phi_{yy})\vartheta + (\Phi_{x\bar{y}} + y'\Phi_{y\bar{y}})\bar{\vartheta} + \Phi_y\varphi = 0. \end{aligned} \tag{33}$$

Thus,

$$d\bar{x} = (\frac{\Phi_y^2\Phi_{\bar{y}}}{\Delta})\varphi, \qquad \mod \vartheta.$$

As Δ is unchanged when barred and unbarred quantities are interchanged, dx can be expressed in a similar way,

$$dx = \frac{\Phi_y\bar{\Phi}_{\bar{y}}^2}{\Delta}\bar{\varphi}.$$

Thus to briefly summarize,

$$d\bar{x} = \frac{\Phi_y^2\Phi_{\bar{y}}}{\Delta}\varphi, \qquad \bar{\vartheta} = -\frac{\Phi_y}{\Phi_{\bar{y}}}\vartheta, \qquad \bar{\vartheta} = \frac{\Delta}{\Phi_y\Phi_{\bar{y}}}\,dx \qquad \mod \vartheta. \tag{34}$$

It is desired to reduce $\bar{\omega}$ to a form as close as possible to the standard one for a projective connection by means of a gauge transformation. This should involve the interchange of the positions of the dx and $k\varphi + m\vartheta$ terms in a matrix of the form (13). However, for normal projective connections we have $k = 1$. From the formulas of a gauge transformation derived before, this is impossible as it would require taking A, B and C the diagonal entries in the gauge transformation matrix to satisfy inconsistent equations. For definiteness we suppose that

$$\frac{A}{C} = \frac{\Phi_{\bar{y}}}{\Phi_y}, \qquad \frac{A}{B} = \frac{\Delta}{\Phi_y^2 \Phi_{\bar{y}}}, \qquad \frac{B}{C} = \frac{\Phi_y \Phi_{\bar{y}}^2}{\Delta}. \tag{35}$$

The other coefficients of the gauge transformation are chosen so that the ϑ component of $\bar{\omega}_2^1$ is eliminated as well as the dx and ϑ components of $\bar{\omega}_2^2$. There is then a unique gauge transformation matrix h such that it has a lower triangular form,

$$h^{-1} \bar{\omega} h + h^{-1}\, dh = \begin{pmatrix} \varphi + m\vartheta & & \\ \vartheta & dx & \kappa\varphi \end{pmatrix} \tag{36}$$

Given a trace-free matrix valued one-form ϖ with $\varpi^2 = -\vartheta \ \varpi_1^2 = dx$ and ϖ a multiple of φ, the remaining elements of ϖ are uniquely determined by the following conditions on its curvature form Π: it must be strictly upper-triangular and Π_2^1 is a multiple of $dx \wedge \vartheta$. The strategy is to compute the componets of Π and look at the consequences of taking them to be zero. To calculate Π, we have to calculate

$$\begin{pmatrix} \tilde{\omega}_0^0 & \tilde{\omega}_1^0 & \tilde{\omega}_2^0 \\ \tilde{\omega}_0^1 & \tilde{\omega}_1^1 & \tilde{\omega}_2^1 \\ \tilde{\omega}_0^2 & \tilde{\omega}_1^2 & \tilde{\omega}_2^2 \end{pmatrix} \wedge \begin{pmatrix} \tilde{\omega}_0^0 & \tilde{\omega}_1^0 & \tilde{\omega}_2^0 \\ \tilde{\omega}_0^1 & \tilde{\omega}_1^1 & \tilde{\omega}_2^1 \\ \tilde{\omega}_0^2 & \tilde{\omega}_1^2 & \tilde{\omega}_2^2 \end{pmatrix}$$

Suppose μ, λ, ν are functions, then we note that

$$d\nu = \Gamma(\nu)\, dx + \nu_y \vartheta + \nu_{y'}\, \varphi, \tag{37}$$

and moreover,

$$d\vartheta = -\varphi \wedge dx, \quad d\varphi = -df \wedge dx = -(f_y \vartheta + f_{y'} \varphi) \wedge dx. \tag{38}$$

The notation for the components of ϖ and Π follows the same system as for Ω. First we have

$$\Pi_1^2 = d\varpi_1^2 + \varpi^2 \wedge \varpi_1^0 + \varpi_1^2 \wedge \varpi_1^1 + \varpi_2^2 \wedge \varpi_1^2 = -\vartheta \wedge \varpi_1^0 + dx \wedge (\varpi_1^1 \wedge \varpi_2^2). \tag{39}$$

This will vanish provided that,

$$\varpi_1^0 = \lambda \wedge dx, \qquad \varpi_1^1 = \varpi_2^2 + \lambda_0\, dx - \lambda\, \vartheta. \tag{40}$$

As noted λ and μ are functions which are arbitrary at first. The next component is

$$\Pi^2 = d\varpi^2 + \varpi^2 \wedge \varpi_0^0 + \varpi_1^2 \wedge \varpi^1 + \varpi_2^2 \wedge \varpi^2 = (\varphi - \varpi^1) \wedge dx - \vartheta \wedge (\varpi_0^0 - \varpi_2^2). \tag{41}$$

