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Radon measure-valued solutions for nonlinear strongly degenerate parabolic equations with measure data

Quincy Stévène Nkombo[1,*], Fengquan Li[1]

[1] *School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, China*

Abstract. In this paper, we prove the existence of Radon measure-valued solutions for nonlinear strongly degenerate parabolic equations with nonnegative bounded Radon measure as initial data. Moreover, we show the uniqueness of the Radon measure-valued solutions when the Radon measure as a forcing term is diffuse with respect to the parabolic capacity and the Radon measure as a initial value is diffuse with respect to the Newtonian capacity. We also deduce that the concentrated part of the Radon measure-valued solution with respect to the Newtonian capacity depends on time.

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1. Introduction

In this work we address the nonhomogeneous nonlinear strongly degenerate parabolic equations having the nonnegative bounded Radon measure on the right-hand side with the nonnegative bounded Radon measure as initial data. This problem is described as follows

$$\begin{cases} u_t - \Delta\psi(u) = \mu & \text{in } \; Q := \Omega \times (0,T), \\ u = 0 & \text{on } \; \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \; \Omega, \end{cases} \tag{P}$$

where $T > 0$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is an open bounded domain with smooth boundary $\partial\Omega$, the initial value data u_0 is a nonnegative bounded Radon measure on Ω and μ is a nonnegative bounded Radon measure on Q.

The nonlinear strongly degenerate parabolic equations (P) is the special case derived

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[*] Corresponding author.

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Email addresses: `quincysnk@yahoo.fr` (Q. S. Nkombo), `fqli@dlut.edu.cn` (F. Li)

from the study of quasilinear parabolic equations with degenerate coercivity involving a quadratic gradient term (see [4, 7]). The general model of the problem (P) is given by

$$\begin{cases} u_t - \operatorname{div}(\alpha(u)\nabla u) = \beta(u) \mid \nabla u \mid^2 + f(x,t) & \text{in } \ Q := \Omega \times (0,T), \\ u = 0 & \text{on } \ \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \ \Omega, \end{cases} \tag{S}$$

where α and β are real continuous functions, moreover α is positive bounded and may vanish at $\pm\infty$, $u_0 \in L^\infty(\Omega)$ and $f \in L^m(\Omega)(m > 1 + \frac{N}{2})$ (see [4]). For the problem (S), the typical example of functions α and β are expressed as follows

$$\alpha(s) = \frac{1}{\sqrt{1+s^2}} \quad \text{and} \quad \beta(s) = \frac{1}{\sqrt{(1+s^2)^3}}.$$

In [7], the authors studied the problem (S) with more general assumptions in which (S) is a nonlinear degenerate parabolic equation. Meanwhile, in [32] Bogelein, Duzaarr and Gianazza dealt with nonhomogenous porous medium type equations related to Cauchy-Dirichlet problem in a space-time cylinder $Q := \Omega \times (0,T)$ (see also [13]). Likewise, Fiorenza, Mercaldo and Rakotoson [1] studied some regularity and uniqueness results of the evolution N-Laplacian equation with right hand term $\mu \in L^1((0,T), \mathcal{M}(\Omega))$. Furthermore Porzio, Smarrazzo and Tesei [23] introduced the definition of Radon measure-valued solutions to quasilinear parabolic equations with initial value as measure data. More precisely, in [23] authors proved the existence, uniqueness and qualitative properties of Radon measure-valued solutions to the following problem

$$\begin{cases} u_t = \Delta\varphi(u) & \text{in } \ Q, \\ u = 0 & \text{on } \ \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \ \Omega, \end{cases} \tag{F}$$

where $u_0 \in \mathcal{M}^+(\Omega)$ is a bounded Radon measure and

$$\varphi(s) = \gamma \left[1 - \frac{1}{(1+s)^\sigma}\right] \tag{A.1}$$

with $\gamma \in (0, +\infty)$, $\sigma > 0$. Since φ increases monotonically to limiting value γ as $s \to +\infty$. Therefore, $\varphi'(s) \to 0$, thus the problem (F) is strongly degenerate parabolic equation at infinity.

Another interesting problems similar to the problem (P) has been investigated in [18, 22, 24, 28, 30, 31] in which authors showed the existence and uniqueness of Radon measure valued solutions to nonlinear parabolic equations.

To obtain the problem (P), we replace the function φ by ψ which is defined by

$$\psi(s) = \int_0^s e^{-|z|^m} dz \quad (0 < m \leq 1) \tag{1.1}$$

The function ψ increases monotonically to limiting value γ as $s \to +\infty$. Therefore, the problem (P) is nonlinear strongly degenerate parabolic equation at infinity and the function ψ is given by Oleinik-Kruzhkov in [26].

The choice of the special function ψ in (1.1) is motivated by the connection with the function φ in (A.1), such as $\psi' \leq \varphi'$ in $\mathbb{R}_+$. This comparison leads to the connection of the problem (P) with the previous study problem (F).

In order to construct the problem (P), we add a Radon measure as a forcing term $\mu \in \mathcal{M}^+(Q)$(a nonnegative bounded Radon measure with respect to the parabolic capacity) to the problem (F).

The first difficulty when studying the problem (P) is due to the presence of a forcing term μ and the second difficulty is a lack of coercivity of the differential operator $u \to \operatorname{div}(\psi'(u)\nabla u)$.

In the study of degenerate parabolic equations, a physical model may be imagined in which the degenerate parabolic equations described arise in nonlinear fluid mechanics, heat transfer or diffusion. Moreover the Radon measures involved as data describe the distribution of mass in the length area, and volume.

The last decades some authors studied the parabolic and elliptic equations involving measure data, but the solutions of these equations are not measures (see [2, 17, 25]). Due to this reason, the main purpose of this paper is to study the degenerate parabolic equations with measure data which the solutions of such equations are measures as well. This result is possible because of the definition of weak Radon measure-valued solutions introduced in [23], hence the main motivation to study of the problem (P).

The unique point of the novelty of this paper is the study of the uniqueness of the Radon measure-valued solutions when the Radon measure as a forcing term is diffuse with respect to the parabolic capacity and the Radon measure as initial data is diffuse with respect to the Newtonian capacity.

To the best of our knowledge there is no existing results of the problem (P) are known in the literature. Hence, this interesting case will be discussed in this paper.

The plan of this paper is organized as follows. In the next section, we recall some preliminaries about capacity and Radon measures. Then in Section 3, we state the main results, while in Section 4-6, we prove the main results.

2. Preliminaries

2.1 About capacity and measures

For any Borel set $E \subset \Omega$, the C_2-capacity of E in Ω is defined as

$$C_2(E) = \inf \left\{ \int_\Omega | \nabla u |^2 dx / u \in \mathbb{Z}_\Omega^E \right\}$$

where $\mathbb{Z}_\Omega^E$ denotes the set of u belongs to $H_0^1(\Omega)$ such that $0 \leq u \leq 1$ almost everywhere in Ω, and $u = 1$ almost everywhere in a neighborhood E (see [23]).

Let $W = \left\{ u \in L^2((0,T), H_0^1(\Omega)) \text{ and } u_t \in L^2((0,T), H^{-1}(\Omega)) \right\}$ endowed with its natural norm $\| u \|_W = \| u \|_{L^2((0,T),H_0^1(\Omega))} + \| u_t \|_{L^2((0,T),H^{-1}(\Omega))}$ a Banach space. For any open set $U \subset Q$, we define the parabolic capacity as

$$\operatorname{Cap}(U) = \inf \left\{ \| u \|_W / u \in \mathbb{V}_Q^U \right\}$$

where $\mathbb{V}_Q^U$ denotes the set of u belongs to W such that $0 \leq u \leq 1$ almost everywhere in Q, and $u = 1$ almost everywhere in a neighborhood U (see [16]). Let $\mathcal{M}(\Omega)$ be the space of bounded Radon measures on Ω, and
$\mathcal{M}^+(\Omega) \subset \mathcal{M}(\Omega)$ the cone of nonnegative bounded Radon measures on Ω. For any $\mu \in \mathcal{M}(\Omega)$ a bounded Radon measure on Ω, we set

$$\| \mu \|_{\mathcal{M}(\Omega)} := | \mu | (\Omega)$$

where $| \mu |$ stands for the total variation of μ.
The duality map $\langle \cdot, \cdot \rangle_\Omega$ between the space $\mathcal{M}(\Omega)$ and $C_c(\Omega)$ is defined by

$$\langle \mu, \varphi \rangle_\Omega = \int_\Omega \varphi d\mu.$$

For any $\mu \in \mathcal{M}(\Omega)$ and any Borel set $B \subseteq \Omega$, the restriction $\mu \llcorner B$ of μ to B is defined by setting

$$(\mu \llcorner B)(A) := \mu(B \cap A) \quad \text{for every Borel set} \quad A \subseteq \Omega.$$

It is worth observing that $(\mu \llcorner B)(\emptyset) = 0$.
$\mathcal{M}_s^+(\Omega)$ denotes the set of nonnegative measures *singular with respect to the Lebesgue measure*, namely

$$\mathcal{M}_s^+(\Omega) := \left\{ \mu \in \mathcal{M}^+(\Omega) / \exists \text{ a Borel set } E \subseteq \Omega \; ; \; | E | = 0 \; , \mu = \mu \llcorner B \right\}$$

we will consider $| \cdot |$ the Lebesgue measure on $\mathbb{R}^N$. Similarly, $\mathcal{M}_{ac}^+(\Omega)$ the set of nonnegative measures *absolutely continuous with respect to the Lebesgue measure*, namely

$$\mathcal{M}_{ac}^+(\Omega) := \left\{ \mu \in \mathcal{M}^+(\Omega) / \mu(E) = 0, \text{ for every Borel set} \, E \subseteq \Omega \, ; | E | = 0 \right\}.$$

Recall that $\mathcal{M}_s^+(\Omega) \cap \mathcal{M}_{ac}^+(\Omega) = \{0\}$. Moreover, by the Lebesgue decomposition and Radon-Nikodym theorem (see [9]), for any $\mu \in \mathcal{M}^+(\Omega)$:

(i) there exists a unique couple $\mu_{ac} \in \mathcal{M}_{ac}^+(\Omega)$, $\mu_s \in \mathcal{M}_s^+(\Omega)$ such that

$$\mu = \mu_{ac} + \mu_s \tag{2.1}$$

(ii) there exist a unique nonnegative function $u_r \in L^1(\Omega)$ called the density of the measure μ_{ac} such that

$$\mu_{ac}(E) = \int_E u_r \mathrm{dx}, \quad \text{for every Borel set } E \subseteq \Omega. \tag{2.2}$$

Let $\mathcal{M}_{c,2}^+(\Omega)$ be the set of nonnegative measures on Ω which are *concentrated with respect to the Newtonian capacity*

$$\mathcal{M}_{c,2}^+(\Omega) := \left\{ \mu \in \mathcal{M}^+(\Omega) / \exists \text{ a Borel set } E \subseteq \Omega \; ; \; \mu = \mu \llcorner E \text{ and } C_2(E) = 0 \right\}.$$

Notice that $\mathcal{M}^+_{c,2}(\Omega)$ can be also defined as the set of all measures μ in $\mathcal{M}^+(\Omega)$ which are *singular with respect to the Newtonian capacity*, i.e.

$$\mathcal{M}^+_{c,2}(\Omega) := \left\{\mu \in \mathcal{M}^+_s(\Omega) / \exists \text{ a Borel set } E \subseteq \Omega \;;\; C_2(E) = 0\right\}.$$

It is clear to observe that $\mathcal{M}^+_{c,2}(\Omega) \subseteq M^+_s(\Omega)$ (see [12]).
$\mathcal{M}^+_{d,2}(\Omega)$ denotes the set of nonnegative measures on Ω which are *diffuse with respect to the Newtonian capacity*

$$\mathcal{M}^+_{d,2}(\Omega) := \left\{\mu \in \mathcal{M}^+(\Omega)/\mu(E) = 0, \text{ for every Borel set } E \subseteq \Omega \;; C_2(E) = 0\right\}.$$

