Markov Modeling of Battery Cell Behavior Taking into account Pulsed Discharge Recovery

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Abstract. In this work, we modeled the behavior of a battery. After having formulated a Markovian model, we evaluated the delivered capacity as well as the gained capacity. We, likewise, evaluated the mean number of pulses and studied the asymptotic behavior and the variance of this mean number. As a last resort, we introduced an extension of the Markov model.

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1. Introduction

In this work we are concerned with modeling the capacity level dynamics in battery cells under discharge. Depending on usage patterns, batteries are commonly subject to variations in life-length and other measures of performance due to variable disload mechanisms of the electrochemically stored charges. The nominal capacity of a battery is the amount of charge which is delivered if the cell is put under constant use and drained of its load under maximal discharge conditions. It is often the case, however, that the discharge process occurs on slower time scales, which allows for recovery mechanisms to take place in which bound charge becomes available and hence adds to the nominal charge level [2]. The theoretical capacity of the battery is a measure of the maximal charge which in principle could be obtained where the battery discharged arbitrarily slow, restricted only by the total amount of electrochemical material contained in the cell.

Over the years, many battery models have been developed for different application areas (see for instance [1, 3, 5–12] and references therein).

Chiasserini and Rao in [3] have introduced a battery model with recovery effect. Kaj and Konané, in [8] have studied the so-called kinetic battery model, KIBAM with some extensions like the spatial diffusion KiBAM and diffusive model KiDiBAM. Chen and al, in [1] have studied an electric model of the battery capable of capturing the runtime of the battery.

Our main approach to the modeling of simple batteries will be to focus on the interplay between the remaining nominal capacity and the remaining theoretical capacity during discharge evolution. In particular, we will study the discharge process by performing a phase-plane analysis of nominal and remaining capacities. This will allow us to consider performance measures such as gained capacity and delivered capacity as functions of basic model parameters.

To introduce the main ideas of our approach, we consider a battery which is initially fully loaded with nominal capacity N and which has the theoretical capacity T, at time $t = 0$. Realistically, $N \leq T$. For $t \geq 0$, let $x(t)$ denote the level of available charge and $v(t)$ the level of remaining theoretical capacity of the battery at time t. Then $(v(0), x(0)) = (T, N)$. Charge is drawn from the battery either continuously, or, such that the charge level drops instantly from one discrete level to a lower level. The discharge process acts randomly or in a deterministic fashion and it acts continuously in time or at discrete time epochs. In either case, we may let λ denote an average discharge rate per time unit, and write $(v_\lambda(t), x_\lambda(t))$ to emphasize the dependence on the discharge rate λ. The battery is empty and stops functioning at the first instance of time t when either $x_\lambda(t) = 0$ or $v_\lambda(t) = 0$. It is natural to consider the trajectory of the system $(v_\lambda(t), x_\lambda(t))$, $t \geq 0$, as a path in

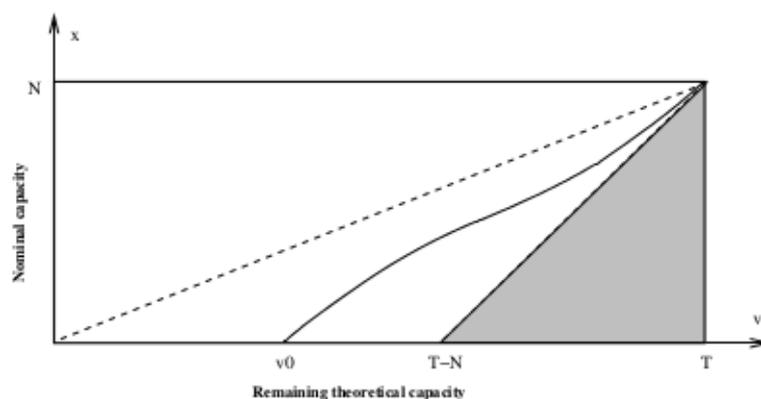
Figure 1: Phase-plane trace of nominal and theoretical capacities

the (v, x) phase-plane, which starts in (T, N) at time $t = 0$. Initially, the path moves downwards and to the left in the (v, x) plane as the nominal, and hence the remaining theoretical, capacity decreases. While the remaining theoretical capacity continues its descent with the same average rate as the discharge process, it is reasonable to expect that the battery may recover some nominal charge capacity because of chemical transport in the electrolytes enabling previously stored material to become available, at least if λ is not too large. This effect is likely to be less effective at lower levels of nominal charge. The battery goes empty whenever $x(v)$ hits one of the axis $x = 0$ or $v = 0$. If the path exits the positive phase-plane through the axis $v = 0$ for a large x, this indicates that the initial theoretical capacity is insufficient. If the path exits through $x = 0$ for large v, then