This is zero if and only if the coefficient of φ in ϖ^1 is one and also

$$\varpi_0^0 = \varpi_2^2 - \mu\, dx + \nu\vartheta, \qquad \varpi^1 = \varpi + \mu\vartheta. \tag{42}$$

The trace of the matrix of forms ϖ is to be zero. This constraint implies that

$$\varpi_0^0 + \varpi_1^1 + \varpi_2^2 = \varpi_2^2 - \mu\, dx + \nu\vartheta + \varpi_2^2 + \lambda_0\, dx - \lambda\vartheta + \varpi_2^2 = (\lambda - \mu)\, dx + (\nu - \lambda)\,\vartheta + 3\varpi_2^2$$

$$= (\lambda_0 - \mu)\, dx + (\nu - \lambda)\,\vartheta + 3\kappa\varphi. \tag{43}$$

Since dx, ϑ, and φ are independent it follows that

$$\kappa = 0, \qquad \lambda_0 = \mu, \qquad \lambda = \nu, \qquad \varpi_2^2 = 0, \qquad \varpi_0^0 = -\varpi_1^1 = -\mu\, dx + \nu\vartheta. \tag{44}$$

The next component of Π to consider is

$$\Pi^1 = d\varpi^1 + \varpi^1 \wedge \varpi_0^2 + \varpi_1^1 \wedge \varpi^1 + \varpi_2^1 \wedge \varpi^2$$

$$= d(\varphi+\mu\vartheta)+\varpi^1\wedge(\varpi_0^0-\varpi_1^1)+\varpi_2^1\wedge\varpi^2 = d\varphi+d\mu\wedge\vartheta+\mu d\vartheta+2(\varphi+\mu\vartheta)\wedge(-\mu\, dx+\nu\vartheta)-\varpi_2^1\wedge\vartheta$$

$$= d\varphi + d\mu \wedge \vartheta - \mu\,\varphi \wedge dx + 2(\varphi + \mu\vartheta) \wedge (-\mu dx + \nu\vartheta) - \varpi_2^1 \wedge \vartheta. \tag{45}$$

The differentials $d\mu$ and df are given by

$$d\mu = \Gamma(\mu)\, dx + \mu_y\vartheta + \mu_{y'}\,\varphi, \qquad df = f_x\, dx + f_y(\vartheta + y'\, dx) + f_{y'}(\varphi + f\, dx). \tag{46}$$

Consequently, Π^1 is

$$\Pi^1 = -(f_y(\vartheta + y'\, dx) + f_{y'}(\varphi + f\, dx)) \wedge dx + (\Gamma(\mu)dx + \mu_y\vartheta + \mu_{y'}\varphi) \wedge \vartheta$$

$$-\mu\varphi \wedge dx - 2\mu\varphi \wedge dx + 2\nu\varphi \wedge \vartheta - 2\mu^2\vartheta \wedge dx - \varpi_2^1 \wedge \vartheta \tag{47}$$

$$= (f_{y'} + 3\mu)\, dx \wedge \varphi + (f_y + \Gamma(\mu) + 2\mu^2)\, dx \wedge \vartheta + (\mu_{y'} + 2\nu)\varphi \wedge \vartheta - \varpi_2^1 \wedge \vartheta.$$

Necessary and sufficient conditions for Π^1 to be zero are the following

$$\mu = -\frac{1}{3}\, f_{y'}, \qquad \varpi_2^1 = (f_y + \frac{2}{9}f_{y'}^2 - \frac{1}{3}\Gamma(f_{y'}))\, dx + (2\nu - \frac{1}{3}f_{y'y'})\varphi + \rho\vartheta. \tag{48}$$

The second diagonal element of Π is

$$\Pi_2^2 = \varpi_2 \wedge \varpi_2^0 + \varpi_1^2 \wedge \varpi_2^1 = -\vartheta \wedge \varpi_2^0 + dx \wedge \varpi_2^1 = (\varpi_2^0 + \rho\, dx) \wedge \vartheta + (2\nu - \frac{1}{3}f_{y'y'})dx \wedge \varphi. \tag{49}$$

The component Π_2^2 vanishes if and only if

$$\varpi_2^0 = -\rho\, dx + \sigma\vartheta, \qquad \nu = \frac{1}{6}f_{y'y'}. \tag{50}$$

Notice that the latter condition makes the φ component vanish in ϖ_2^1.

The next component to consider is Π^0_0 which is given by

$$\Pi^0_0 = d\varpi^0_0 + \varpi^0_1 \wedge \varpi^1 + \varpi^0_2 \wedge \varpi^2 = d\varpi^0_0 + \varpi^0_1 \wedge \varpi^1 + \varpi^0_2 \wedge \vartheta. \tag{51}$$

Using the fact that

$$df_{y'y'} = \Gamma(f_{y'y'})\, dx + f_{y'y'y}\, \vartheta + f_{y'y'y'}\, \varphi, \tag{52}$$