Due to $C_2(E) = 0$ implies that $\mid E \mid = 0$ (see [9]), we observe that $\mathcal{M}^+_{ac}(\Omega) \subseteq \mathcal{M}^+_d(\Omega)$.
It is known that a measure $\mu_{d,2} \in \mathcal{M}^+_{d,2}(\Omega)$ if there exist $f_0 \in L^1(\Omega)$ and $G_0 \in \left[L^2(\Omega)\right]^N$ such that

$$\mu_{d,2} = f_0 - \text{div} G_0 \quad \text{in} \quad D'(\Omega). \tag{2.3}$$

For any $\mu \in \mathcal{M}^+(\Omega)$, if there exists a unique couple $\mu_{d,2} \in \mathcal{M}^+_{d,2}(\Omega)$, $\mu_{c,2} \in \mathcal{M}^+_{c,2}(\Omega)$ such that

$$\mu = \mu_{d,2} + \mu_{c,2}. \tag{2.4}$$

Notice that $\mu_{c,2} = [\mu]_{c,2}$ and $\mu_{d,2} = [\mu]_{d,2}$.
For the above assertions we can also refer to ([18, 23, 30] and references therein).
Let $\mathcal{M}(Q)$ be the space of bounded Radon measures on Q, and
$\mathcal{M}^+(Q) \subset \mathcal{M}(Q)$ the cone of nonnegative bounded Radon measures on Q.
For any $\mu \in \mathcal{M}(Q)$, we set

$$\parallel \mu \parallel_{\mathcal{M}(Q)} := \mid \mu \mid (Q)$$

where $\mid \mu \mid$ denotes the total variation of μ.
For any diffuse measure $\mu_0 \in \mathcal{M}^+_{d,2}(Q)$, there exist $f \in L^1(Q)$, $g \in L^2((0,T), H^1_0(\Omega))$ and $G \in \left[L^2(Q)\right]^N$

$$\mu_0 = f - \text{div} G + g_t \quad \text{in} \quad D'(Q) \tag{2.5}$$

(see [10, 11, 16]). The rest of statements of $\mathcal{M}(Q)$ can be deduce from the properties of $\mathcal{M}(\Omega)$.
Let E be a Borel subset of Ω, for $t_0 \in (0,T)$ fixed, one has $\text{Cap}(E \times \{t_0\}) = 0$ if and only if $\mid E \mid = 0$ and for any $0 \le t_0 < t_1 \le T$, there holds $\text{Cap}(E \times (t_0, t_1)) = 0$ if and only if $C_2(E) = 0$ (see [16]).
The relationship between parabolic capacity and Newtonian capacity is given in [27] such that :
(i) There exist positive constants $0 < k_1 < k_2$ such that

$$k_1 C_2(E) \le \text{Cap}(E \times \{t_0\}) \le k_2 C_2(E).$$

(ii) For any $0 < t_0 < t_1$, there exist positive constants $0 < l_1 < l_2$ such that

$$l_1 C_2(E) \le \text{Cap}(E \times (t_0, t_1)) \le l_2 C_2(E).$$

Let $U \subset Q$ an open set and $K \subset Q$ a compact set with $\mathrm{Cap}(K) = 0$, then there exists $\varphi_n \in C_c^\infty(U)$ such that
(iii) $0 \leq \varphi_n \leq 1$ a.e in Q, (iv) $\varphi_n = 1$ a.e in K, (v) $\varphi_n \to 0$ in W, (vi) φ_n converges to zero Cap-quasi continuous (see [27, Proposition 2.2]).
On the other hand, assume that $V \subset \Omega$ an open set and $\mathbb{K} \subset \Omega$ a compact set with $\mathrm{Cap}(\mathbb{K}) = 0$, then there exists $\phi_n \in C_c^\infty(V)$ such that (vii) $0 \leq \phi_n \leq 1$ a.e in Ω, (viii) $\phi_n = 1$ a.e in $\mathbb{K}$, (ivx) $\phi_n \to 0$ in $H_0^1(\Omega)$, (x) φ_n converges to zero Cap-quasi continuous (see [15, Lemma 4.E.1].
By $L^\infty\left((0,T), \mathcal{M}^+(\Omega)\right)$, the set of nonnegative Radon measures $u \in \mathcal{M}^+(Q)$ which satisfy the following property: For almost every $t \in (0,T)$, there exists a measure $u(\cdot,t) \in \mathcal{M}^+(\Omega)$ such that

(a) for every $\xi \in C(\overline{Q})$, the map $t \mapsto \langle u(\cdot,t), \xi(\cdot,t)\rangle_\Omega$ is Lebesgue measurable and there holds

$$\langle u, \xi\rangle_Q = \int_0^T \langle u(\cdot,t), \xi(\cdot,t)\rangle_\Omega dt \tag{2.6}$$

(b) for every Borel set $E \subseteq \Omega$, the map $t \mapsto u(\cdot,t)(E^t)$ is Lebesgue measurable and there holds

$$u(E) = \int_0^T u(\cdot,t)(E^t)dt$$

where $E^t = \{x \in \Omega/(x,t) \in E\}$

(c) there exists a constant $C > 0$ such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \parallel u(\cdot,t) \parallel_{\mathcal{M}(\Omega)} \leq C.$$

In the following, we will use the notation

$$\parallel u \parallel_{L^\infty((0,T),\mathcal{M}(\Omega))} = \operatorname*{ess\,sup}_{t\in(0,T)} \parallel u(\cdot,t) \parallel_{\mathcal{M}(\Omega)} .$$

If $u \in L^\infty((0,T), \mathcal{M}(\Omega))$, it is easily seen that $u_{ac}, u_s \in L^\infty((0,T), \mathcal{M}(\Omega))$ as well and that $u_r \in L^\infty((0,T), L^1(\Omega))$.
Moreover, the inequality (2.6) implies that for every $\xi \in C(\overline{Q})$

$$\langle u_{ac}, \xi\rangle_Q = \int_Q u_r \xi dxdt$$

and

$$\langle u_s, \xi\rangle_Q = \int_0^T \langle u_s(\cdot,t), \xi(\cdot,t)\rangle_\Omega dt$$

Notice that $u_{ac}(\cdot,t) = [u(\cdot,t)]_{ac}$, $u_r(\cdot,t) = [u(\cdot,t)]_r$ and $u_s(\cdot,t) = [u(\cdot,t)]_s$ (see [18, 23, 30]). Assume that the function ψ satisfies the following conditions:

$$(I) \qquad \begin{cases} (i) \quad \psi \in L^\infty(\mathbb{R}_+) \cap C^2(\mathbb{R}_+), \quad \psi(0) = 0, \quad \psi' > 0 \ \text{ in } \ \mathbb{R}_+, \\ (ii) \quad \psi^{(j)} \in L^\infty(\mathbb{R}_+^*), \ \text{ for any } j = 1,2,\dots,n \ \text{ if } \ 0 < m \le 1, \\ (iii) \quad \psi(s) \to \gamma \text{ as } s \to +\infty, \end{cases}$$

where $\mathbb{R}_+ \equiv [0,+\infty)$ and $\gamma \in \mathbb{R}_+^* \equiv (0,+\infty)$. By ψ' and $\psi^{(j)}$ we denote the first and $j\,th$ derivative of the function ψ. The assumption (I)-(iii) stems from (I)-(i), hence we extend the function ψ in $[0,+\infty]$ defining $\psi(+\infty) = \gamma$.

To prove the well-posedness of (P) (if $N \ge 2$) we will need further assumption

$$(J) \qquad \begin{cases} \text{There exist } \gamma > 0, \ \underline{s} < \overline{s} \text{ and } l_1, l_2 > 0, \ l_1 < l_2 \text{ such that} \\ (i) \quad \psi'(s) \ge l_1 e^{-|s|^m} \ , \\ (ii) \quad \psi'(s) \le l_2 e^{-|s|^m} \ , \\ \quad \text{for any} \quad \underline{s} < s < \overline{s}. \end{cases}$$

where l_1, l_2 can be expressed as follows

$$l_1 = \min_{s \in [\underline{s},\overline{s}]} \psi'(s) e^{|s|^m} \quad \text{and} \quad l_2 = \max_{s \in [\underline{s},\overline{s}]} \psi'(s) e^{|s|^m} \qquad (0 < m \le 1).$$

3. Statement of main results

Definition 3.1. For any $u_0 \in \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(Q)$, a measure u is called a *weak solution* of the problem (P), if $u \in \mathcal{M}^+(Q)$ such that
(i) $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$
(ii) $\psi(u_r) \in L^1((0,T), W_0^{1,1}(\Omega))$
(iii) for every $\xi \in C^1([0,T], C_0^1(\Omega))$, $\xi(\cdot,T) = 0$ in Ω, u satisfies the identity

$$\int_0^T \langle u(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega dt = \int_Q \nabla\psi(u_r)\nabla\xi dxdt - \int_Q \xi d\mu - \langle u_0, \xi(\cdot,0)\rangle_\Omega \tag{3.1}$$

where u_r is the density of the absolutely continuous part of the Radon-measure with respect to the Lebesgue measure such that $0 \le u_r \in L^\infty((0,T), L^1(\Omega))$.

Remark 3.1 In (3.1), we can choose test functions ξ in $C^1(\overline{Q})$ which vanish on $\partial\Omega \times [0,T]$ and $t = T$.

The following theorem gives necessary conditions on the measures μ and u_0 for the existence of weak solutions to the problem (P) with respect to the parabolic capacity and Newtonian capacity respectively.

Theorem 3.1. Assume that (I), (J), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ hold. If u is a weak solution to the problem (P). Then μ and $u_0 \otimes \delta_{\{t=0\}}$ are absolutely continuous measures with respect to the parabolic capacity.

Since Newtonian capacity and parabolic capacity are equivalent, then μ and $u_0 \otimes \delta_{\{t=0\}}$ are absolutely continuous measures with respect to the C_2-capacity as well.

Theorem 3.2. Assume that the hypothesis (I) holds. Let u be a weak solution to the problem (P). Then there exist a set $F \subset (0,T)$ with zero Lebesgue measure and $\nu^t \in \mathcal{M}^+(\Omega)$ such that

$$[u(\cdot,t)-u_0]_{c,2} = \left[\nu^t\right]_{c,2} \tag{3.2}$$

for every $t \in (0,T) \setminus F$.

Remark 3.2. Theorem 3.2 improves Theorem 2.4 in [23].

To prove the existence of solutions to the problem (P), we will consider the approximating problems

$$\begin{cases} u_{nt} = \Delta\psi_n(u_n) + \mu_n & \text{in } \; Q := \Omega \times (0,T), \\ u_n = 0 & \text{on } \; \partial\Omega \times (0,T), \\ u(x,0) = u_{0n} & \text{in } \; \Omega, \end{cases} \tag{P_n}$$

where $\{u_{0n}\} \subseteq C_0^\infty(\Omega)$ and $\{\mu_n\} \subseteq C_c^\infty(Q)$ satisfy

$$\begin{cases} u_{0n} \overset{*}{\rightharpoonup} u_0 \quad \text{in} \quad \mathcal{M}^+(\Omega), \\ u_{0n} \to u_{0r} \quad \text{a.e in } \; \Omega, \\ \| u_{0n} \|_{L^1(\Omega)} \leq \| u_0 \|_{\mathcal{M}^+(\Omega)} . \end{cases} \tag{3.3}$$

And

$$\begin{cases} \mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in} \quad \mathcal{M}^+(Q), \\ \| \mu_n \|_{L^1(Q)} \leq \| \mu \|_{\mathcal{M}^+(Q)} . \end{cases} \tag{3.4}$$

The approximating function ψ_n is such that

$$\psi_n(u) = \psi(u) + \frac{1}{n} \tag{3.5}$$

for every $n \in \mathbb{N}$.