a fraction of the initially stored capacity remains in the battery at the end of its life and is hence wasted. In both cases the battery could be considered not properly designed. If there exists a solution $v_0 > 0$ of $x(v) = 0$, then v_0 is the remaining capacity at the battery charge expiration time. For this case we note that $D = T - v_0$ is the delivered capacity of the battery. We expect, based on the brief discussion above, that the delivered capacity tends to N if $\lambda \to \infty$ and to T if $\lambda \to 0$, c.f. [13], Figure 6. Moreover, we define the gain G of the battery to be the quantity $G = T - N - v_0$. This is the capacity which is gained during the life of the battery and measures the amount of bound charge that the battery was able to convert into available charge and deplete during its time of operation. Figure 1 indicates a typical trace in the phase-plane starting from (T, N) and ending in $(v_0, 0)$. Clearly, the path can not visit the shadowed area in the lower right corner. In some natural models, like the kinetic battery model (see [8]) the path $x(v)$, $0 \leq v \leq T$, can not cross the line $x = cv$, where $c = N/T$ is the initial ratio of nominal to theoretical capacity.

2. Markov chain model of pulsed discharge recovery

In this section, we formulate our Markov model and analyze the delivered and gained capacities.

2.1. Model formulation

Consider a discrete time Markov process (X_n) with finite state space $E = \{0, \ldots, N\}$ and jump transition probabilities

$$
\begin{cases}
p_{0,0} = 1 \\
p_{i,i-1} = q, & 1 \leq i \leq N \\
p_{i,i} = p(1 - e^{-\alpha(N-i)}), & 1 \leq i \leq N - 1 \\
p_{i,i+1} = pe^{-\alpha(N-i)}, & 1 \leq i \leq N - 1 \\
p_{N,N} = p
\end{cases}
$$

where $0 \leq p < 1$, $q = 1 - p$. For example, the transition probability matrix for $N = 4$ is given by

$$
\mathbf{P} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
q & p(1 - e^{-3\alpha}) & pe^{-3\alpha} & 0 & 0 \\
0 & q & p(1 - e^{-2\alpha}) & pe^{-2\alpha} & 0 \\
0 & 0 & q & p(1 - e^{-\alpha}) & pe^{-\alpha} \\
0 & 0 & 0 & q & p
\end{pmatrix}
$$

For this model, introduced in [3, 4], the state of X_n signifies the current nominal charge level of a battery. α is a parameter that depends on the battery technology characteristics: smaller α is, greater is the recovery capability of the battery. We assume that the level may only change in steps of size one. Initially the battery is fully loaded in state $X_0 = N$. Each discrete time point a request is received with probability q to discharge one unit load. With probability p the battery either remains in the same state or is able to recharge one

load unit, as long it has not reached the minimum level 0, which is an absorbing state. Fix an integer T, which is assumed to be the maximal theoretical capacity of the battery in the sense that there is enough electrochemical material available in the battery to discharge one unit of load a total of T times. The battery stops functioning either at the time of absorption in state 0, or at the first time the battery has delivered T discharge events, whichever occur first.

2.2. Delivered and gained capacities

Let Q_n denote the number of discharge events after n steps of the Markov chain. Then $Q_n = \sum_{k=1}^{n} Z_i$, $n \geq 1$, is a random walk associated with the i.i.d. sequence (Z_k) such that $P(Z_k = 1) = 1 - P(Z_k = 0) = q$. Hence for each n, Q_n has a binomial distribution $\text{Bin}(n, q)$ with parameters n and q. Also, $V_n = T - Q_n$ counts the remaining number of units of charge in the battery at time n. In analogy with our previous analysis, we consider now the path $(V_n, X_n)_{n \geq 0}$, $(V_0, X_0) = (T, N)$, evolving on the lattice points $\{1 \leq v \leq T, 1 \leq x \leq N\}$ until its exit time at the first passage of $v = 0$ or $x = 0$. To begin we notice that $EV_n = T - qn$ and

$$E(X_{n+1} - X_n | X_n = x) = -q + (1 - q)e^{-\alpha(N-x)}.$$

To obtain a continuous approximation (v_t, x_t) of the average behavior of the path we apply $v_t = T - qt$ and solve the equation

$$\frac{d}{dt}x_t = -q + (1-q)e^{-\alpha(N-x_t)}, \qquad x_0 = N,$$

which yields

$$x_t = N - \frac{1}{\alpha}\ln\left(e^{\alpha q t} - \frac{1-q}{q}(e^{\alpha q t} - 1)\right).$$