The element Π^0_0 is explicitly calculated to be

$$\Pi^0_0 = d(\frac{1}{3} f_{y'}\, dx + \frac{1}{6} f_{y'y'}\, \vartheta) + (\frac{1}{6} f_{y'y'}\, dx + \lambda' \vartheta) \wedge (\varphi - \frac{1}{3} f_{y'} \vartheta) + \rho\, dx \wedge \vartheta$$

$$= \frac{1}{3}(f_{yy'} \vartheta + f_{y'y'}\, \varphi) \wedge dx + \frac{1}{6}(\Gamma(f_{y'y'}) dx + f_{y'y'y} \vartheta + f_{y'y'y'} \varphi) \wedge \vartheta - \frac{1}{6} f_{y'y'}\, \varphi + \frac{1}{6} f_{y'y'}\, dx \wedge \varphi$$

$$-\frac{1}{18} f_{y'} f_{y'y'}\, dx \wedge \vartheta + \lambda' \vartheta \wedge \varphi + \rho\, dx \wedge \vartheta \tag{53}$$

$$= (\rho - \frac{1}{3} f_{y'y} - \frac{1}{18}\, f_{y'} f_{y'y'} + \frac{1}{6} \Gamma(f_{y'y'}))\, dx \wedge \vartheta + (\lambda' - \frac{1}{6} f_{y'y'y'})\, \vartheta \wedge \varphi.$$

For Π^0_0 to vanish, it is required that

$$\rho = \frac{1}{3} f_{yy'} + \frac{1}{18} f_{y'} f_{y'y'} - \frac{1}{6} \Gamma(f_{y'y'})) \quad \lambda' = \frac{1}{6}\, f_{y'y'y'}. \tag{54}$$

When these conditions hold, the component Π^1_1 will also be zero since it is required that the trace of Π vanish.

All of ϖ has been fixed, but with the exception of the coefficient σ in ϖ^0_2. This is determined by imposing that Π^1_2 be a multiple of $dx \wedge \vartheta$ hence semi-basic. Since

$$\frac{\partial}{\partial y'}\big(f_y + \frac{2}{9} f^2_{y'} - \frac{1}{3}(\frac{\partial f_{y'}}{\partial x} + y' \frac{\partial f_{y'}}{\partial y} + f \frac{\partial f_{y'}}{\partial y'})\big) = f_{yy'} + \frac{4}{9} f_{y'} f_{y'y} - \frac{1}{3} \Gamma(f_{y'y'}) - \frac{1}{3} f_{yy'} - \frac{1}{3} f_{y'} f_{y'y'}$$

$$= \frac{2}{3} f_{yy} + \frac{1}{9} f_{y'} f_{y'y'} - \frac{1}{3} \Gamma(f_{y'y'}) = 2\rho. \tag{55}$$

Using this, we find that $\partial/\partial y' \rfloor \Pi^1_2 = (\rho_{y'} + \sigma)\vartheta$, so Π^1_2 is semi-basic if and only if $\sigma = -\rho_{y'}$.

This completes the determination of ϖ. In fact, ϖ is given in terms of the first normal projective connection by

$$\varpi^0_1 = -\omega^2_2, \quad \varpi^0_1 = \omega^1_2 \quad \varpi^0_2 = -\omega^0_0 \quad \varpi^1 = \omega^2_1 \quad \varpi^1_1 = -\omega^1_1 \quad \varpi^1_2 = \omega^0_1, \quad \varpi^2 = -\omega^2 \quad \varpi^2_1 = \omega^1 \quad \varpi^2_2 = -\omega^0_0. \tag{56}$$

This can be summarized concisely in matrix form as follows

$$\varpi = -K \omega^T K, \tag{57}$$

where ω^T is the transpose of ω and matrix K is defined to be

$$K = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{58}$$

It may be verified that when ϖ and ω are related as in (57), their curvatures Π and Ω are related in the same way. The minus signs are important for this to hold. There would be no obvious relation between the components if that were the case. The crucial point is that the mapping $M \to -KM^TK = M'$ is a homomorphism of the matrix Lie algebra so in fact

$$[M_1', M_2'] = K[M_1^T, M_2^T]K = -K[M_1, M_2]^T K = [M_1, M_2]'. \tag{59}$$

This holds for any K such that K^2 is the identity. Without the minus sign an antihomomorphism results instead and then

$$d\omega' + \frac{1}{2}\,[\omega' \wedge \omega'] = d\omega' + \frac{1}{2}\,[\omega \wedge \omega]'. \tag{60}$$

If Ω has the upper triangular form,

$$\begin{pmatrix} 0 & b\,dx \wedge \vartheta & \Omega_2^0 \\ 0 & 0 & a\vartheta \wedge \varphi \\ 0 & 0 & 0 \end{pmatrix} \tag{61}$$

Then the fact that $\Pi = -K\Omega^T K$ implies that Π is given as

$$\Pi = -K\Omega^T K = \begin{pmatrix} 0 & \vartheta \wedge \varphi & -\Omega_2^0 \\ 0 & 0 & b\,dx \wedge \vartheta \\ 0 & 0 & 0 \end{pmatrix}. \tag{62}$$

The gauged version of the second normal projective connection $h^{-1}\bar{\omega} h + h^{-1}\,dh$ can be reconsidered. By calculation,

$$h^{-1}\bar{\Omega} h = \begin{pmatrix} 0 & A^{-1}B\bar{b}\,d\bar{x} \wedge \bar{\vartheta} & A^{-1}E\bar{b}d\bar{x} \wedge \bar{\vartheta} + A^{-1}C\bar{\Omega}_2^0 + (DE - BF)C\bar{a}\bar{\vartheta} \wedge \bar{\varphi} \\ 0 & 0 & B^{-1}C\bar{a}\bar{\vartheta} \wedge \bar{\varphi} \\ 0 & 0 & 0 \end{pmatrix} \tag{63}$$

Thus the gauge transformed version $h^{-1}\bar{\omega} h + h^{-1}\,dh$ satisfies the conditions that uniquely determine ϖ and therefore must be $\varpi = h^{-1}\bar{\omega} h + h^{-1}\,dh$.