By [3, 20], the approximating problem (P_n) has a solution u_n in $C((0,T),L^1(\Omega)) \cap L^\infty(Q)$.

Theorem 3.3. Assume that (I), $\mu \in \mathcal{M}^+(Q)$ and $u_0 \in \mathcal{M}^+(\Omega)$ hold. Then there exists a weak solution u to the problem (P) obtained as a limiting point of the sequence $\{u_n\}$ of solutions to the problem (P_n) such that for every $t \in (0,T) \setminus H^*$, there holds

$$\| u(\cdot,t) \|_{\mathcal{M}^+(\Omega)} \leq C \left(\| \mu \|_{\mathcal{M}^+(Q)} + \| u_0 \|_{\mathcal{M}^+(\Omega)} \right). \tag{3.6}$$

Moreover, there exists a Radon measure $\nu^t \in \mathcal{M}^+(\Omega)$ such that

$$[u_s(\cdot,t)]^\pm \leq [u_{0s}]^\pm + [\nu_s^t]^\pm \quad \text{in} \quad \mathcal{M}^+(\Omega) \tag{3.7}$$

where C is positive constant and H^* a zero Lebesgue measure set.

To get the uniqueness of the solution to the problem (P), we define the notion of *very* *weak solutions* as follows.

Definition 3.2. For any $\mu \in \mathcal{M}^+_{d,2}(Q)$ and $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$, a measure u is called a very weak solution to the problem (P) if $u \in L^\infty((0,T), \mathcal{M}^+(\Omega))$ such that

$$\int_0^T \langle u(\cdot,t), \xi_t(\cdot,t)\rangle_\Omega dt = -\int_Q \psi(u_r)\Delta\xi dxdt - \int_Q \xi d\mu - \langle u_0, \xi(0)\rangle_\Omega \tag{3.8}$$

for every $\xi \in C^{2,1}(\overline{Q})$, which vanishes on $\partial\Omega \times [0,T]$, for $t = T$.

The notion of *very weak solutions* adapted to our study can be found in [18, 33].

Definition 3.3. Let $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$ and $\mu \in \mathcal{M}^+_{d,2}(Q)$ such that

$$u_0 = f_0 - \mathrm{div}G_0\ ,\quad f_0 \in L^1(\Omega)\ \text{ and }\ G_0 \in \left[L^2(\Omega)\right]^N .$$

$$\mu = f - \mathrm{div}G + g_t\ ,\ f \in L^1(Q)\ ,\ G \in \left[L^2(Q)\right]^N \quad \text{and} \quad g \in L^2((0,T), H^1_0(\Omega)).$$

A measure u is called *very weak solutions obtained as limit of approximation*, if

$$u_n \overset{*}{\rightharpoonup} u \ \text{ in } \mathcal{M}^+(Q) \tag{3.9}$$

where $\{u_n\} \subseteq L^\infty(Q) \cap L^2((0,T), H^1_0(\Omega))$ is a sequence of weak solutions to the problem (P_n) and satisfy

$$\begin{cases} \mu_n = f_n - F_n + g_{nt} \in C^\infty_0(Q), \\ u_{0n} = f_{0n} - F_{0n} \in C^\infty_0(\Omega), \\ f_n \to f \quad \text{in } L^1(Q), \\ F_n \to \mathrm{div}G \quad \text{in } L^2((0,T), H^{-1}(\Omega)), \\ g_n \to g \quad \text{in } L^2((0,T), H^1_0(\Omega)), \\ F_{0n} \to \mathrm{div}G_0 \quad \text{in } H^{-1}(\Omega), \\ f_{0n} \to f_0 \quad \text{in } L^1(\Omega). \end{cases} \tag{3.10}$$

Notice that

$$\mu_n \overset{*}{\rightharpoonup} \mu \quad \text{in } \mathcal{M}^+(Q) \quad \text{and} \quad u_{0n} \overset{*}{\rightharpoonup} u_0 \quad \text{in } \mathcal{M}^+(\Omega).$$

Theorem 3.4. Under assumptions of (I) and (J), then for every $\mu \in \mathcal{M}^+_{d,2}(Q)$ and $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$, there exists a unique *very weak solution obtained as limit of approximation* u of the problem (P).

Notice that a *very weak solution* is also *weak solution* to the problem (P), therefore the problem (P) possesses a unique weak solution obtained as limit of approximation.

4. Approximating problems and the persistence

Now we establish some technical statements which will be used in the proof of the existence solution.

Lemma 4.1. Assume that (I) and (J) are satisfied and u_n is the solution of the approximation problem (P_n). Then there exists a zero Lebesgue measure set $F^* \subset (0,T)$ such that

$$\| u_n(\cdot,t) \|_{L^1(\Omega)} \leq \| u_0 \|_{\mathcal{M}^+(\Omega))} + \| \mu \|_{\mathcal{M}^+(Q)} \tag{4.1}$$

for every $t \in (0,T) \setminus F^*$ and $n \in \mathbb{N}$.

Proof. Assuming that any sequence $\{\Omega_j\}$ of smooth open sets such that

$$\overline{\Omega}_j \subset \Omega_{j+1} \subset \overline{\Omega}_{j+1} \subset \Omega \ , \ \ \Omega = \bigcup_{j=1}^{\infty} \Omega_j \ , \ \ \mathrm{dist}(\overline{\Omega}_j, \partial\Omega) \leq \frac{1}{j}.$$

Let $\{\rho_j\} \subseteq C_c^\infty(\Omega)$ be any function such that

$$0 \leq \rho_j \leq 1 \ \text{ in } \Omega \ , \ \ \rho_j = 1 \text{ in } \ \Omega_j \ , \ \ | \nabla \rho_j | \leq j \ \text{ in } \Omega \setminus \overline{\Omega}_j.$$

Then for any

$$| \nabla \rho_j | \leq j \leq \frac{1}{d(x)}$$

where $d(x) := \mathrm{dist}(x, \partial\Omega) \leq \mathrm{dist}(\overline{\Omega}_j, \partial\Omega)$ (see [24]).

Let us consider the truncated function η such that for any $0 \leq t_1 < t_2 \leq T$

$$\eta(s) = \begin{cases} 0 & \text{if } \ 0 \leq s \leq t_1, \\ 1 & \text{if } \ t_1 < s < t_2, \\ 0 & \text{if } \ s \geq t_2. \end{cases}$$

For any fixed $j \in \mathbb{N}$, we choose $\xi_j(x,s) = \eta(s)\rho_j(x)$ as a test function in the problems (P_n) gives

$$\int_\Omega u_n(x,t_2)\rho_j(x)dx - \int_\Omega u_n(x,t_1)\rho_j(x)dx = -\int_{t_1}^{t_2}\int_\Omega \eta(s)\nabla\psi(u_n)\nabla\rho_j(x)dxds + \\ + \int_{t_1}^{t_2}\int_\Omega \eta(s)\rho_j(x)\mu_n(x)dx. \tag{4.2}$$

It is worth observing that

$$\left| \int_\Omega \nabla\psi(u_n)\nabla\rho_j(x)dx \right| \leq | \Omega \setminus \overline{\Omega}_j \, | \| \nabla\psi(u_n) \|_{L^2(\Omega)} \, .$$

By letting j to infinity, we deduce that

$$\lim_{j\to\infty} \int_\Omega \nabla\psi(u_n)\nabla\rho_j(x)dx = 0. \tag{4.3}$$

By the properties of the sequence functions $\{\rho_j\}$, we set $t_2 = t \ , \ t_1 = 0$ and then combining together (4.2) with (4.3), there holds

$$\int_\Omega u_n(x,t)dx \leq \int_\Omega u_{0n}(x)dx + \int_0^t \int_\Omega d\mu_n. \tag{4.4}$$

Hence the estimate (4.1) follows. □

To show the existence of the solutions to the problems (P) we need a priori estimates of sequences $\{\psi(u_n)\}$.

Proposition 4.1. Under the assumptions of $(I)-(J)$ and u_n be the solution of the approximation problem (P_n). Then we obtain

$$\| \nabla\psi(u_n) \|_{L^2(Q)} \leq C. \tag{4.5}$$

$$\| \psi(u_n) \|_{L^\infty((0,T),H_0^1(\Omega))} \leq C. \tag{4.6}$$

Proof. Since $\psi(u_n) \geq 0$ in Q and $\psi(u_n) = 0$ on $\partial\Omega \times (0,T)$ for every $t \in (0,T)$. The fact that $u_n = \psi(\psi^{-1}(u_n)) \in C^1([0,T], H_0^1(\Omega))$. Take $\psi(u_n)$ as a test function in (P_n), we get

$$\int_Q | \nabla\psi(u_n) |^2 \, dxdt = \int_\Omega \left(\int_0^{u_{0n}(x)} \psi(s)ds \right) dx - \int_\Omega \left(\int_0^{u_n(x,T)} \psi(s)ds \right) dx + \int_Q \mu_n \psi(u_n) dxdt.$$

It follows that

$$\int_Q | \nabla\psi(u_n) |^2 \, dxdt \leq \int_\Omega \left(\int_0^{u_{0n}(x)} \psi(s)ds \right) dx + \int_Q \mu_n \psi(u_n) dxdt.$$

By (I)-(i) and the assumption (3.3), there exists a positive constant C such that (4.5) holds.
Assume that $\{\eta_j\}$ a sequence such that $\| \eta_j \|_{L^1(\Omega)} \leq C$ and $\eta_j \overset{*}{\rightharpoonup} \delta_{t_0}(t)$ in $\mathcal{M}^+(0,T)$. Suppose that $\xi(x,t) = \psi(u_n)(T-t)^\alpha \int_t^T \eta_j(s)ds \quad (1 < T-t < \tau \ , \ \alpha > 1)$ as a test function in the approximating problem (P_n), there holds

$$-\int_\Omega \left(\int_0^{u_{0n}(x)} \psi(s)ds \right) T^\alpha \int_0^T \eta_j(s)ds + $$

$$+ \int_\Omega \left(\int_0^{u_n(x,t)} \psi(s)ds \right) \left\{ (T-t)^\alpha \int_0^T \eta_j(s)ds + \int_0^T \eta_j(s)(T-t)^\alpha dt \right\} =$$

$$= \frac{1}{1+\alpha} \int_\Omega | \nabla\psi(u_n) |^2 \, dx \left(\int_0^T \eta_j(s)\chi_{(0,T)}(s)ds \right) (T-t)^\alpha - \int_Q \mu_n \psi(u_n)(T-t)^\alpha \int_T^t \eta_j(s)ds. \tag{4.7}$$

This leads to the following result

$$\left(\int_0^T \eta_j(s)\chi_{(0,T)}(s)ds \right) \int_\Omega | \nabla\psi(u_n) |^2 \, dx \leq C \left(\| u_0 \|_{\mathcal{M}^+(\Omega)} + \| \mu \|_{\mathcal{M}^+(Q)} \right)$$

Letting $j \to +\infty$ the assertion (4.6) holds true. □

Proposition 4.2. Suppose that $(I)-(J)$ and (1.1) hold. Let u_n be the solution of the problem (P_n) and $\phi \in C^1(\mathbb{R}_+)$ be the function defined by

$$\phi(s) = \int_0^s \psi(z)dz. \tag{4.8}$$

Then the sequence $[\phi(T_k(\psi(u_n))]_t$ is bounded in $L^2((0,T), H^{-1}(\Omega)) + L^1(Q)$. Where $T_k(s) = \min\{s, k\}$.