Hence, the average nominal charge x as a function of remaining theoretical charge v obeys the approximation

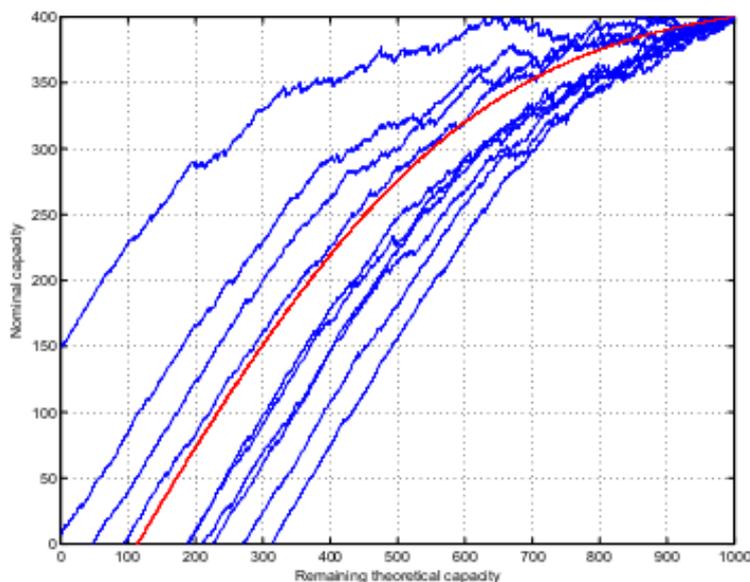$$x(v) = N - \frac{1}{\alpha}\ln\left(e^{\alpha(T-v)} - \frac{1-q}{q}(e^{\alpha(T-v)} - 1)\right). \tag{1}$$

Figure 2: Markovchain with $T = 1000$, $N = 400$, $\alpha = 0.005$, $q = 0.52$; 10 simulated paths and associated average discharge profile (1)

Figure 2 reveals that the discharge profile of the pulsed Markov model is subject to substantial fluctuations. Moreover, equation $x(v) = 0$ has a unique solution $v_0 > 0$, if $q > q_0$, where

$$q_0 = (1 + \frac{e^{\alpha T} - e^{\alpha N}}{e^{\alpha T} - 1})^{-1} > 1/2.$$

In this case,

$$v_0 = T - \frac{1}{\alpha} \ln \left(\frac{qe^{\alpha N} - 1 + q}{2q - 1} \right)$$

Thus, we have the following result:

Proposition 1. *The delivered capacity and the gain capacity are given by*

$$
\begin{aligned}
D(q, T, c) &= \frac{1}{\alpha} \ln \left(\frac{qe^{\alpha cT} + q - 1}{2q - 1} \right) \\
&\text{and} \\
G(q, T, c) &= \frac{1}{\alpha} \ln \left(\frac{qe^{\alpha cT} + q - 1}{2q - 1} \right) - cT.
\end{aligned}
\tag{2}
$$

Proof. The proof of this proposition is based on the definitions of both Delivered and gained capacity given in the introduction. We recall it: $D = T - v_0$ and $G = T - v_0 - N$ where v_0 is solution of equation $x(v) = 0$.
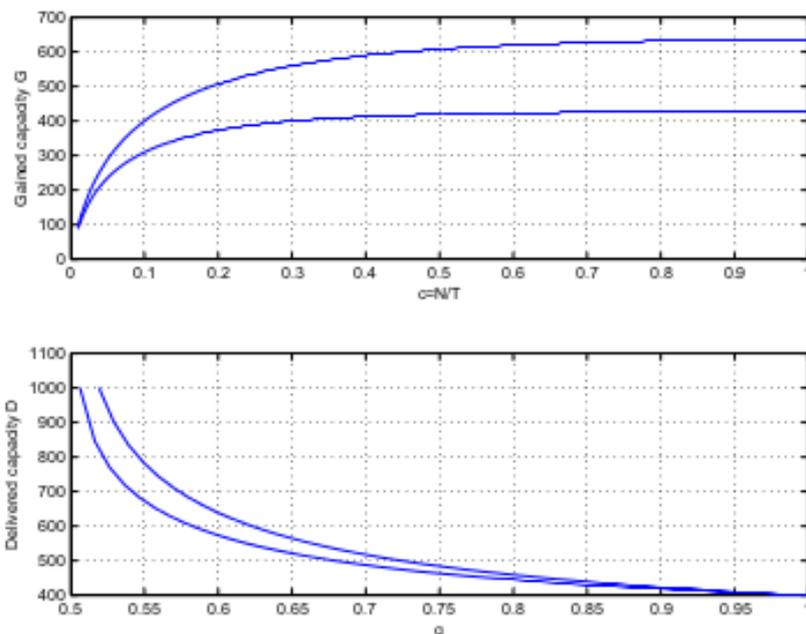
Figure 3: Gainedcapacity $G(0.52, 1000, c), 0 \leq c \leq 1$,and delivered capacity $D(q, 1000, 0.4), q_0 \leq q \leq 1$,