4. Duality

The condition $\Delta \neq 0$ inposed previously may be thought of in another way. It states that the one-form $\theta = \Phi_x\,dx + \Phi_y\,dy = -(\Phi_{\bar{x}}\,d\bar{x} + \Phi_{\bar{y}}\,d\bar{y})$ satisfies the condition $\theta \wedge d\theta$ on S. It may be said it defines a contact structure on this three-dimensional manifold.

Let S be a three-dimensional manifold endowed with a contact structure which it may be convenient to think of as a two-dimensional distribution $\mathcal{D}$ which is nonintegrable in the following sense. For any pair of linearly independent vector fields $X, Y \in \mathcal{D}$, $[X, Y] \notin \mathcal{D}$. Any one-form θ on S which is an annihilator of $\mathcal{D}$ satisfies the condition $\theta \wedge d\theta \neq 0$. Suppose a basis has been given for $\mathcal{D}$, and these basis vectors are denoted as X, $\bar{X}$. This means that the set $\{X, \bar{X}, [X, \bar{X}]\}$ is a basis for vector fields on S. Let the set $\{\phi, \bar{\phi}, \theta\}$ be the dual basis of one-forms.

In this case, we would take X to be tangent to one of the fibers of the double fibration of S and $\bar{X}$ to the other. Then D would be the distribution spanned by $X, \bar{X}$ and θ would be a scalar multiple of $\Phi_x\, dx + \Phi_y\, dy$. The purpose of this discussion is to reexamine the the effect of the Cartan connection form of interchanging the roles of the fibrations while treating them on an equal footing. This is accomplished by working in terms of the dual basis just proposed. When X and $\bar{X}$ are interchanged, the new basis of one-forms becomes $\{\bar{\phi}, \phi, -\theta\}$. For the normal Cartan projective connection already described,

$$X = \Gamma, \qquad \bar{X} = \frac{\partial}{\partial y}. \tag{64}$$

It is then possible to calculate the following bracket,

$$[X, \bar{X}] = -\frac{\partial}{\partial y} - f_{y'} \frac{\partial}{\partial y'}. \tag{65}$$

The one-forms which reside in the lower triangular portion of the connection matrix dx, $\phi - \frac{1}{3} f_{y'} \theta$ and θ are not dual to the basis of vector fields. This is the main difference between what has been discussed and what comes next.

The dual one-form basis is actually given as $\{dx, \phi - f_y d\theta, -\theta\}$ The new approach requires us to take the Cartan connection form in the lower triangular part of the connection form matrix as

$$\begin{pmatrix} & & \\ \phi & & \\ -\theta & \bar{\phi} & \end{pmatrix} \tag{66}$$

In the case just discussed, this would be gauge-equivalent to the version used previously. It is to be emphasized that now X may be any vector field tangent to the first fibration and $\bar{X}$ any vector field tangent to the second. Thus in the present version of the theory, transformations of the form $X \to \lambda X$ and $\bar{X} \to \bar{\lambda} \bar{X}$ for any nonvanishing functions λ, $\bar{\lambda}$, will be allowed: such transformations induce gauge transformations of the kind discussed previously, with coefficients given in terms of $\lambda, \bar{\lambda}$ and their derivatives.

It is a consequence of these definitions that the exterior derivatives of the basis one-forms can be written as

$$d\phi = \psi \wedge \theta, \quad d\bar{\phi} = -\bar{\psi} \wedge \theta, \quad d\theta = -\phi \wedge \bar{\phi} + \chi \wedge \bar{\theta}, \tag{67}$$

where ψ, $\bar{\psi}$ and χ are certain one-forms which are linear combinations of ϕ and $\bar{\phi}$.

Changing the notation proves to be useful, so the connection form has the structure,

$$\omega = \begin{pmatrix} \alpha & \beta & \gamma \\ \phi & -\alpha - \alpha' & \beta' \\ -\theta & \bar{\phi} & \alpha' \end{pmatrix} \tag{68}$$

It is assumed that X and $\bar{X}$ and hence $\phi, \bar{\phi}$ and θ have been fixed. Therefore, the only remaining gauge freedom is the one coming from a gauge transformation of the form

$$h = \begin{pmatrix} 1 & 0 & F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h^{-1} = \begin{pmatrix} 1 & 0 & -F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{69}$$

For such a choice of h, it is determined that

$$h^{-1}\omega h + h^{-1}\, dh$$

$$= \begin{pmatrix} \alpha - F\theta & \beta - F\bar{\phi} & F(\alpha - F\theta) \\ \phi & -\alpha - \alpha' & F\phi - \beta' \\ \theta & \bar{\phi} & F\theta + \alpha' \end{pmatrix} \tag{70}$$