Proof. We choose $\psi(u_n)\varphi$ as a test function in (P_n), with $\varphi \in C_c^{2,1}(Q)$, there holds

$$[\phi(T_k(\psi(u_n))]_t - \text{div}\,[\psi(T_k(\psi(u_n)))\nabla\psi(T_k(\psi(u_n)))] + \mid \nabla\psi(T_k(\psi(u_n))) \mid^2 = \psi(T_k(\psi(u_n)))\mu_n.$$

It follows that

$$\| \, [\phi(T_k(\psi(u_n)))]_t \, \|_{L^2((0,T),H^{-1}(\Omega))+L^1(Q)}$$

$$\leq \| \, \psi(T_k(\psi(u_n)))\nabla\psi(T_k(\psi(u_n))) \, \|_{L^2(Q)} + \| \, \nabla\psi(T_k(\psi(u_n))) \, \|^2_{L^1(Q)} + \| \, \psi(T_k(\psi(u_n)))\mu_n \, \|_{L^1(Q)} \,.$$

By the condition (I), we obtain the sequence $\{[\phi(T_k(\psi(u_n))]_t\}$ is bounded in $L^2((0,T), H^{-1}(\Omega)) + L^1(Q)$. □

Proof of Theorem 3.1. This proof is similar to ([21, Theorem 1.1]). As in ([27, Proposition 3.1]), it is enough to show that for any compact $K \subset Q$ such that $\mu^-(K) = 0$, $\left(u_0^- \otimes \delta_{\{t=0\}}\right)(K) = 0$ and $\text{Cap}(K) = 0$, then $\mu^+(K) = 0$ and $\left(u_0^+ \otimes \delta_{\{t=0\}}\right)(K) = 0$. By the equivalence of the capacity, we have $\text{Cap}(E \times \{t = 0\}) = 0$, where E a compact set of Ω with $u_0^-(E) = 0$. Let $\epsilon > 0$ and we choose an open set U such that $\left(\mid \mu \mid + \mid u_0 \mid \otimes \delta_{\{t=0\}}\right)(U \setminus K) < \epsilon$ and $K \subset U \subset Q$. Then there exists a sequence $\{\varphi_n\} \subseteq C_0^\infty(Q)$ such that

$$\text{(i)} \quad 0 \leq \varphi_n \leq 1 \quad \text{in } Q, \quad \varphi_n \equiv 1 \quad \text{in } K.$$

$$\text{(ii)} \quad \| \, \Delta\varphi_n \, \|_{L^1(Q)} \to 0 \quad \text{as} \quad n \to \infty.$$

In particular, $\varphi_n \to 0$ in W, indeed

$$\int_Q \mid \nabla\varphi_n \mid^2 dxdt = -\int_Q \varphi_n \Delta\varphi_n dxdt \leq \int_Q \mid \Delta\varphi_n \mid dxdt.$$

Let us consider φ_n as a test function in (P), there holds

$$\int_Q \varphi_n d\mu + \int_\Omega \varphi_n(0) du_0 = -\int_Q \psi(u_r)\Delta\varphi_n dxdt. \tag{4.9}$$

On the other hand, we get

$$\int_Q \varphi_n d\mu + \int_\Omega \varphi_n(0) du_0 \geq \mu^+(K) + \left(u_0^+ \otimes \delta_{\{t=0\}}\right)(K)$$

$$- \left(\mid \mu \mid + \mid u_0 \mid \otimes \delta_{\{t=0\}} \right) (U \setminus K).$$

It follows that

$$\int_Q \varphi_n d\mu + \int_\Omega \varphi_n(0) du_0 \geq \mu^+(K) + \left(u_0^+ \otimes \delta_{\{t=0\}} \right) (K) - \epsilon. \tag{4.10}$$

Combining (4.11) with (4.12), we obtain that

$$\mu^+(K) + \left(u_0^+ \otimes \delta_{\{t=0\}} \right) (K) \leq \parallel \psi \parallel_{L^\infty(Q)} \parallel \Delta\varphi_n \parallel_{L^1(Q)} + \epsilon.$$

Letting n to infinity, we infer that

$$\mu^+(K) = \left(u_0^+ \otimes \delta_{\{t=0\}} \right) (K) = 0. \qquad \square$$

Proof of Theorem 3.2. Let $\mathbb{K} \subseteq \Omega$ be any compact set such that $C_2(\mathbb{K}) = 0$, there exists a sequence $\{\phi_n\} \subseteq C_c^\infty(\Omega)$ satisfying (iv) and (viii) as stated in preliminaries, Section 2. Furthermore, $\rho_V \in C_c^\infty(V)$ be any smooth function such that

$$\text{(iii)} \quad 0 \leq \rho_V \leq 1 \quad \text{in } \Omega\,, \qquad \rho_V \equiv 1 \quad \text{in } \mathbb{K}.$$

By standard regularization argument, we consider $\phi_\tau(x,s) = \rho(x)\eta_\tau(s)$ as a test function in (3.8), where

$$\eta_\tau(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq t, \\ \frac{1}{\tau}(t + \tau - s) & \text{if } t \leq s \leq t + \tau, \\ 0 & \text{if } s \geq t + \tau, \end{cases}$$

for any $\rho \in C_0^2(\Omega)$ and $\tau > 0$. There holds

$$\frac{1}{\tau} \int_t^{t+\tau} \langle u(s), \rho \rangle_\Omega \, ds - \langle u_0, \rho \rangle_\Omega = \int_0^T \eta_\tau(s) ds \int_\Omega \psi(u_r) \Delta\rho dx + \int_0^T \eta_\tau(s) \int_\Omega \rho d\mu.$$

Since $\eta_\tau(s) \to \chi_{(0,t]}$ for every $s \in (0,T)$ as $\tau \to 0$ and we replace the test function ρ by $\phi_n(x)\rho_V(x)$.
Then we infer that

$$\langle u(\cdot,t), \phi_n\rho_V \rangle_\Omega - \langle u_0, \phi_n\rho_V \rangle_\Omega = \int_0^t \int_\Omega \psi(u_r) \Delta(\phi_n\rho_V) dx ds + \int_0^t \int_\Omega (\phi_n\rho_V) d\mu.$$

By ([14, Theorem 8, p.85]), the measure $\mu \in \mathcal{M}^+(Q)$ can be decomposed as $\lambda \in \mathcal{M}^+(0,T)$ and $\nu^t \in \mathcal{M}^+(\Omega)$ such that for $\phi_n\rho_V \in C(\Omega)$, there holds

$$\langle \mu, \phi_n\rho_V \rangle_Q = \int_{(0,T)} d\lambda(s) \int_\Omega \phi_n\rho_V d\nu^t$$

with $\lambda(s) := \delta_{(0,T)}(s)$, where $\delta_{(0,T)}$ a Dirac measure on (0,T). Therefore,

$$\langle [u(\cdot,t)]_{c,2}, \phi_n\rho_V \rangle_\Omega + \langle [u(\cdot,t)]_{d,2}, \phi_n\rho_V \rangle_\Omega =$$

$$= \int_0^t \int_\Omega \psi(u_r)\Delta(\phi_n\rho_V)dxds + \left\langle [\nu^t]_{c,2}, \phi_n\rho_V \right\rangle_\Omega + $$
$$+ \left\langle [\nu^t]_{d,2}, \phi_n\rho_V \right\rangle_\Omega + \langle [u_0]_{c,2}, \phi_n\rho_V \rangle_\Omega + \langle [u_0]_{d,2}, \phi_n\rho_V \rangle_\Omega . \tag{4.11}$$

By the assumptions stated above, we infer that

$$\lim_{n\to\infty} \int_0^t \int_\Omega \psi(u_r)\Delta(\phi_n\rho_V)dxds = 0.$$

Moreover, since $[u(\cdot,t)]_{d,2}$, $[\nu^t]_{d,2}$, $[u_0]_{d,2}$ belong to $L^1(\Omega)+H^{-1}(\Omega)$ and $\phi_n \overset{*}{\rightharpoonup} 0$ in $L^\infty(\Omega)$, $\phi_n \to 0$ in $H_0^1(\Omega)$ so that

$$\lim_{n\to\infty} \langle [u(\cdot,t)]_{d,2}, \phi_n\rho_V \rangle_\Omega = \lim_{n\to\infty} \left\langle [\nu^t]_{d,2}, \phi_n\rho_V \right\rangle_\Omega = \lim_{n\to\infty} \langle [u_0]_{d,2}, \phi_n\rho_V \rangle_\Omega = 0.$$

It follows that (4.11) can be rewritten as

$$\langle [u(\cdot,t)]_{c,2}, \phi_n\rho_V \rangle_\Omega = \left\langle [\nu^t]_{c,2}, \phi_n\rho_V \right\rangle_\Omega + \langle [u_0]_{c,2}, \phi_n\rho_V \rangle_\Omega . \tag{4.12}$$

Since K is a subset compact of Ω, then

$$[u(\cdot,t)-u_0]_{c,2}(K) \le \limsup_{n\to\infty} \langle [u(\cdot,t)-u_0]_{c,2}, \phi_n\rho_V \rangle_\Omega = \limsup_{n\to\infty} \left\langle [\nu^t]_{c,2}, \phi_n\rho_V \right\rangle_\Omega \le \left[\nu^t\right]_{c,2}(K).$$

On the other hand, we get

$$\left[\nu^t\right]_{c,2}(K) \le \limsup_{n\to\infty} \left\langle [\nu^t]_{c,2}, \phi_n\rho_V \right\rangle_\Omega = \limsup_{n\to\infty} \langle [u(.,t)-u_0]_{c,2}, \phi_n\rho_V \rangle_\Omega \le [u(.,t)-u_0]_{c,2}(K).$$

The above inequality implies that

$$[u(\cdot,t)-u_0]_{c,2}(K) \le \inf\left\{ \left[\nu^t\right]_{c,2}(V) \mid K \subset V, \text{ open} \right\} = \left[\nu^t\right]_{c,2}(K).$$

Similarly, we have

$$\left[\nu^t\right]_{c,2}(K) \le \inf\left\{ [u(\cdot,t)-u_0]_{c,2}(V) \mid K \subset V, \text{ open} \right\} = [u(.,t)-u_0]_{c,2}(K).$$

Whence, the following statement

$$\left[\nu^t\right]_{c,2}(K) = [u(\cdot,t)-u_0]_{c,2}(K) \tag{4.13}$$

holds true. According to the arbitrariness of K, (4.13) is satisfied for every Borel set $E \subseteq \Omega$ with $C_2(E)=0$. By the definition of concentrated measure with respect to the Newtonian capacity, we have for any $t \in (0,T) \setminus F$,

$$[u(\cdot,t)]_{c,2} = [u(\cdot,t)]_{c,2} \llcorner B_1(t) \ , \ \left[\nu^t\right]_{c,2} = \left[\nu^t\right] \llcorner B_2(t) \ \text{ and } \ [u_0]_{c,2} = [u_0]_{c,2} \llcorner A$$

for some Borel sets $B_1(t)$, $B_2(t)$, and A is a zero Newtonian capacity, then (4.13) yields

$$[u(\cdot,t)]_{c,2}((B_1(t)\cup B_2(t))\setminus A) = \left[\nu^t\right]_{c,2}((B_1(t)\cup B_2(t))\setminus A) =$$

$$= [u_0]_{c,2}\left((B_1(t) \cup B_2(t)) \setminus A\right) = 0.$$

Therefore for every $t \in (0,T) \setminus F$, $[u(\cdot,t)]_{c,2}$, $\left[\nu^t\right]_{c,2}$, $[u_0]_{c,2}$ are concentrated measures on the set $B^*(t)$ such that $B^*(t) = (B_1(t) \cap A) \cup (B_2(t) \cap A)$. Therefore, for every set $E \subseteq \Omega$ and $t \in (0,T) \setminus F$, there holds

$$[u(\cdot,t) - u_0]_{c,2}(E) = \left([u(\cdot,t) - u_0]_{c,2} \llcorner B^*(t)\right)(E) = [u(\cdot,t) - u_0]_{c,2} \llcorner (B^*(t) \cap E) =$$

$$= \left[\nu^t\right]_c \llcorner (B^*(t) \cap E) = \left(\left[\nu^t\right]_{c,2} \llcorner B^*(t)\right)(E).$$

Hence, the proof is achieved. □

5. Existence results

We prove the existence result of the problem (P).

