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A Breadth-first Search Tree Construction for Multiplicative Circulant Graphs

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Abstract. In this paper, we give a recursive method in constructing a breadth-first search tree for multiplicative circulant graphs of order power of odd. We then use the proposed construction in reproving some results concerning multiplicative circulant graph's diameter, average distance and distance spectral radius. We also determine the graph's Wiener index, vertex-forwarding index, and a bound for its edge-forwarding index. Finally, we discuss some possible research works in which the proposed construction can be applied.
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1. Introduction

Let Γ be a simple connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The number $d_\Gamma(v_i, v_j)$ denotes the distance between two vertices v_i and v_j of Γ, which is the number of edges in a shortest path between the vertices. For a fix $v_i \in V(\Gamma)$ and for any $v_j \in V(\Gamma)$, $d_\Gamma(v_i, v_j)$ can be determined using the *Breadth-first Search Method* or simply called bfs method. The pseudo-code for bfs method is given in the next page.

When bfs method is applied to a particular vertex $v_i \in V(\Gamma)$ of the graph Γ, the result is a rooted tree with vertex v_i as the root. This tree is called a *bfs tree* with root v_i and is denoted by $bfs_{v_i}(\Gamma)$. The rooted tree $bfs_0(C_5)$ is shown in the right part of Figure 1.

In a rooted tree, we call a vertex v_i the *parent* of vertex v_j and vertex v_j a *child* of vertex v_i if the edge (v_i, v_j) is an edge in a rooted tree; where the naming of an edge (v_i, v_j) is with respect to their level relative to the root. Also, a vertex v_i is said to be an *ancestor* of vertex v_j and vertex v_j is a *descendant* of vertex v_i if there is a path from v_i to v_j whose edges all go from parent to child.

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Breadth-first Search Algorithm [5]

Input: Undirected graph $\Gamma = (V(\Gamma), E(\Gamma))$ and a vertex $s \in V(\Gamma)$
Output: Breadth-first tree T from s.
    $V_i = \{\text{all vertices at distance } i \text{ from } s\}$
$V_0 = \{s\}$
make s the root of T
$i = 0$
while $V_i \neq \emptyset$ do construct V_{i+1}
    $V_{i+1} = \emptyset$
    for each vertex $v \in V_i$ do
    "scan v"
        for each edge (v, w) do
            if $w \notin \bigcup_j V_j$ then
            make w the next child of v in T
            add w to V_{i+1}
    $i = i + 1$

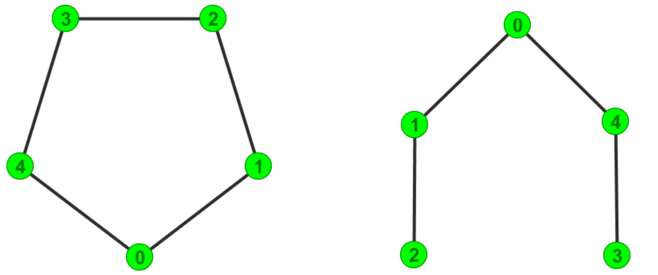

Figure 1: The graph C_5 and its corresponding bfs tree for vertex 0.

In the rooted tree of Figure 1, vertex 1 is the parent of vertex 2 and hence, vertex 2 is a child of vertex 1. Also, the vertices 1,2,3, and 4 are descendants of the root vertex 0.

The bfs tree contains the distance information between the root and all the other vertices in $V(\Gamma)$. For instance, for the graph C_5 in the left part of Figure 1, its corresponding bfs tree rooted from 0-vertex shown in the right part of Figure 1 reveals that $d_{C_5}(0, v) = 1$ if $v = 1, 4$ and $d_{C_5}(0, v) = 2$ if $v = 2, 3$.

For a tree with vertical axial symmetry such as the tree in Figure 1, we classify its vertices as to whether it is located on the left part or on the right part of the tree. For instance, the left part of the bfs tree of C_5 with root vertex 0 denoted by $L[bfs_0(C_5)]$ contains the vertices 1 and 2; while the right part of the bfs tree ofC_5 with root vertex 0 denoted by $R[bfs_0(C_5)]$ contains the vertices 3 and 4.

The main goal of this paper is to present a method on constructing a bfs tree for multiplicative circulant graphs of order power of odd. We formally define multiplicative circulant graph in the next paragraph.

Multiplicative circulant graphs are special type of *Cayley graphs*. By definition, given a group G and a subset S of $G - \{e\}$, a graph Γ is a *Cayley graph* of G with connection (or jump) set S, written $\Gamma = Cay(G, S)$ if $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} : g \in G, s \in S\}$. If $G = \langle \mathbb{Z}_n, +_n \rangle$, then the graph $\Gamma = Cay(G, S)$ is called the *circulant graph* with connection set S. If a circulant graph $Cay(\mathbb{Z}_n, S)$ is such that $n = m^h$ and $S = \{m^0, m^1, \ldots, m^{h-1}\}$ where m and h are integers with bounds $m > 1$ and $h \geq 0$, then $Cay(\mathbb{Z}_n, S)$ is called a *multiplicative circulant graph* or MC graph for short. MC graphs will be denoted by $MC(m^h)$ or $\Gamma_{(}m^h)$.