By pursuing an argument similar to the one used before, that $\alpha, \alpha', \beta, \beta'$ and γ are uniquely determined in terms of $\psi, \bar{\psi}$ and χ and their derivatives by the requirement that the curvature Ω of ω takes the form,

$$\Omega = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & B' \\ 0 & 0 & 0 \end{pmatrix} \tag{71}$$

with B a multiple of $\phi \wedge \theta$. The constraints that require the torsion to vanish amount to the following system:

$$d\phi - (2\alpha + \alpha') \wedge \phi - \beta' \wedge \theta = 0, \qquad d\bar{\phi} + (\alpha + 2\alpha') \wedge \bar{\gamma} + \beta \wedge \theta = 0, \qquad d\theta - (\alpha - \alpha') \wedge \theta + \phi \wedge \bar{\phi} = 0. \tag{72}$$

It follows from the last of these together with the gauge-fixing assumption that $\alpha - \alpha' = \chi$. The first two equations in (72) determine the ϕ and $\bar{\phi}$ components of α. Therefore α' is in terms of ψ, $\bar{\psi}$ and χ with the ϕ, $\bar{\phi}$ components of β and β' in terms of α and α'. The conditions that the diagonal elements of Ω must vanish results in the pair

$$d\alpha + \beta \wedge \phi + \gamma \wedge \theta = 0, \qquad d\alpha' - \beta' \wedge \bar{\phi} + \gamma \wedge \theta = 0. \tag{73}$$

This are equivalent under linear combinations to the following,

$$d(\alpha + \alpha') + \beta \wedge \phi - \beta' \wedge \bar{\phi} = 0, \qquad d(\alpha - \alpha') + \beta \wedge \phi + \beta' \wedge \bar{\phi} = 2\gamma \wedge \theta. \tag{74}$$

The $\phi \wedge \bar{\phi}$ component of these determines the θ component of $\alpha + \alpha'$ and therefore of α and α' since they have the same θ component. The remaining components determine the θ components of β and β'. The $\phi \wedge \bar{\phi}$ component of the second equation is satisfied identically, and the other two components yield the ϕ and $\bar{\phi}$ components of γ. Given that ω has the form (68) the curvature form Ω is given by

$$\begin{pmatrix} d\alpha & d\beta & d\gamma \\ d\phi & -d(\alpha + \alpha') & d\beta' \\ -d\theta & d\bar{\phi} & d\alpha' \end{pmatrix}$$

$$+ \begin{pmatrix} \beta \wedge \phi - \gamma \wedge \theta & \alpha \wedge \beta - \beta \wedge (\alpha + \alpha') + \gamma \wedge \bar{\phi} & \alpha \wedge \gamma + \beta \wedge \beta' + \gamma \wedge \alpha' \\ \phi \wedge \alpha - (\alpha + \alpha') \wedge \phi - \beta' \wedge \theta & \phi \wedge \beta + \beta' \wedge \bar{\phi} & \phi \wedge \gamma - (\alpha + \alpha') \wedge \beta' + \beta' \wedge \alpha' \\ -\theta \wedge \alpha + \bar{\phi} \wedge \phi - \alpha' \wedge \theta & -\theta \wedge \beta - \bar{\phi} \wedge (\alpha + \alpha') + \alpha' \wedge \phi & -\theta \wedge \beta - \bar{\phi} \wedge (\alpha + \alpha') + \alpha' \wedge \phi \end{pmatrix} \tag{75}$$

It is required that the curvature have the following structure

$$\Omega = \begin{pmatrix} 0 & B & 0 \\ 0 & 0 & B' \\ 0 & 0 & 0 \end{pmatrix} \tag{76}$$

The requirement that the lower left entries vanish gives rise to the following three equations

$$d\phi + \phi \wedge (2\alpha \wedge \alpha') - \beta' \wedge \theta = 0, \quad d\bar{\phi} + (\alpha + 2\alpha') \wedge \bar{\phi} + \beta \wedge \theta = 0, \quad d\theta - (\alpha - \alpha') \wedge \theta + \phi \wedge \bar{\phi} = 0. \tag{77}$$

These may be referred to as the torsion equations.

The diagonal elements of the curvature Ω yield the constraints

$$d\phi + \beta \wedge \phi - \gamma \wedge \theta = 0, \quad d\alpha' - \theta \wedge \gamma + \bar{\phi} \wedge \beta' = 0, \quad -d(\alpha + \alpha') + \phi \wedge \beta + \beta' \wedge \phi = 0, \tag{78}$$

Adding together the first two in (78) the negative of the third one results and everything is consistent. As well,

$$d(\alpha - \alpha') + \beta \wedge \phi - \bar{\phi} \wedge \beta' = 2\gamma \wedge \theta. \tag{79}$$

It follows from their basic definitions that the exterior derivatives of the basic one-forms can be expressed as

$$d\phi = \psi \wedge \theta, \qquad d\bar{\phi} = -\bar{\psi} \wedge \theta, \qquad d\theta = -\phi \wedge \bar{\phi} + \chi \wedge \theta. \tag{80}$$

The one-forms ψ, $\bar{\psi}$ and χ are linear combinations of ϕ and $\bar{\phi}$.