In figure 3 we plot the gain and delivered capacity functions for different parameters value.

3. Analysis of the Mean number of discharge pulses

In this present section we provide some additional analysis of the original Markov chain model, which we have not found in the literature. To this aim, we introduce

$$
\begin{aligned}
D_i &= \text{number of downward jumps before absorption, if } X(0) = i \\
A_i &= \text{number of steps before absorption, if } X(0) = i
\end{aligned}
$$

and put

$$d_i = E(D_i) = E(\text{number of downward jumps until absorption}|X(0) = i)$$

and

$$a_i = E(A_i) = E(\text{number of steps until absorption}|X(0) = i).$$

We have $d_0 = 0$ and for $1 \leq i \leq N - 1$, by conditioning on the first jump,

$$d_i = q(1 + d_{i-1}) + p\,(1 - e^{-\alpha(N-i)})d_i + p\,e^{-\alpha(N-i)}d_{1+1}.$$

Hence

$$q(d_i - d_{i-1}) = q + pe^{-\alpha(N-i)}(d_{i+1} - d_i), \quad 0 \leq i \leq N - 1.$$

Clearly, $d_N - d_{N-1} = 1$. Introducing the parameter $\kappa = p/q > 0$ and putting $u_i = d_i - d_{i-1}$, we obtain

$$\begin{cases} u_i = 1 + \kappa\, e^{-\alpha(N-i)} u_{i+1}, & 1 \le i \le N-1 \\ u_N = 1 \end{cases}$$

The recursive system is readily solved to give

$$u_i = \sum_{j=0}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}, \quad i = 1, \ldots N.$$

Therefore

$$d_k = \sum_{i=1}^{k} u_i = k + \sum_{i=1}^{k}\sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \tag{3}$$

so

$$d_k = k + \sum_{j=1}^{N-k}\sum_{i=1}^{k} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} + \sum_{j=N-k+1}^{N-1}\sum_{i=1}^{N-j} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}.$$

Thus, $d_k - k$ equals

$$\sum_{j=1}^{N-k} \kappa^j e^{\alpha(j-1)j/2}\frac{e^{-\alpha j(N-k)} - e^{-\alpha jN}}{1 - e^{-\alpha j}} + \sum_{j=N-k+1}^{N-1} \kappa^j e^{\alpha(j-1)j/2}\frac{e^{-\alpha j^2} - e^{-\alpha jN}}{1 - e^{-\alpha j}},$$

which yields

$$d_k = k + \sum_{j=1}^{N-1} \kappa^j e^{-\alpha(j+1)j/2}\, \frac{e^{-\alpha j(\max(N-k-j,0))} - e^{-\alpha j(N-j)}}{1 - e^{-\alpha j}}.$$

In particular,

$$d_N = N + \sum_{j=1}^{N-1} \kappa^j e^{-\alpha(j+1)j/2}\, \frac{1 - e^{-\alpha j(N-j)}}{1 - e^{-\alpha j}}. \tag{4}$$

Similarly,

$$q(a_i - a_{i-1}) = 1 + pe^{-\alpha(N-i)}(a_{i+1} - a_i), \quad 0 \le i \le N-1,$$

and $a_N - a_{N-1} = 1$. Hence $\tilde{u}_i = a_i - a_{i-1}$ satisfies

$$\begin{cases} \tilde{u}_i = 1/q + \kappa\, e^{-\alpha(N-i)}\tilde{u}_{i+1}, & 1 \le i \le N-1 \\ \tilde{u}_N = 1 \end{cases}$$

It follows that $\tilde{u}_k = u_k/q$ hence $a_k = d_k/q$ for $1 \le k \le N-1$. Thus,

$$d_N = 1 + d_{N-1} = 1 + q\, a_{N-1} = q\, a_N + p.$$

3.1. Asymptotic behavior of the mean number of pulses

We study, in this section, the asymptotic behavior of d_N.