Proposition 5.1. Assume that (I) and (J) hold. Let u_n be the solution to the approximation problem (P_n), then there exist a subsequence $\{u_{n_j}\} \subseteq \{u_n\}$ and $v \in L^2((0,T), H_0^1(\Omega)) \cap L^\infty((0,T), H_0^1(\Omega)) \cap L^\infty(Q)$ with $0 \le v \le \gamma$ in Q such that

$$\psi(u_{n_j}) \overset{*}{\rightharpoonup} v \text{ in } L^\infty(Q). \tag{5.1}$$

$$\nabla\psi(u_{n_j}) \rightharpoonup \nabla v \text{ in } \left[L^2(Q)\right]^N. \tag{5.2}$$

$$\psi(u_{n_j}) \to v \text{ a.e in } Q. \tag{5.3}$$

Proof. By the assumption (I)-(ii), the sequence $\{\psi(u_n)\}$ is uniformly bounded in $L^\infty(Q)$, then from [5] there exists a function $v \in L^\infty(Q)$ such that the convergence in (5.1) holds true. Furthermore, the convergence (5.2) stems from estimate (4.5).

By (4.6), we have

$$\begin{aligned} \mid \nabla\phi(T_k(\psi(u_n))) \mid &= \mid \nabla T_k(\psi(u_n)) \mid\mid \psi(T_k(\psi(u_n))) \mid \\ &\le \gamma \mid \nabla T_k(\psi(u_n)) \mid . \end{aligned} \tag{5.4}$$

It follows that,

$$\int_Q \mid \nabla\phi(T_k(\psi(u_n))) \mid dxdt \le \gamma \mid Q \mid \left[\int_Q \mid \nabla T_k(\psi(u_n)) \mid^2 dxdt\right]^{\frac{1}{2}}.$$

Since $T_k(\psi(u_n)) \in L^2((0,T), H_0^1(\Omega))$ then there exists a positive constant C such that

$$\int_Q \mid \nabla\phi(T_k(\psi(u_n)) \mid dxdt \le C. \tag{5.5}$$

By Proposition 4.2, the sequence $[\phi(T_k(\psi(u_n)))]_t$ is bounded in $L^2((0,T), H^{-1}(\Omega)) + L^1(Q)$. According to the compactness theorem in [29], then there exists a subsequence denoted again $\left\{\psi(u_{n_j})\right\}$ (possibly for $\overline{k} > 0$, $T_{\overline{k}}(\psi(u_n)) = \psi(u_{n_j})$ and $\mid \psi(u_{n_j}) \mid \leq \overline{k}$) and a function $\overline{v} \in L^1((0,T), W_0^{1,1}(\Omega)) \cap L^1(Q)$ such that

$$\phi(\psi(u_{n_j})) \to \overline{v} \qquad \text{a.e in} \quad Q. \tag{5.6}$$

Therefore, we get

$$\psi(u_{n_j}) \to \phi^{-1}(\overline{v}) \quad \text{a.e in} \quad Q. \tag{5.7}$$

Combining (5.6) with (5.1) gives $\phi^{-1}(\overline{v}) = v$, this proves (5.3). □

We recall the following sclicing property of the bounded Radon measure
$u \in \mathcal{M}(Q)$. The proof is omitted since it follows from the more general result in ([14, Theorem 8, p.35]).

Proposition 5.2. Assume that $\mu \in \mathcal{M}^+(Q)$. Then there exists a measure $\lambda \in \mathcal{M}^+(0,T)$ and for λ almost everywhere $t \in (0,T)$, there exists a probability $\nu^t \in \mathcal{M}^+(\Omega)$ with the following properties

(i) for any Borel set $E \subseteq Q$

$$\mu(E) = \int_{(0,T)} \nu^t(E^t) d\lambda(t) \tag{5.8}$$

where $E^t = \{x \in \Omega / (x,t) \in E\}$

(ii) for every $\xi \in C(\overline{Q})$

$$\langle \mu, \xi \rangle_Q = \int_{(0,T)} d\lambda(t) \int_\Omega \xi(x,t) d\nu^t(x). \tag{5.9}$$

Proposition 5.3. Let $\left\{u_{n_j}\right\}$ and v as in Proposition 5.1. Then the following assertions hold

(i) $\psi^{-1}(u_{n_j}) \in L^1(Q)$ and we have

$$u_{n_j}(x,t) \to [\psi^{-1}(v)](x,t) \quad \text{a.e} \quad (x,t) \in Q. \tag{5.10}$$

(ii) There exist $\lambda_1, \lambda_2 \in L^\infty((0,T), \mathcal{M}^+(\Omega)))$ and we can extract a subsequence still denoted $\left\{u_{n_j}\right\}$ such that

$$u_{n_j}^+ \overset{*}{\rightharpoonup} [\psi^{-1}(v)]^+ + \lambda_1 \quad \text{in} \quad \mathcal{M}^+(Q), \tag{5.11}$$

$$u_{n_j}^- \overset{*}{\rightharpoonup} [\psi^{-1}(v)]^- + \lambda_2 \quad \text{in} \quad \mathcal{M}^+(Q), \tag{5.12}$$

$$u_{n_j} \overset{*}{\rightharpoonup} [\psi^{-1}(v)] + \lambda \quad \text{in} \quad \mathcal{M}^+(Q), \tag{5.13}$$

where $\lambda := \lambda_1 - \lambda_2$ in $L^\infty((0,T), \mathcal{M}^+(\Omega))$.

Proof. From (5.3), (4.1) and $\psi^{-1}(u_{n_j}) \in L^1(Q)$, then by Fatou's Lemma, we get

$$\int_Q [\psi^{-1}(v)](x,t) dxdt \leq \liminf_{j \to \infty} \int_Q u_{n_j}(x,t) dxdt. \tag{5.14}$$

By (5.3) the convergence (5.11) is satisfied.
Since the sequence $\left\{u_{n_j}\right\}$ is uniformly bounded in $L^1(Q)$ and by (4.6), there exist a subsequence $\left\{u_{n_j}\right\}$ which still denote $\left\{u_{n_j}\right\}$ and Radon-measures $\overline{u}\,,\ \widetilde{u} \in \mathcal{M}^+(Q)$ such that

$$u_{n_j}^{+} \overset{*}{\rightharpoonup} \overline{u} \quad \text{in} \ \ \mathcal{M}^+(Q). \tag{5.15}$$

$$u_{n_j}^{-} \overset{*}{\rightharpoonup} \widetilde{u} \quad \text{in} \ \ \mathcal{M}^+(Q). \tag{5.16}$$

Let us prove that $\overline{u}\ ,\ \widetilde{u} \in L^\infty((0,T),\mathcal{M}^+(\Omega))$. To prove this, we consider $\lambda_i \in \mathcal{M}^+(0,T)$ and λ_i-a.e $t \in (0,T)$. Let $\nu_i^t \in \mathcal{M}^+(\Omega)$ be the measure given by Proposition 5.2 in correspondence with each $\overline{u}\ ,\ \widetilde{u}$. Let us show that the measures $\lambda_i \in \mathcal{M}^+(0,T)$ are absolutely continuous with respect to the Lebesgue measure over $(0,T)$. In this direction, fix arbitrarily $\overline{t} \in (0,T)$ and choose $r, s > 0$ such that $J_{r,s} \equiv (\overline{t}-r-2s, \overline{t}+r+2s) \subseteq (0,T)$. Then for every function $\eta_{r,s} \in C_c^1(0,T)$ such that

$$\eta_{r,s} \equiv 1 \ \text{ in } \ [\overline{t}-r-2s, \overline{t}+r+2s]\ , \quad 0 \leq \eta_{r,s} \leq 1\ , \quad \operatorname{supp}\eta_{r,s} \subseteq J_{r,s}.$$

By the estimate (4.1), we have

$$\int_Q u_{n_j}^{\pm}\eta_{r,s}(t)dxdt \leq 2(r+2s) \parallel \mu \parallel_{\mathcal{M}^+(Q)} + 2(r+2s) \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}\ . \tag{5.17}$$

By (5.15), (5.16) and (5.17), there holds

$$\int_{[\overline{t}-r,\overline{t}+r]} d\lambda_i(t) \leq \int_{(\overline{t}-r-2s,\overline{t}+r+2s)} \nu_i^t(\Omega)d\lambda_i(t) \leq \liminf_{k\to\infty} \int_Q u_{n_j}^{\pm}(x,t)\eta_{r,s}(t)dxdt.$$

Thus

$$\int_{[\overline{t}-r,\overline{t}+r]} d\lambda_i(t) \leq 2(r+2s) \parallel \mu \parallel_{\mathcal{M}^+(Q)} + 2(r+2s) \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}\ .$$

Noting s is arbitrary, thus we divide both sides of the above inequality by $2r$, we obtain

$$\frac{1}{2r}\int_{[\overline{t}-r,\overline{t}+r]} d\lambda_i(t) \leq \parallel \mu \parallel_{\mathcal{M}^+(Q)} + \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}\ .$$

Therefore there exists $h_i \in L^1(0,T)$, $h_i \geq 0$ such that $d\lambda_i(t) = h_i(t)dt$, this means that the Radon-measure $\mathcal{M}^+(0,T)$ is regular (e.g, [9]).
Since $\overline{u}\ ,\ \widetilde{u} \in \mathcal{M}^+(Q)$ are nonnegative Radon-measures, letting $r \to 0$ in the previous inequality yields

$$0 \leq h_i(t) \leq C\left(\parallel \mu \parallel_{\mathcal{M}^+(Q)} + \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}\right)$$

for almost every $t \in (0,T)$. Finally, defining

$$\overline{u}(t) = h_1(t)\nu_1^t \ \text{ and } \ \widetilde{u}(t) = h_2(t)\nu_2^t \quad \text{for almost everywhere } t \in (0,T).$$

From (5.7) and (5.8) we obtain that $\overline{u},\ \widetilde{u} \in L^\infty((0,T),\mathcal{M}^+(\Omega))$.
Since $u_{n_j} \to \psi^{-1}(v)$ almost everywhere in Q, then $u_{n_j}^{\pm} \to [\psi^{-1}(v)]^{\pm}$ almost everywhere in

Q.
By (5.14) and (5.16), then we infer from Fatou's Lemma

$$\int_Q [\psi^{-1}(v)]^+\xi(x,t)dxdt \le \liminf_{j\to\infty}\int_Q u_{n_j}^+\xi(x,t)dxdt \le \langle \overline{u},\xi\rangle_Q\,.$$

Similarly, we have

$$\int_Q [\psi^{-1}(v)]^-\xi(x,t)dxdt \le \liminf_{j\to\infty}\int_Q u_{n_j}^-\xi(x,t)dxdt \le \langle \widetilde{u},\xi\rangle_Q$$

for every $\xi \in C_c(Q)$, $\xi \ge 0$, thus defining

$$\lambda_1 = \overline{u} - [\psi^{-1}(v)]^+ \quad \text{and} \quad \lambda_2 = \widetilde{u} - [\psi^{-1}(v)]^-.$$