Substituting these derivatives into the torsion equations, the following are obtained

$$(2\alpha + \alpha') \wedge \phi + (\beta' - \psi) \wedge \theta = 0, \qquad (\alpha + 2\alpha') \wedge \bar{\phi} + (\beta - \bar{\phi}) \wedge \theta = 0, \qquad (\chi - (\alpha - \alpha')) \wedge \theta = 0. \tag{81}$$

From the first two of these, $2\alpha + \alpha'$ can be taken proportional to ϕ, $\beta' - \psi$ proportional to θ, $\alpha + 2\alpha'$ proportional to $\bar{\phi}$ and $\beta - \bar{\psi}$ proportional to θ, hence

$$2\alpha + \alpha' = c_1 \phi, \qquad \alpha + 2\alpha' = c_2 \bar{\phi}. \tag{82}$$

Taking the constants $c_1 = c_2 = 1$, these imply that

$$\alpha = \frac{2}{3}\phi - \frac{1}{3}\bar{\phi}, \qquad \alpha' = -\frac{1}{3}\phi + \frac{2}{3}\bar{\phi}. \tag{83}$$

The third constraint in (81) can be satisfied by taking $\chi - (\alpha - \alpha') = 0$ independent of θ as a gauge condition so

$$\chi = \alpha - \alpha' = \phi - \bar{\phi}. \tag{84}$$

Using (80), it is found that this implies that

$$d\chi = d(\alpha - \alpha') = d\phi - d\bar{\phi} = (\psi - \bar{\psi}) \wedge \theta. \tag{85}$$

It makes sense to take $\beta' = \psi$ and $\beta = \bar{\psi}$ in which case, the other two constraints are satisfied.

Returning to Ω in (74) the components B and B' are given as

$$B = d\beta + (2\alpha + \alpha') \wedge \beta + \gamma \wedge \bar{\phi}, \qquad B' = d\beta' - (\alpha - 2\alpha') \wedge \beta' + \phi \wedge \gamma. \tag{86}$$

Upon differentiating the equation $d\bar{\phi} + (\alpha + 2\alpha') \wedge \bar{\phi} + \beta \wedge \theta = 0$ it follows that

$$d(\beta \wedge \theta) + d((\alpha + 2\alpha') \wedge \bar{\phi}) = 0,$$

$$d\beta \wedge \theta - \beta \wedge d\theta + d(\alpha + 2\alpha') \wedge \bar{\phi} - (\alpha - 2\alpha') \wedge d\bar{\phi} = 0,$$

$$d\beta \wedge \theta - \beta \wedge d\theta + (-\beta \wedge \phi + \gamma \wedge \theta + 2\beta' \wedge \bar{\phi} - 2\gamma \wedge \theta) \wedge \bar{\phi} - (\alpha + 2\alpha') \wedge d\bar{\phi} = 0,$$

$$d\beta \wedge \theta - \beta \wedge d\theta - \beta \wedge \phi \wedge \bar{\phi} - \gamma \wedge \theta \wedge \bar{\phi} - (\alpha + 2\alpha') \wedge d\bar{\phi} = 0, \tag{87}$$

$$(d\beta + (2\alpha + \alpha') \wedge \beta + \gamma \wedge \bar{\phi}) \wedge \theta = 0.$$

The expression which remains inside the brackets is exactly B, so this result can be written more concisely as

$$B \wedge \theta = 0. \tag{88}$$

Differentiate exteriorly $d\phi = \beta' \wedge \theta - \phi \wedge (2\alpha + \alpha)$ to obtain

$$d\beta' \wedge \theta - \beta' \wedge \theta - d\phi \wedge (2\alpha + \alpha') + \phi \wedge (-2\beta \wedge \phi + 2\gamma \wedge \theta + \theta \wedge \gamma - \bar{\phi} \wedge \beta') = 0.$$

$$d\beta' \wedge \theta - \beta' \wedge (d\theta + \theta \wedge (2\alpha + \alpha') + \phi \wedge \bar{\phi}) + \phi \wedge \gamma \wedge \theta = 0.$$

Finally after simplifying, we have,

$$(d\beta' + \beta' \wedge (\alpha + 2\alpha') + \phi \wedge \gamma) \wedge \theta = 0. \tag{89}$$

The form inside the bracket is exactly B', so (89) is just the equation

$$B' \wedge \theta = 0. \tag{90}$$

Suppose B is a scalar multiple of $\phi \wedge \theta$ so

$$B = C\, \phi \wedge \theta = d\beta + (2\alpha + \alpha') \wedge \beta + \gamma \wedge \bar{\phi}. \tag{91}$$

This holds if and only if

$$d\beta \wedge \phi + (2\alpha + \alpha') \wedge \beta \wedge \phi = \gamma \wedge \phi \wedge \bar{\phi}. \tag{92}$$

However, $d\beta \wedge \phi = \beta \wedge d\phi + d(\gamma \wedge \theta)$ and so consequently,

$$d\beta \wedge \phi + (2\alpha + \alpha') \wedge \beta \wedge \phi = \beta \wedge d\phi + d(\gamma \wedge \theta) + (2\alpha + \alpha') \wedge \beta \wedge \phi = \beta \wedge \beta' \wedge \theta + d(\gamma \wedge \theta). \tag{93}$$