Proposition 2. *For fixed N and $\alpha \to 0$,*

$$
d_N \to N + \sum_{j=1}^{N-1} \kappa^j (N-j) =
\begin{cases}
N\frac{1}{1-\kappa} - \frac{\kappa}{(1-\kappa)^2}(1-\kappa^N), & \kappa < 1 \\
N(N+1)/2, & \kappa = 1 \\
\frac{\kappa}{(\kappa-1)^2}(\kappa^N - 1) - N\frac{1}{\kappa-1}, & \kappa > 1,
\end{cases}
$$

with equality for the case $\alpha = 0$.

    Proof. From relation (4), we have

$$
\begin{aligned}
d_N &= N + \sum_{j=1}^{N-1} \kappa^j \left(1 - \frac{\alpha j(j+1)}{2}\right) \frac{\alpha j(N-j)}{\alpha j} \\
&\to N + \sum_{j=1}^{\infty} \kappa^j (N-j).
\end{aligned}
$$

From this relation we conclude that $\kappa = 1$, that is $p = q = 1/2$, represents a transitional regime. If the arrival probability is high in the sense $q > 1/2$, and so $\kappa < 1$, then the number of recharge events behaves on average linearly with N. Under low utilization, meaning $q < 1/2$ and $\kappa > 1$, then in contrast d_N has exponential increase as a function of N. It appears unrealistic however to expect the pulsed discharge mechanism to run effectively on batteries with such a large theoretical capacity T that $\ln T \sim N$ represents normal operation. At the critical parameter value reaches $\kappa = 1$ from below we see that the typical duration of battery operation before absorption given by d_N, and disregarding for the moment the role of T, changes from linear in N to quadratic in N. This suggests that in order to achieve recharge events and keep the battery running until all T unit loads have been discharged, one should try to operate the battery with an arrival probability $q \leq 1/2$.

We also have the following result.

Proposition 3. *For fixed $\alpha > 0$ and $N \to \infty$,*

$$
d_N - N \to \sum_{j=1}^{\infty} \frac{\kappa^j e^{-\alpha(j+1)j/2}}{1 - e^{-\alpha j}},
$$

which is finite for any $\kappa, \alpha > 0$.

    Proof. To establish the result, we need to apply the dominated convergence in relation (4).

Now we consider the simultaneous convergence of $\alpha \to 0$ and $N \to \infty$, which allows us to analyze the onset regime of non-linear increase of the expected absorbtion time.

Proposition 4. *(i) Fix $\beta > 0$ and take $\alpha = \beta/N \to 0$ as $N \to \infty$. Then, if $\kappa < 1$,*

$$\frac{d_N - N}{N} \to \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{\kappa^j}{j}(1 - e^{-\beta j}) = \frac{1}{\beta} \ln\left(\frac{1 - \kappa e^{-\beta}}{1 - \kappa}\right).$$

(ii) For $\gamma > 0$ and δ a real number, consider $\alpha = \gamma/N^2 \to 0$ and

$$\kappa = 1 + \frac{\delta}{N}, \quad q = \frac{1}{2 + \delta/N}$$

as $N \to \infty$. For small α and large N,

$$
\begin{aligned}
d_N &= N + \sum_{j=1}^{N-1}(1 + \frac{\delta}{N})^j e^{-\alpha(j+1)j/2} \frac{1 - e^{-\alpha j(N-j)}}{1 - e^{-\alpha j}} \\
&\sim N^2 \frac{e^{\gamma} - 1 - \gamma}{\gamma^2} - \alpha N^4 \left(\frac{(e^{\gamma} - 1 - \gamma - \frac{\gamma^2}{2})(\gamma - 2)}{2\gamma^3} + \frac{1}{4}\right) + o(N^6 \alpha^2).
\end{aligned}
$$