Hence, λ_1 , $\lambda_2 \in L^\infty((0,T),\mathcal{M}^+(\Omega))$ hods true. □

Proposition 5.4. Let u and v be in Proposition 5.3 and Proposition 5.1. Then for almost every $t \in (0,T)$, we have

$$u_{n_j}(\cdot,t) \to [\psi^{-1}(v)](\cdot,t) \quad \text{a.e in} \quad \Omega. \tag{5.18}$$

$$u_{n_j}^+(t) \overset{*}{\rightharpoonup} [\psi^{-1}(v)]^+(\cdot,t) + \lambda_1(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega). \tag{5.19}$$

$$u_{n_j}^-(\cdot,t) \overset{*}{\rightharpoonup} [\psi^{-1}(v)]^-(\cdot,t) + \lambda_2(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega). \tag{5.20}$$

$$u_{n_j}(\cdot,t) \overset{*}{\rightharpoonup} [\psi^{-1}(v)](\cdot,t) + \lambda(\cdot,t) \quad \text{in} \quad \mathcal{M}^+(\Omega). \tag{5.21}$$

Proof. This proof is similar to that given in [18, 24]. Let us recall the statement of the function $\mathcal{F}$ which belongs to $C^2(\mathbb{R}_+)$ (see [18, Proposition 4.3]. Let u_n be the solution of the problem (P_n), and $\mathcal{F} \in C^2(\mathbb{R}_+)$, then for any $\rho \in C_c^1(\Omega)$, $\rho(x) \ge 0$ and there exists a zero Lebesgue measure set H such that $(0,T)\setminus H$, the following identity is satisfied

$$\int_\Omega \mathcal{F}(u_n)(x,t)\rho(x)dx - \int_\Omega \mathcal{F}(u_n)(x,0)\rho(x)dx =$$

$$= \int_0^T\int_\Omega \left\{-\mathcal{F}'(u_n)\nabla\psi(u_n)\nabla\rho dx - \frac{\mathcal{F}''(u_n)}{\psi'(u_n)} \mid \nabla\psi(u_n)\mid^2 \rho\right\} dxdt+$$

$$+\int_0^T\int_\Omega \mu_n\mathcal{F}'(u_n)\rho dxdt. \tag{5.22}$$

The convergence (5.18) immediately follows from (5.3). Next let us fix $J > 1$ and we consider the functions $\mathcal{F}_J$, $\mathcal{R}_J \in C^2(\mathbb{R}_+)$ defined as follows

$$\mathcal{F}_J(s) = \begin{cases} 0 & \text{if } 0 \le s \le J, \\ s-J & \text{if } J \le s \le J+1, \\ s-J & \text{if } s \ge J+1, \end{cases}$$

and $\mathcal{R}_J(s) = s - \mathcal{F}_J(s)$ $(s \in \mathbb{R}_+)$ and $\mathcal{R}_J(s)\chi_{\{s\geq J+1\}} = J$.
Let us consider the function $\mathcal{H}_n$ belongs to $C^1(\mathbb{R}_+)$ by setting

$$\mathcal{H}_{n,\rho}(t) = \int_\Omega \mathcal{F}_J(u_n(x,t))\rho(x)dx.$$

By (4.1), there exists a positive constant C such that

$$\int_0^T \mid \mathcal{H}_{n,\rho}(t) \mid dt \leq \parallel \rho \parallel_{L^\infty(\Omega)} \int_0^T \int_\Omega u_n^+(x,t)dxdt \leq C$$

where $C = C\left[T, \parallel \rho \parallel_{L^\infty(\Omega)}, \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}, \parallel \mu \parallel_{\mathcal{M}^+(Q)}\right] > 0$.
Thus $\mathcal{H}_{n,\rho} \in L^1(0,T)$ for every $\rho \in C_c^1(\Omega)$. Furthermore by (5.22) yields

$$\int_0^T \left|\frac{d\mathcal{H}_{n,\rho}(t)}{dt}\right| dt \leq \int_0^T \int_\Omega \mathcal{F}_J'(u_n) \mid \nabla\psi(u_n) \mid^2 \mid \nabla\rho \mid dxdt +$$

$$+ \int_0^T \int_\Omega \frac{\mathcal{F}''(u_n)}{\psi'(u_n)} \mid \nabla\psi(u_n) \mid^2 \rho dxdt + \int_0^T \int_\Omega \mu_n \mathcal{F}'(u_n)\rho dxdt. \tag{5.23}$$

By properties of sequence $\{\mathcal{F}_J(u_n)\}_{J>1}$ mentioned above and $\rho \in C_c^1(\Omega)$, there exists a positive constant
$C = C\left[\parallel \rho \parallel_{L^\infty(\Omega)}, \parallel u_0 \parallel_{\mathcal{M}^+(\Omega)}, \parallel \mu \parallel_{\mathcal{M}^+(Q)}\right] > 0$ such that

$$\int_0^T \left|\frac{d\mathcal{H}_{n,\rho}(t)}{dt}\right| dt \leq C.$$

Thus the family $\mathcal{H}_{n,\rho}$ is uniformly bounded in $W^{1,1}(0,T)$.
Hence there exist a subsequence $\left\{\mathcal{H}_{n_j,\rho}\right\} \subseteq \{\mathcal{H}_{n,\rho}\}$ and a function
$\mathcal{H}_\rho \in L^1(0,T)$ such that

$$\mathcal{H}_{n_j,\rho} \to \mathcal{H}_\rho \quad \text{in} \quad L^1(0,T). \tag{5.24}$$

By the properties of the function $\mathcal{F}_J$, the function $\mathcal{R}_J$ is continuous and bounded in $\mathbb{R}_+$, then the convergence (5.10) and the dominated convergence theorem imply that

$$\mathcal{R}_J(u_{n_j}) \to \mathcal{R}_J\left(\psi^{-1}(v)\right) \quad \text{in} \quad L^1(Q). \tag{5.25}$$

By (5.10), (5.11) and the definition of $\mathcal{R}_J$, we have

$$\mathcal{F}_J(u_{n_j}) = u_{n_j}^+ - \mathcal{R}_J(u_{n_j}) \overset{*}{\rightharpoonup} \left[\psi^{-1}(v)\right]^+ + \lambda_1 - \mathcal{R}_J\left(\psi^{-1}(v)\right) = \mathcal{F}_J\left(\psi^{-1}(v)\right) + \lambda_1 \quad \text{in} \quad \mathcal{M}^+(Q). \tag{5.26}$$

In view of (5.24) and (5.26), for any $h \in C_c(0,T)$ and $\rho \in C_c^1(\Omega)$ we get

$$\int_0^T \mathcal{H}_\rho(t)h(t)dt = \lim_{j\to\infty} \int_0^T \mathcal{H}_{n_j,\rho}(t)h(t)dt = \lim_{j\to\infty} \int_Q \mathcal{F}_J(u_{n_j})\rho(x)h(t)dxdt =$$

$$= \int_0^T h(t)\left\langle \mathcal{F}_J\left(\psi^{-1}(v)(\cdot,t)\right) + \lambda_1(\cdot,t), \rho\right\rangle_\Omega dt.$$

Then by the above equality, we deduce that

$$\mathcal{H}_\rho(t) = \left\langle \mathcal{F}_J\left(\psi^{-1}(v)(\cdot,t)\right) + \lambda_1(\cdot,t), \rho\right\rangle_\Omega$$

for almost every $t \in (0,T)$ and

$$\mathcal{H}_{j,\rho} \to \left\langle \mathcal{F}_J\left(\psi^{-1}(v)(\cdot,t)\right) + \lambda_1(\cdot,t), \rho\right\rangle_\Omega \quad \text{in} \quad L^1(0,T)$$

for any $\rho \in C_c^1(\Omega)$. □

Proof of Theorem 3.3. Let us show that for every $\rho \in C_c^1(\Omega)$, $\rho \geq 0$ and for almost every $\tau \in (0,T)$, there exists a Radon measure $\nu^\tau \in \mathcal{M}^+(\Omega)$ such that

$$\langle \lambda_1(\tau), \rho\rangle_\Omega \leq \left\langle [u_{0s}]^+ + [\nu_s^\tau]^+, \rho\right\rangle_\Omega, \tag{5.27}$$

$$\langle \lambda_2(\tau), \rho\rangle_\Omega \leq \left\langle [u_{0s}]^- + [\nu_s^\tau]^-, \rho\right\rangle_\Omega. \tag{5.28}$$

We prove the first inequality (5.27) and the second one follows by similar argument. Fix any $\rho \in C_c^1(\Omega)$, $\rho \geq 0$ and we consider the sequence $\{\mathcal{F}_J(u_n)\}$ as mentioned above and we use it in (5.22), then we obtain for every $\tau \in (0,T)$

$$\begin{aligned}&\int_\Omega \mathcal{F}_J(u_n)(x,\tau)\rho(x)dx - \int_\Omega \mathcal{F}_J(u_{0n})(x)\rho(x)dx\\ &\leq -\int_0^\tau\int_\Omega \mathcal{F}_J'(u_n)\nabla\psi(u_n)\nabla\rho dxdt + \int_0^\tau\int_\Omega \mu_n\mathcal{F}_J'(u_n)\rho dxdt.\end{aligned} \tag{5.29}$$

Let us consider $\left\{u_{n_j}\right\}$ the sequence given in Proposition 5.1 and Proposition 5.2 and let us take the limit as j tends to infinity in (5.29) (with $n = n_j$). By (5.2), (5.3) and the fact that $\left\{\mathcal{F}_J'(u_{n_j})\right\}$ is bounded in $L^\infty(Q)$, there holds

$$\lim_{j\to\infty}\int_0^\tau\int_\Omega \mathcal{F}_J'(u_{n_j})\nabla\psi(u_{n_j})\nabla\rho dxdt = \int_0^\tau\int_\Omega \mathcal{F}_J'(\psi^{-1}(v))\nabla v\nabla\rho dxdt.$$

In view of the definition of the sequence $\left\{\mathcal{F}_J'(u_{n_j})\right\}$, yields

$$0 \leq \mathcal{F}_J'(u_{n_j}) \leq 1\ ,\ \ \mathcal{F}_J'(u_{n_j}) \to 0 \text{ as } J \to \infty \text{ and }\ \psi^{-1}(v) \in L^1(Q).$$

It follows that

$$\lim_{J\to\infty}\lim_{j\to\infty}\int_0^\tau\int_\Omega \mathcal{F}_J'(\psi^{-1}(v))\nabla v\nabla\rho dxdt = 0. \tag{5.30}$$

On the other hand, by (5.26) one has

$$\lim_{j\to\infty}\int_\Omega \mathcal{F}_J(u_{n_j}(x,\tau))\rho(x)dx = \int_\Omega \mathcal{F}_J(\psi^{-1}(v))(x,\tau)dx + \langle\lambda_1(\tau),\rho\rangle_\Omega.$$

Referring to the definition of the sequence $\{\mathcal{F}_J(u_n)\}_{J>1}$, we infer that

$$0 \leq \mathcal{F}_J(u_{n_j}) \leq 1\ ,\ \ \mathcal{F}_J(u_{n_j}) \to 0 \text{ as } J \to \infty \text{ and }\ \psi^{-1}(v) \in L^1(Q).$$

Then we obtain

$$\lim_{J\to\infty}\lim_{j\to\infty}\int_{\Omega}\mathcal{F}_J(u_n(\tau))\rho(x)dx = \langle\lambda_1(\tau),\rho\rangle_{\Omega}\,. \tag{5.31}$$

Let us consider the sequence $\{u_{0n}(x)\}$ satisfies (3.3), then

$$\mathcal{F}_J(u_{0n_j}) = [u_{0n_j}]^+ - \mathcal{R}_J(u_{0n_j}) \leq [u_{0rn_j}]^+ + [u_{0sn}]^+ - \mathcal{R}_J(u_{0n_j}).$$