Since B' is a multiple of $\bar{\phi} \wedge \theta$, or explicitly,

$$B' = \tilde{C}\, \bar{\phi} \wedge \theta = d\beta' - (\alpha + 2\alpha') \wedge \beta' - \gamma \wedge \phi. \tag{94}$$

This occurs if and only if

$$(d\beta' - (\alpha + 2\alpha') \wedge \beta') \wedge \bar{\phi} = \gamma \wedge \phi \wedge \bar{\phi}. \tag{95}$$

Since $d\beta' \wedge \bar{\phi} = \beta' \wedge d\bar{\phi} + d(\gamma \wedge \theta)$ we have

$$\beta' \wedge d\bar{\phi} + d(\gamma \wedge \theta) - (\alpha + 2\alpha') \wedge \beta' \wedge \bar{\phi} = \gamma \wedge \phi \wedge \bar{\phi}. \tag{96}$$

Therefore, it follows that

$$\begin{aligned} d\beta' \wedge \phi - (\alpha + 2\alpha') \wedge \beta \wedge \bar{\phi} &= -\beta' \wedge (\alpha + 2\alpha') \wedge \bar{\phi} - \beta' \wedge \beta \wedge \theta + d(\gamma \wedge \theta) - (\alpha + 2\alpha') \wedge \beta' \wedge \bar{\phi} \\ &= -\beta' \wedge \beta \wedge \theta + d(\gamma \wedge \theta). \end{aligned} \tag{97}$$

The same condition arises both ways

$$\beta \wedge \beta' \wedge \theta + d(\gamma \wedge \theta) = \gamma \wedge \phi \wedge \bar{\phi}. \tag{98}$$

Therefore, it holds that $B \wedge \phi = 0$ if and only if $B' \wedge \bar{\phi} = 0$, so the condition that B be a multiple of $\phi \wedge \theta$ and B' a multiple of $\bar{\phi} \wedge \theta$ are the same.

Finally, let us discuss the effects of the duality transformation $X \to \bar{X}$ and $\bar{X} \to X$. This is supposed to suggest the idea that this action look like complex conjugation so θ can be thought of as purely imaginary and χ as real. The dual connection form is $\bar{\omega}$ and takes the form

$$\bar{\omega} = \begin{pmatrix} \bar{\alpha} & \bar{\beta} & \bar{\gamma} \\ \bar{\phi} & -\bar{\alpha} - \bar{\alpha}' & \bar{\beta}' \\ \theta & \phi & \bar{\alpha}' \end{pmatrix} \tag{99}$$

It is assumed to be gauged so that $\bar{\alpha} - \bar{\alpha}'$ is independent of θ as before. Using (99) the curvature form is found to be

$$\begin{pmatrix} d\bar{\alpha} & d\bar{\beta} & d\bar{\gamma} \\ d\bar{\phi} & -d(\bar{\alpha} + \bar{\alpha}') & d\bar{\beta}' \\ d\theta & d\phi & d\bar{\alpha}' \end{pmatrix}$$

$$+ \begin{pmatrix} \bar{\beta} \wedge \bar{\phi} + \bar{\gamma} \wedge \theta & \bar{\alpha} \wedge \bar{\beta} - \bar{\beta} - \bar{\beta} \wedge (\bar{\alpha} + \bar{\alpha}') + \bar{\gamma} \wedge \phi & \bar{\alpha} \wedge \bar{\gamma} + \bar{\beta} \wedge \bar{\beta}' + \bar{\gamma}' \wedge \bar{\alpha}' \\ \bar{\phi} \wedge \bar{\alpha} - (\bar{\alpha} + \bar{\alpha}') \wedge \phi + \bar{\beta}' \wedge \theta & \bar{\phi} \wedge \bar{\beta} + \bar{\beta}' \wedge \phi & \bar{\phi} \wedge \bar{\gamma} - (\bar{\alpha} + \bar{\alpha}') \wedge \beta + \bar{\beta}' \wedge \bar{\alpha}' \\ \theta \wedge \bar{\alpha} + \phi \wedge \bar{\phi} + \bar{\alpha}' \wedge \theta & \theta \wedge \bar{\beta} - \phi \wedge (\bar{\alpha} + \bar{\alpha}') + \bar{\alpha}' \wedge \phi & \theta \wedge \bar{\gamma} + \phi \wedge \bar{\beta}' \end{pmatrix} \tag{100}$$

The connection is then uniquely determined by the conditions for the torsion to vanish

$$d\bar{\phi} + \bar{\phi} \wedge \bar{\alpha} \wedge \bar{\phi} + \bar{\beta}' \wedge \theta = 0, \qquad d\phi + \theta \wedge \bar{\beta} - \phi \wedge (\bar{\alpha} + 2\bar{\alpha}') = 0, \qquad d\theta + \theta \wedge \bar{\alpha} + \phi \wedge \bar{\phi} + \bar{\alpha}' \wedge \theta = 0. \tag{101}$$

Vanishing of the diagonal elements of $\bar{\Omega}$ gives

$$d\bar{\alpha} + \bar{\beta} \wedge \bar{\phi} + \bar{\gamma} \wedge \theta = 0, \qquad d\alpha' - \beta' \wedge \phi - \gamma \wedge \theta = 0. \tag{102}$$

At last, we have

$$\bar{\gamma} \wedge \phi \wedge \bar{\phi} = -(d\bar{\beta} + (2\bar{\alpha} + \bar{\alpha}') \wedge \bar{\phi} = -(d\bar{\beta}' - (\bar{\alpha} + 2\bar{\alpha}') \wedge \beta') \wedge \phi. \tag{103}$$

This is just $\bar{B} \wedge \bar{\phi} = 0$ or equivalently, $\bar{B}' \wedge \phi = 0$.