With $\alpha = \gamma/N^2$ as $N \to \infty$, Then

$$
\begin{aligned}
\frac{d_N - N}{N^2} &\to \int_0^1 e^{\delta x} e^{-\gamma x^2} \frac{1 - e^{-\gamma x(1-x)}}{\gamma x} dx \\
&\sim \frac{e^{\delta} - 1 - \delta}{\delta^2} - \frac{\delta + 2 + e^{\delta}\delta - 2e^{\delta}}{2\delta^3}\gamma + O(\gamma^2)
\end{aligned}
$$

Proof. The proof of Proposition 4 is done by using the Taylor series expansion of order 1 in the neighborhood of 0 and the approximations method of a sum by an integral. From relation 4, we have

$$
\begin{aligned}
\frac{d_N - N}{N} &= \frac{1}{N} \sum_{j=1}^{N-1} \kappa^j e^{-\alpha(j+1)j/2} \frac{1 - e^{-\alpha(N-j)j}}{1 - e^{-\alpha j}} \\
&= \frac{1}{N} \sum_{j=1}^{N-1} \kappa^j (1 - \alpha(j+1)j/2) \frac{1 - e^{-\frac{\beta}{N}(N-j)j}}{1 - (1 - \frac{\beta}{N}j)} \\
&= \sum_{j=1}^{N-1} \kappa^j (1 - \frac{\beta}{N}(j+1)j/2) \frac{1 - e^{-\beta(1-\frac{j}{N})j}}{\beta j} \\
&= \frac{1}{\beta} \sum_{j=1}^{N-1} \frac{\kappa^j}{j}(1 - \frac{\beta}{N}(j+1)j/2)(1 - e^{-\beta(1-\frac{j}{N})j}).
\end{aligned}
$$

As N goes to infinity, we get

$$\frac{d_N - N}{N} \to \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{\kappa^j}{j}(1 - e^{-\beta j})$$

$$= \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{\kappa^j}{j} - \frac{1}{\beta} \sum_{j=1}^{\infty} \frac{(\kappa e^{-\beta})^j}{j}.$$

The series expansion of $x \mapsto \ln(1-x)$ allow us to get point (i) of proposition 4, which is

$$\frac{d_N - N}{N} \to \frac{1}{\beta} \ln \left(\frac{1 - \kappa e^{-\beta}}{1 - \kappa} \right).$$

We have

$$
\begin{aligned}
\frac{d_N - N}{N^2} &= \frac{1}{N^2} \sum_{j=1}^{N-1} \kappa^j e^{-\alpha(j+1)j/2} \frac{1 - e^{-\alpha(N-j)j}}{1 - e^{-\alpha j}} \\
&= \frac{1}{N} \sum_{j=1}^{N-1} (1 + \frac{\delta}{N})^j e^{-\gamma \frac{(j+1)}{N} \frac{j}{N}/2} \frac{1 - e^{-\gamma(1-\frac{j}{N})\frac{j}{N}}}{\gamma \frac{j}{N}} \\
&= \frac{1}{N} \sum_{j=1}^{N-1} e^{\delta \frac{j}{N}} e^{-\gamma \frac{(j+1)}{N} \frac{j}{N}/2} \frac{1 - e^{-\gamma(1-\frac{j}{N})\frac{j}{N}}}{\gamma \frac{j}{N}} \\
&\to \int_0^1 e^{\delta x} e^{-\frac{\gamma}{2}x^2} \frac{1 - e^{-\gamma(1-x)x}}{\gamma x} dx.
\end{aligned}
$$

The Maclaurin expansion of order 2 of maps $\gamma \mapsto e^{-\frac{\gamma}{2}x^2}$ and $\gamma \mapsto e^{-\gamma x(1-x)}$ leads to get

$$\frac{d_N - N}{N^2} \sim \frac{e^\delta - \delta - 1}{\delta^2} - \frac{\delta + 2 + e^\delta \delta - 2e^\delta}{2\delta^3} \gamma + O(\gamma^2).$$

3.2. Variance of the discharge number of pulses

In this section, we study the variance of the discharge number.
Let

$$\sigma_i^2 = \text{Var}(D_i), \quad 1 \le i \le N.$$

Set $t_0 = 0$ and we introduce

$$t_i = E(D_i^2), \quad y_i = t_i - t_{i-1}, \quad 1 \le i \le N.$$

By conditioning on the first jump,

$$
\begin{cases}
y_i = 1 + 2d_{i-1} + \kappa e^{-\alpha(N-i)} y_{i+1}, & 1 \le i \le N - 1 \\
y_N = 1 + 2d_{N-1}
\end{cases}
$$

so

$$y_i = u_i + 2 \sum_{j=0}^{N-i} d_{i-1+j} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}, \quad i = 1, \dots N.$$