Since $u_{0rn_j}\to u_{0r}$ in $L^1(\Omega)$ and the sequence $\left\{\mathcal{R}_J(u_{0rn_j})\right\}$ is bounded in $L^\infty(\Omega)$, we obtain

$$[u_{0rn_j}]^+ - \mathcal{R}_J(u_{0n_j}) \to [u_{0r}]^+ - \mathcal{R}_J(u_{0r}) = \mathcal{F}_J(u_{0r}) \quad \text{in} \quad L^1(\Omega)$$

which leads to

$$\lim_{J\to\infty}\limsup_{j\to\infty}\int_{\Omega}\mathcal{F}_J(u_{0n_j})\rho(x)dx \leq \left\langle [u_{0s}]^+,\rho\right\rangle_{\Omega}\,. \tag{5.32}$$

Let us now consider the function $\overline{\eta}_{r,s}$ constructs from the function $\eta_{r,s}$ given in Proposition 5.2 as follows

$$\overline{\eta}_{r,s}(t) = \int_{t+r+2s}^{t}\eta_{r,s}(\theta)d\theta \quad \text{for every } \theta\in(0,T)$$

we deduce that

$$\int_0^\tau\int_\Omega \mu_{n_j}\mathcal{F}'_J(u_{n_j})\rho dxdt = \int_0^\tau\int_\Omega \mu_{n_j}\left(1-\overline{\eta}_{r,s}(t)\right)\mathcal{F}'_J(u_{n_j})\rho dxdt +$$

$$+\int_0^\tau\int_\Omega \mu_{n_j}\overline{\eta}_{r,s}(t)\mathcal{F}'_J(u_{n_j})\rho dxdt. \tag{5.33}$$

Since $\left\{\mu_{n_j}\right\}$ is a nonnegative bounded Radon-measure, and the function $1-\overline{\eta}_{r,s}(t)$ is bounded in $\mathbb{R}_+$, there holds

$$\limsup_{j\to\infty}\int_0^\tau\int_\Omega \mu_{n_j}\left(1-\overline{\eta}_{r,s}(t)\right)\mathcal{F}'_J(u_{n_j})\rho dxdt \leq \int_0^\tau \left\langle \mu,\rho\mathcal{F}_J(\psi^{-1}(v))\left(1-\overline{\eta}_{r,s}(t)\right)\right\rangle dt.$$

Letting J to infinity, we obtain

$$\lim_{J\to\infty}\limsup_{j\to\infty}\int_0^\tau\int_\Omega \mu_{n_j}\left(1-\overline{\eta}_{r,s}(t)\right)\mathcal{F}'_J(u_{n_j})\rho dxdt = 0. \tag{5.34}$$

By [11, Theorem 8, p.85], there exist $\nu_{n_j}^t\in\mathcal{M}^+(\Omega)$ and $\delta_0\in\mathcal{M}^+(0,T)$ for $\mu_{n_j}\in\mathcal{M}^+(Q)$ such that (5.33), becomes

$$\int_0^\tau\int_\Omega \mu_{n_j}\overline{\eta}_{r,s}(t)\mathcal{F}'_J(u_{n_k})\rho dxdt \leq \overline{\eta}_{r,s}(0)\int_\Omega \nu_{n_j}^\tau\mathcal{F}_J(u_{n_j})\rho dx \leq (4r+2s)\int_\Omega \nu_{n_j}^\tau\mathcal{F}_J(u_{n_j})\rho dx.$$

Setting $r=\frac{1}{8}$ and $s=\frac{1}{4}$, then

$$\int_0^\tau\int_\Omega \mu_{n_j}\overline{\eta}_{r,s}(t)\mathcal{F}'_J(u_{n_j})\rho dxdt \leq \int_\Omega [\nu_s^\tau]_{n_j}^+\rho dx + \int_\Omega [\nu_r^\tau]_{n_j}^+\mathcal{F}_J(u_{n_j})\rho dx.$$

Therefore,

$$\lim_{J\to\infty}\limsup_{j\to\infty}\int_0^{\tau}\int_{\Omega}\mu_{n_j}\mathcal{F}'_J(u_{n_j})\rho dxdt \le \left\langle [\nu_s^{\tau}]^+,\rho\right\rangle_{\Omega}. \tag{5.35}$$

Combining (5.30), (5.31), (5.32), (5.34) and (5.35) together. Hence (5.27) holds true. □

Remark 5.1. By the assumptions (I) and (J), it has been proved that

(i) the set

$$\widetilde{S} = \{(x,t)\in\Omega/\psi(u_r)(x,t)=\gamma\}$$

has zero Lebesgue measure (see [23, Proposition 5.2]).

(ii) There hold

$$\text{supp}(u(x,t))\subseteq\widetilde{S} \quad \text{and} \quad u_r=\psi^{-1}(v) \ \text{ a.e in } \ Q\setminus\widetilde{S}$$

(see [30, Proposition 4.1]).

6. Monotonicity and Uniqueness Results

Lemma 6.1. Under assumption (I). If u is a weak solution of the problem (P). Then

(i) there exist a zero Lebesgue measure set $D\subseteq(0,T)$ and a positive constant c such that

$$\operatorname*{ess\,lim}_{t\to0^+}\int_{\Omega}u(\cdot,t)dx = c \tag{6.1}$$

(ii) for any $\rho\in C_0^2(\Omega)$, $\rho\ge0$, there holds

$$\operatorname*{ess\,lim}_{t\to0^+}\langle u(\cdot,t),\rho\rangle_{\Omega} = \langle u_0,\rho\rangle_{\Omega} \tag{6.2}$$

for almost every $t\in(0,T)\setminus D$.

Proof. Let us consider for every $\tau>0$, the smooth function $\eta_\tau\in C_0^1(0,T)$, $0\le\eta_\tau\le1$ such that

$$\eta_\tau(t)=\begin{cases}0 & \text{if } 0\le t\le t_1-\tau,\\ \frac{1}{\tau}(t+\tau-t_1) & \text{if } t_1-\tau\le t\le t_1,\\ 1 & \text{if } t_1\le t\le t_2,\\ \frac{1}{\tau}(-t+\tau+t_2) & \text{if } t_2\le t\le t_2+\tau,\\ 0 & \text{if } t_2+\tau\le t\le T.\end{cases}$$

Let us choose $\rho_j(x)\eta_\tau(t)$ as a test function in (P), there holds

$$\int_0^T\int_{\Omega}\left\{-u\rho_j(x)\eta'_\tau(t)-\psi(u_r)\eta_\tau(t)\Delta\rho_j(x)\right\}dxdt = \int_0^T\int_{\Omega}\mu\rho_j(x)\eta_\tau(t)dxdt.$$

It is worth observing that the first term of the left hand side of the above equality becomes

$$\int_0^T\int_{\Omega}-u\rho_j(x)\eta'_\tau(t)dxdt = -\frac{1}{\tau}\int_{t_1-\tau}^{t_1}\int_{\Omega}u(x,t)\rho_j(x)dxdt+\frac{1}{\tau}\int_{t_2}^{t_2+\tau}\int_{\Omega}u(x,t)\rho_j(x)dxdt.$$

Let us consider a zero Lebesgue measure set D_j in $(0,T)$ such that for any $t_1,\ t_2\in(0,T)\setminus D_j$, one has

$$\lim_{\tau\to0}\int_0^T\int_{\Omega}-u\rho_j(x)\eta'_\tau(x,t)dxdt = -\int_{\Omega}u(x,t_1)\rho_j(x)dx+\int_{\Omega}u(x,t_2)\rho_j(x)dx.$$

We use a sequence $\{\rho_j(x)\}_{j\in\mathbb{N}}$ of test functions in Ω such that $\rho_j(x) \in C_0^2(\overline{\Omega})$, $0 \leq \rho_j(x) \leq 1$, $\rho_j(x) \to 1$ in Ω and $-\Delta\rho_j(x) \geq 0$ (for instance, $\rho_j(x) = 1-(1-\phi)^j$, where ϕ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, with normalization $\max\phi = 1$)(see [6] reference therein). For every $s \in (0,T) \setminus D_j$, there holds

$$\int_\Omega u(x,t)\rho_j(x)dx - \int_{Q_t} \psi(u_r)\Delta\rho_j(x)dxds = \int_{Q_t} \rho_j(x)d\mu + \int_\Omega \rho_j(x)du_0.$$

Let j goes to infinity, then we get that

$$D \equiv \bigcup_{j\in N} D_j$$

which leads to

$$\int_\Omega u(x,t)dx \leq \int_{Q_t} d\mu + \int_\Omega du_0.$$

Now let us consider $\{\phi_k\}$ be a sequence of $C_0(\Omega)$ functions such that $0 \leq \phi_j \leq 1$, $\phi_k \to 1$ as $j \to \infty$.
By [12, Lemma 5.1], the following statement hold

$$\int_\Omega \phi_j du_0 \leq \frac{1}{j} \quad \text{and} \quad \int_{Q_t} \phi_j d\mu \leq \frac{1}{j}$$

then

$$\int_{Q_t} d\mu + \int_\Omega du_0 - \int_\Omega u(x,t)dx = \int_{Q_t} (1-\phi_j)\,d\mu + \int_{Q_t} \phi_j d\mu + \int_\Omega (1-\phi_j)\,du_0 + \int_\Omega \phi_j du_0 -$$
$$- \int_\Omega u(x,t)\phi_j dx + \int_\Omega u(x,t)\,(\phi_j - 1)\,dx.$$

Since $\phi_j \leq 1$ yields

$$\int_{Q_t} d\mu + \int_\Omega du_0 - \int_\Omega u(x,t)dx \leq \int_{Q_t} (1-\phi_j)\,d\mu + \int_\Omega (1-\phi_j)\,du_0 - \int_\Omega u(x,t)\phi_j dx + \frac{2}{j}.$$

Since $u(x,t)$ converges to δ_x, we get

$$\limsup_{t\to 0^+} \left| \int_\Omega du_0 - \int_\Omega u(x,t)dx \right| \leq \int_\Omega (1-\phi_j)\,du_0 + \frac{2}{j}.$$

Let j to infinity, there exists a positive constant c such that (6.1) holds. Using the same method as the previous, it is obvious that for every $\rho \in C_0^2(\Omega)$

$$\operatorname{ess}\lim_{t\to 0^+} \langle u(x,t), \rho\rangle_\Omega = \langle u_0, \rho\rangle_\Omega\,.$$

Hence (6.2) is satisfied. □
For every $g \in C^1(\mathbb{R})$

$$G(s) = \int_0^s g(\psi(z))dz. \tag{6.3}$$

Assuming (I) holds. Let us state the following definition.
Definition 6.1. For any $\mu \in \mathcal{M}^+_{d,2}(Q)$ and $u_0 \in \mathcal{M}^+_{d,2}(\Omega)$, a measure u is called a weak entropy solution, if u is a weak solution of (P) such that for every $g \in C^1(\mathbb{R})$, $g' \geq 0$, $g(\gamma) = 0$, the inequality holds

$$\int_Q \left\{g'(\psi(u_r)) \mid \nabla\psi(u_r) \mid^2 \phi + g(\psi(u_r))\nabla\psi(u_r)\nabla\phi - G(u_r)\phi_t\right\} dxdt$$

$$\leq \int_Q g(\psi(u_r))\phi d\mu + \int_\Omega G(u_{0r})\phi(0)dx \tag{6.4}$$

for every $\phi \in C^1([0,T], C^1_0(\Omega))$, $\phi(.,T) = 0$ in Ω and $\phi \geq 0$.
By the Definition 6.1, the existence of weak entropy solutions of problem (P) is the same as stated in [23, Theorem 2.8]. For that we use entropy inequality to prove the monotonicity of solutions given by the following proposition.