Subtracting the two expressions $d\theta - (\alpha - \alpha') \wedge \theta + \phi \wedge \bar{\phi} = 0$ and $d\theta - (\bar{\alpha} - \bar{\alpha}') \wedge \theta + \phi \wedge \bar{\phi} = 0$, it is found that

$$(\alpha - \alpha') \wedge \theta = (\bar{\alpha} - \bar{\alpha}') \wedge \theta. \tag{104}$$

This implies that $\alpha - \alpha' = \bar{\alpha} - \bar{\alpha}'$ which is equivalent to $\alpha + \bar{\alpha}' = \bar{\alpha} + \alpha'$. Grouping from both sets, there are four torsion equations which remain, and these are

$$\begin{aligned} &d\phi - (2\alpha + \alpha') \wedge \phi - \beta' \wedge \theta = 0 \quad d\bar{\phi} + (\alpha + 2\alpha') \wedge \bar{\phi} + \beta \wedge \theta = 0 \\ &d\phi + (\bar{\alpha} + 2\bar{\alpha}') \wedge \phi - \bar{\beta} \wedge \theta = 0 \quad d\bar{\phi} - (2\bar{\alpha} + \bar{\alpha}') \wedge \bar{\phi} + \bar{\beta}' \wedge \theta = 0. \end{aligned} \tag{105}$$

Subtracting the equations in (105) pairwise, the following linear combinations are produced,

$$(2(\alpha + \bar{\alpha}') + \alpha' + \bar{\alpha}) \wedge \phi + (\beta' - \bar{\beta}) \wedge \theta = 0, \qquad (\alpha + \bar{\alpha}' + 2(\alpha' + \bar{\alpha})) \wedge \bar{\phi} + (\beta + \bar{\beta}') \wedge \theta = 0. \tag{106}$$

Forming the wedge product with θ, the results in (106) imply that

$$(2(\alpha + \bar{\alpha}') + \alpha' + \bar{\alpha}) \wedge \phi \wedge \theta = 0, \qquad (\alpha + \bar{\alpha}' + 2(\alpha' + \bar{\alpha})) \wedge \bar{\phi} \wedge \theta = 0. \tag{107}$$

Based on the fact that $\alpha + \alpha'$ is a multiple of θ we can write this as $\kappa\,\theta$, κ a scalar. Also reasoning in a similar way, $\alpha + \bar{\alpha}' = \kappa\theta$. Applying these new results in (106), it is found that

$$(\beta' - \beta) \wedge \theta = -3\kappa\theta \wedge \phi. \tag{108}$$

On the other hand , it follows that

$$d(\alpha' + \bar{\alpha}) = d\kappa \wedge \theta + \kappa(\phi \wedge \bar{\phi} - \chi \wedge \theta) = (\beta' - \beta) \wedge \bar{\phi} - (\gamma + \bar{\gamma}) \wedge \theta. \tag{109}$$

Taking the exterior product of this with θ we obtain $\kappa\,\phi \wedge \bar{\phi} \wedge \theta = 0$ which leads to $\kappa = 0$. Therefore, it is concluded that

$$\alpha' = -\bar{\alpha}, \qquad \bar{\alpha}' = -\alpha. \tag{110}$$

Thus $\beta' - \bar{\beta}$ is in fact a scalar multiple of θ, and similarly as well, so is $\beta - \bar{\beta}'$. Since $\alpha' + \bar{\alpha} = 0$, it is concluded that

$$(\beta' - \bar{\beta}) \wedge \bar{\phi} = (\gamma + \bar{\gamma}) \wedge \theta = (\beta - \bar{\beta}') \wedge \phi. \tag{111}$$

It follows that $\beta' = \bar{\beta}$ and $\bar{\beta}' = \beta$ with $\gamma \wedge \theta = -\bar{\gamma} \wedge \theta$. The conditions on B and B' yield the final conclusion,

$$\bar{\gamma} \wedge \phi \wedge \bar{\phi} = -(d\beta + \alpha' + 2\alpha) \wedge \beta) \wedge \phi = -\gamma \wedge \phi \wedge \bar{\phi}. \tag{112}$$

This implies that $\bar{\gamma} = -\gamma$. Finally, it follows that the form ω can be summarized in matrix form,

$$\omega = \begin{pmatrix} \alpha & \beta & \gamma \\ \phi & -(\alpha - \bar{\alpha}) & \bar{\beta} \\ -\theta & \bar{\phi} & -\bar{\alpha} \end{pmatrix} \tag{113}$$

where $\bar{\gamma} = -\gamma$ and $\bar{\omega}$ is calculated from ω through

$$\bar{\omega} = -K\omega^T K. \tag{114}$$

It may be concluded that $\Omega = -K\,\Omega^T K$, and it follows that $B' = \bar{B}$ and $\bar{B}' = B$ hold.

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