Thus, for $1 \leq k \leq N$,

$$v_k = \sum_{i=1}^{k} y_i = d_k + 2 \sum_{i=0}^{k-1} d_i + 2 \sum_{i=1}^{k} \sum_{j=1}^{N-i} d_{i-1+j} \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}.$$

We will focus on the case $k = N$, where

$$v_N = d_N + 2 \sum_{i=0}^{N-1} d_i + 2 \sum_{i=1}^{N} \sum_{j=1}^{N-i} d_{i-1+j} \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}.$$

Recall

$$d_k = k + \sum_{i=1}^{k} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

Thus

$$2 \sum_{k=1}^{N-1} d_k = 2 \sum_{k=1}^{N-1} k + 2 \sum_{k=1}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

so

$$
\begin{aligned}
v_N &= N + \sum_{i=1}^{N} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&\quad + N(N-1) + 2 \sum_{k=1}^{N-1} \sum_{i=1}^{k} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&\quad + 2 \sum_{i=1}^{N} \sum_{j=1}^{N-i} (i-1+j) \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&\quad + 2 \sum_{i=1}^{N} \sum_{j=1}^{N-i} \sum_{r=1}^{i-1+j} \sum_{s=1}^{N-r} \kappa^s e^{-\alpha(2(N-r)-s+1)s/2} \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&= N^2 + \sum_{i=1}^{N} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&\quad + 2 \sum_{i=1}^{N} \sum_{j=1}^{N-i} (N-1+j) \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2} \\
&\quad + 2 \sum_{i=1}^{N} \sum_{j=1}^{N-i} \sum_{r=1}^{i-1+j} \sum_{s=1}^{N-r} \kappa^s e^{-\alpha(2(N-r)-s+1)s/2} \, \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}
\end{aligned}
$$

This yields for $\sigma_i^2 = t_i - d_i^2$,

$$\sigma_N^2 = \sum_{i=1}^{N} \sum_{j=1}^{N-i} \kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$+2\sum_{i=1}^{N}\sum_{j=1}^{N-i}(j-1)\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$+2\sum_{i=1}^{N}\sum_{j=1}^{N-i}\sum_{r=1}^{i-1+j}\sum_{s=1}^{N-r}\kappa^s e^{-\alpha(2(N-r)-s+1)s/2}\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$-\Big(\sum_{i=1}^{N}\sum_{j=1}^{N-i}\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}\Big)^2$$

$$=\ \sum_{i=1}^{N}\sum_{j=1}^{N-i}(2j-1)\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$+\sum_{i=1}^{N}\sum_{j=1}^{N-i}\sum_{r=1}^{i-1+j}\sum_{s=1}^{N-r}\kappa^s e^{-\alpha(2(N-r)-s+1)s/2}\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$-\sum_{i=1}^{N}\sum_{j=1}^{N-i}\sum_{r=i+j}^{N}\sum_{s=1}^{N-r}\kappa^s e^{-\alpha(2(N-r)-s+1)s/2}\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}.$$

A better writting is

$$\sigma_N^2\ =\ \sum_{i=1}^{N}\sum_{j=1}^{N-i}(2j-1)\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}$$

$$+\Big(\sum_{i=1}^{N}\sum_{j=1}^{N-i}\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}\Big)^2$$

$$-2\sum_{i=1}^{N}\sum_{j=1}^{N-i}\sum_{r=i+j}^{N}\sum_{s=1}^{N-r}\kappa^s e^{-\alpha(2(N-r)-s+1)s/2}\,\kappa^j e^{-\alpha(2(N-i)-j+1)j/2}.$$

The cases studied above in proposition 2 for $E(D_N)$ suggests that for case (i) $\sigma_N^2 \sim$ constN^2 and case (ii) $\sigma_N^2 \sim$ constN^4. It is better to approximate the above sums directly with integrals. Checking cases such as $\alpha \to 0$, $\kappa = 1$ might give some insights.