Proposition 6.1. Suppose that the assumption (I) holds. Let u be a weak entropy solution to the problem (P).
For any $\rho \in H^1_0(\Omega)$, $\rho \geq 0$, then

$$\langle u_s(\cdot, t_2), \rho\rangle_\Omega \leq \langle u_s(\cdot, t_1), \rho\rangle_\Omega \leq \langle u_{0s}, \rho\rangle_\Omega \tag{6.5}$$

hols, for almost every $t_1, t_2 \in (0,T)$; $t_1 < t_2$.
Proof. Let G_j be the function given in (6.3) and we take $g = g_j$ for any $j \in \mathbb{N}$. By the Definition 6.1, we obtain

$$\int_Q \left\{g_j'(\psi(u_r)) \mid \nabla\psi(u_r) \mid^2 \phi + g_j(\psi(u_r))\nabla\psi(u_r)\nabla\phi - G_j(u_r)\phi_t\right\} dxdt$$

$$\leq \int_Q g_j(\psi(u_r))\phi d\mu + \int_\Omega G_j(u_{0r})\phi(0)dx \tag{6.6}$$

for every $\phi \in C^1([0,T], C^1_0(\Omega))$, $\phi(\cdot,T) = 0$ in Ω and $\phi \geq 0$, where

$$g_j(s) = \begin{cases} -1 & \text{if } s \leq \gamma - \frac{1}{j}, \\ j(s-\gamma) & \text{if } \gamma - \frac{1}{j} \leq s \leq \gamma, \\ 0 & \text{if } s \geq \gamma. \end{cases}$$

To avoid repeating the same calculation we refer to the proof of [23, Theorem 2.9].
Then by letting j to infinity, we get

$$\int_Q \{u_r\phi_t - \nabla\psi(u_r)\nabla\phi\} dxdt \leq -\int_Q \phi d\mu - \int_\Omega u_{0r}\phi(0)dx. \tag{6.7}$$

Combining (6.7) with (3.1), we have

$$-\int_0^T \langle u_s(\cdot,t),\phi_t\rangle_\Omega\,dt \le \langle u_{0s},\phi(0)\rangle_\Omega\,. \tag{6.8}$$

For any fix $0\le t_1<t_2\le T$. We consider

$$\chi_r(t)=\begin{cases}\frac{1}{r}\left(t-t_1+\frac{r}{2}\right) & \text{if } t_1-\frac{r}{2}<t<t_1+\frac{r}{2},\\ 1 & \text{if } t_1+\frac{r}{2}<t<t_2-\frac{r}{2},\\ -\frac{1}{r}\left(t-t_2-\frac{r}{2}\right) & \text{if } t_2-\frac{r}{2}<t<t_2+\frac{r}{2},\\ 0 & \text{otherwise,}\end{cases}$$

where $0<r<t_2-t_1$, such that $[t_1-\frac{r}{2},t_2+\frac{r}{2}]\subset(0,T)$ (see [30, Theorem 2.5]. For any $\phi\in C_0^1(\Omega)$, $\rho\ge 0$ we choose $\phi(x,t)=\rho(x)\chi_r(t)$ as a test function in (6.8), one has

$$-\frac{1}{r}\int_{t_1-\frac{r}{1}}^{t_1+\frac{r}{2}}\langle u_s(t),\rho\rangle_\Omega\,dt+\frac{1}{r}\int_{t_2-\frac{r}{2}}^{t_2+\frac{r}{2}}\langle u_s(t),\rho\rangle_\Omega\,dt\le 0$$

for almost every $0<t_1<t_2<T$ and letting $r\to 0$ in the above inequality, there holds

$$\langle u_s(\cdot,t_2),\rho\rangle_\Omega\le\langle u_s(\cdot,t_1),\rho\rangle_\Omega\,.$$

Similarly, let us consider for every fixed $t_1\in(0,T)$

$$\chi_r(t)=\begin{cases}1 & \text{if } 0\le t\le t_1,\\ -\frac{1}{r}\left(t-t_1-r\right) & \text{if } t_1\le t\le t_1+r,\\ 0 & \text{if } t\ge t_1+r.\end{cases}$$

Therefore, we can deduce that

$$\frac{1}{r}\int_{t_1}^{t_1+r}\langle u_s(\cdot,t),\rho\rangle_\Omega\,dt\le\langle u_{0s},\rho\rangle_\Omega\,.$$

Hence the estimate (6.5) holds true. □

Proof of Theorem 3.4. Let u_1 , u_2 be two *very weak solutions obtained as limit of approximation* of (P) with initial data u_{01n} and u_{02n} respectively . Let $\{u_{1n}\}$, $\{u_{2n}\}\subseteq L^\infty(Q)\cap L^2((0,T),H_0^1(\Omega))$ be two approximating sequences of solutions to the approximation problem (P_n) and satisfying the assumption (3.9).
For every $\xi\in C^{2,1}(Q)$ vanishing on $\partial\Omega\times(0,T)$ and $\xi(\cdot,T)=0$ in Ω, there holds

$$\begin{aligned}\int_Q(u_{1n}-u_{2n})\,\xi_t dxdt=-\int_Q(\psi(u_{1n})-\psi(u_{2n}))\,\Delta\xi dxdt-\\ -\int_Q(\mu_{1n}-\mu_{2n})\,\xi dxdt-\int_\Omega(u_{01n}-u_{02n})\,\xi(x,0)dx,\end{aligned}\tag{6.9}$$

where $\{\mu_{1n}\}$, $\{\mu_{2n}\}$, $\{u_{01n}\}$, and $\{u_{02n}\}$ are approximating Radon measures satisfying (3.10).
For almost every $(x,t) \in Q$, we consider the function $a_n(x,t)$ defined by

$$a_n(x,t) = \begin{cases} \frac{\psi(u_{1n}(x,t)) - \psi(u_{2n}(x,t))}{u_{1n}(x,t) - u_{2n}(x,t)} & \text{if } u_{1n}(x,t) \neq u_{2n}(x,t), \\ \psi'(u_{1n}(x,t)) & \text{if } u_{1n}(x,t) = u_{2n}(x,t). \end{cases} \tag{6.10}$$

Obviously $a_n \in L^\infty(Q)$ and for every $n \in \mathbb{N}$ there exists a positive constant C_n such that

$$\text{ess} \inf_{(x,t)\in Q} a_n(x,t) \geq C_n > 0.$$

This ensures that for every $z \in C_c^2(Q)$, the problem

$$\begin{cases} \xi_{nt} + a_n \Delta \xi_n + z = 0 & \text{in } Q \\ \xi_n = 0 & \text{on } \partial\Omega \times (0,T) \\ \xi_n(\cdot, T) = 0 & \text{in } \Omega \end{cases} \tag{6.11}$$

has a unique solution $\xi_n \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H_0^1(\Omega))$ with $\xi_{nt} \in L^2(Q)$ (see [8, 19]).
Moreover, it can be seen that

$$| \xi_n(x,t) | \leq (T-t) \parallel z \parallel_{L^\infty(Q)} . \tag{6.12}$$

Let us consider the function η such that for any $t_1 + 1 < t_2$ and t_1 , $t_2 \in (0,T)$

$$\eta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq t_1, \\ t - t_1 & \text{if } t_1 < t < t_2, \\ t_2 - t_1 & \text{if } t \geq t_2. \end{cases}$$

Choosing $\eta \Delta \xi_n$ as a test function in (6.11), then we obtain

$$\int_Q \xi_{nt} \eta(t) \Delta \xi_n dxdt + \int_Q \eta(t) a_n(x,t) [\Delta \xi_n]^2 dxdt + \int_Q z \eta(t) \Delta \xi_n dxdt = 0. \tag{6.13}$$

It follows that

$$\frac{1}{2} \int_Q | \nabla \xi_n |^2 \, dxdt + \int_Q a_n(x,t) [\Delta \xi_n]^2 dxdt \leq C_0(T,z) \tag{6.14}$$

holds, for some constant $C_0(T,z)$ independent on n.
From (6.12) and (6.14), there exists a constant $C_1(T,z)$ such that

$$\parallel \xi_n \parallel_{L^2((0,T),H_0^1(\Omega))} + \parallel \sqrt{a_n} \Delta \xi_n \parallel_{L^2(Q)} \leq C_1(T,z). \tag{6.15}$$

On the other hand, multiplying (6.11) by $\Delta \xi_n$, we obtain

$$-\int_Q \nabla \xi_n \nabla \xi_{nt} + \int_Q a_n [\Delta \xi_n]^2 dxdt = -\int_Q \xi_n \Delta z dxdt$$

which leads to

$$\frac{1}{2}\int_{\Omega} \mid \nabla\xi_n \mid^2 (x,0)dx + \int_Q a_n[\Delta\xi_n]^2 dxdt \leq C_2(T,z), \tag{6.16}$$

where $C_2(T,z) = \parallel \xi_n \parallel_{L^\infty(Q)} \parallel z \parallel_{C^2(\overline{Q})}$. Therefore, we get

$$\parallel \xi_n(.,0) \parallel_{H_0^1(\Omega)} + \parallel \sqrt{a_n}\Delta\xi_n \parallel_{L^2(Q)} \leq C_2(T,z). \tag{6.17}$$

By standard density argument and for $\xi = \xi_n$ a test function in (6.9). Moreover, by recalling (6.10) and (6.9), there holds

$$\int_Q (u_{1n} - u_{2n})\, zdxdt = \int_Q (\mu_{1n} - \mu_{2n})\, \xi(x,t)dxdt + \int_\Omega (u_{01n} - u_{02n})\, \xi(x,0)dx. \tag{6.18}$$

Letting n to infinity in (6.18). Then it is enough to observe from (6.15), there exists $\xi_n \in L^\infty((0,T), H^2(\Omega)) \cap L^2((0,T), H_0^1(\Omega))$ which is obtained by extracting the subsequence of the sequence $\{\xi_n\}$, such that

$$\xi_n(x,t) \overset{*}{\rightharpoonup} \xi(x,t) \text{ in } L^\infty(Q). \tag{6.19}$$

$$\nabla\xi_n(x,t) \rightharpoonup \nabla\xi(x,t) \text{ in } [L^2(Q)]^N. \tag{6.20}$$

Since $\xi_{nt} \in L^2(Q)$, as stated in [19], we deduce that

$$\xi_{nt}(x,t) \to \xi_t(x,t) \text{ in } L^2(Q), \tag{6.21}$$

$$\xi_n(x,t) \to \xi(x,t) \text{ a.e in } Q. \tag{6.22}$$

On one hand, it is enough to observe that from (6.17), there exists $\xi(\cdot,0) \in L^\infty(\Omega) \cap H_0^1(\Omega)$ such that the following statements

$$\xi_n(x,0) \overset{*}{\rightharpoonup} \xi(x,0) \text{ in } L^\infty(\Omega), \tag{6.23}$$

$$\xi_n(x,0) \rightharpoonup \xi(x,0) \text{ in } H_0^1(\Omega), \tag{6.24}$$

holds true.

Combining (6.18)-(6.24) and (3.10), there holds

$$\lim_{n\to\infty}\int_Q (u_{1n} - u_{2n})\, zdxdt = \lim_{n\to\infty}\int_Q (f_{1n} - f_{2n})\, \xi(x,t)dxdt +$$

$$+ \lim_{n\to\infty}\int_Q (F_{1n} - F_{2n})\, \xi(x,t)dxdt - \lim_{n\to\infty}\int_Q (g_{1n} - g_{2n})\, \xi_t(x,t)dxdt +$$

$$+ \lim_{n\to\infty}\int_\Omega (g_{01n} - g_{02n})\, \xi(x,0)dxdt + \lim_{n\to\infty}\int_\Omega (F_{01n} - F_{02n})\, \xi(x,0)dx = 0.$$

Therefore the following equality holds

$$\langle u_1 - u_2, z\rangle_Q = 0.$$

As we stated above in the previous proof

$$u_{1n} \overset{*}{\rightharpoonup} u_1 \quad \text{in } \mathcal{M}^+(Q) \quad \textit{and} \quad u_{2n} \overset{*}{\rightharpoonup} u_2 \quad \text{in } \mathcal{M}^+(Q).$$

Thus we can deduce $u_1 = u_1$ holds. □

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