4. Variations of the Markov chain model

In this section we introduce variations of the Markov chain model.
Recall that the Markov chain model has the defining properties

$$(X_{n+1}|X_n = x) = \begin{cases} x-1 & \text{with probability} & q \\ x+1 & \text{with probability} & (1-q)e^{-\alpha(N-x)} \\ x & \text{with probability} & (1-q)(1-e^{-\alpha(N-x)})) \end{cases}$$

It is natural to consider the extended, bivariate model $(V_n, X_n)_{n \geq 1}$, with dynamics specified by

$$(v, x) \rightarrow \begin{cases} (v-1, x-1) & \text{with probability} & q \\ (v, x+1) & \text{with probability} & (1-q)e^{-\alpha(N-x)-\beta(T-v)} \\ (v, x) & \text{with probability} & (1-q)(1 - e^{-\alpha(N-x)-\beta(T-v)}) \end{cases}$$

which yields

$$x_t = N + \frac{\beta(T-v)}{\alpha} - \frac{1}{\alpha} \ln \left(e^{\alpha q t} - \frac{1-q}{q} \left(e^{\alpha q t} - 1 \right) \right).$$

Hence, for $q \geq 1/2$ the average nominal charge x as a function of remaining theoretical charge v obeys the approximation

$$x(v) = N - \frac{1}{\alpha} \ln \left(e^{-\alpha(T-v)} - \frac{1-q}{q} \left(e^{-\alpha(T-v)} - 1 \right) \right) \tag{5}$$

Conclusion

In this paper, we presented a Markov modeling approach of the battery based on the work in [3]. We derived results on the delivered and gained capacities. These quantities where studied in [8] for the kinetic battery models. Moreover, we investigated the mean and variance of the number of pulses. We also looked at the asymptotic behavior of the mean number of pulses. We end up the paper by the extended version of the Markov model.

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References

[1] Min Chen and Gabriel A Rincon-Mora. Accurate electrical battery model capable of predicting runtime and iv performance. *IEEE transactions on energy conversion*, 21(2):504–511, 2006.

[2] C-F Chiasserini and Ramesh R Rao. Energy efficient battery management. *IEEE journal on selected areas in communications*, 19(7):1235–1245, 2001.

[3] Carla-Fabiana Chiasserini and Ramesh R Rao. A model for battery pulsed discharge with recovery effect. In *WCNC. 1999 IEEE Wireless Communications and Networking Conference (Cat. No. 99TH8466)*, volume 2, pages 636–639. IEEE, 1999.

[4] CF Chiasserini and Ramesh R Rao. A traffic control scheme to optimize the battery pulsed discharge. In *MILCOM 1999. IEEE Military Communications. Conference Proceedings (Cat. No. 99CH36341)*, volume 2, pages 1419–1423. IEEE, 1999.

[5] Marc Doyle, Thomas F Fuller, and John Newman. Modeling of galvanostatic charge and discharge of the lithium/polymer/insertion cell. *Journal of the Electrochemical society*, 140(6):1526, 1993.

[6] Marijn R Jongerden and Boudewijn R Haverkort. Which battery model to use? *IET software*, 3(6):445–457, 2009.

[7] Marijn R Jongerden and Boudewijn R Haverkort. Battery aging, battery charging and the kinetic battery model: A first exploration. In *International Conference on Quantitative Evaluation of Systems*, pages 88–103. Springer, 2017.

[8] Ingemar Kaj and Victorien Konané. Analytical and stochastic modelling of battery cell dynamics. In *International Conference on Analytical and Stochastic Modeling Techniques and Applications*, pages 240–254. Springer, 2012.

[9] Ingemar Kaj and Victorien Konané. Modeling battery cells under discharge using kinetic and stochastic battery models. *Applied Mathematical Modelling*, 40(17-18):7901–7915, 2016.

[10] Alamgir Ahmad Khattak, Asad Nawaz Khan, Muhammad Safdar, Abdul Basit, and Nauman Ahmad Zaffar. A hybrid electric circuit battery model capturing dynamic battery characteristics. In *2020 IEEE Kansas Power and Energy Conference (KPEC)*, pages 1–6. IEEE, 2020.

[11] LHJ Raijmakers, DL Danilov, R-A Eichel, and PHL Notten. An advanced all-solid-state li-ion battery model. *Electrochimica Acta*, 330:135147, 2020.

[12] Daler Rakhmatov, Sarma Vrudhula, and Deborah A Wallach. A model for battery lifetime analysis for organizing applications on a pocket computer. *IEEE Transactions on Very Large Scale Integration (VLSI) Systems*, 11(6):1019–1030, 2003.

[13] Venkat Rao, Gaurav Singhal, Anshul Kumar, and Nicolas Navet. Battery model for embedded systems. In *18th International Conference on VLSI Design held jointly with 4th International Conference on Embedded Systems Design*, pages 105–110. IEEE, 2005.