MC graphs was originally defined by Stojmenovic [12] in 1997 when he studied a particular class of circulant graph called *recursive circulant graph* or RC graph that was introduced by Park and Chwa [11] in 1994. Both MC and RC graphs are a special class of *generalized recursive circulant graph* or GRC graph defined by Tang et al. [14] in 2012. In particular, MC graphs are GRC graphs in which each dimensions have identical bases. Figure 2 shows some examples of MC graphs.

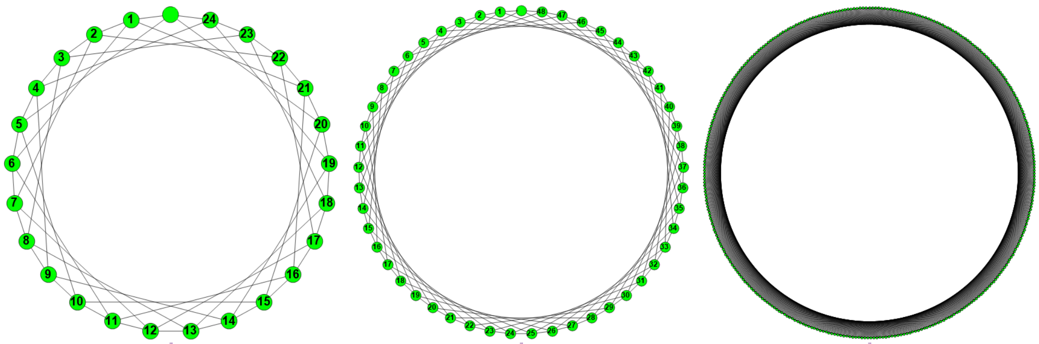

Figure 2: The graphs $MC(5^2)$, $MC(7^2)$ and $MC(7^3)$

MC graphs and in general circulant graphs have vast applications in different fields of study; some of these fields include telecommunication networking [4], VLSI (Very-large-scale integration) design [8], and distributed computing [10].

In the definition of multiplicative circulant graph, let m be odd. The following are important observable properties of Γ_{m^h} whose proofs follow from the definition of MC graph and the bfs method:

(i) $d_{\Gamma_{m^h}}(0, i) = d_{\Gamma_{m^h}}(0, m^h - i)$ for all non-zero $i \in V(\Gamma_{m^h})$.

(ii) Let $A = \{m^{h-1} - m^{h-2}, m^{h-1} - m^{h-3}, \ldots, m^{h-1} - m^{h-h}\}$. For each $a \in A$, we have
$$d_{\Gamma_{m^h}}(0, a) = d_{\Gamma_{m^{h-1}}}(0, a) + 1.$$

(iii) Γ_{m^h} is ancestor-preserving for parents $m^0, m^1, \ldots, m^{h-2}$ in $bfs_0(\Gamma_{m^h})$. That is, for parents $m^0, m^1, \ldots, m^{h-2}$, the ancestor-descendant relationship is the same for $bfs_0(\Gamma_{m^{h-1}})$ and $bfs_0(\Gamma_{m^h})$.

(iv) For $b = 1, 2, \ldots, \frac{m^{h-1}-1}{2}$, we have

$$d_{\Gamma_{m^{h-1}}}(0, b) = d_{\Gamma_{m^h}}(m^{h-1}, m^{h-1} + b).$$

(v) For $b = 1, 2, \ldots, \frac{m^{h-1}-1}{2}$, we have

$$\begin{aligned} d_{\Gamma_{m^h}}(m^{h-1}, m^{h-1} \pm b) &= d_{\Gamma_{m^h}}(2(m^{h-1}), 2(m^{h-1}) \pm b) \\ &= d_{\Gamma_{m^h}}(3(m^{h-1}), 3(m^{h-1}) \pm b) \\ &\vdots \\ &= d_{\Gamma_{m^h}}\left(\frac{m-1}{2}(m^{h-1}), \frac{m-1}{2}(m^{h-1}) \pm b\right). \end{aligned}$$

In this paper, we give a recursive method on constructing the bfs tree for Γ_{m^h} using the listed properties above. We then use the construction to reprove some known results about Γ_{m^h}'s diameter, average distance and distance spectral radius. We also determine the following graph-related properties for Γ_{m^h}: Wiener index, vertex-forwarding index, and bounds for its edge-forwarding index. Finally, we discuss some possible research works in which the proposed construction can be applied.

2. Preliminaries

In this section, we discuss in a brief, the necessary concepts and results that will be used in the discussion of our main results. The discussion includes graphs' distance matrix, distance spectral radius, vertex and edge forwarding index, and Wiener index.

In the following definitions and discussion, we assume that our graph Γ is of n number of vertices. We begin by defining the concept of *distance matrix of a graph.*

Definition 1. *The* **distance matrix** *of* Γ *denoted by* $\mathbf{D}(\Gamma) = [D_{ij}]$ *where*

$$D_{ij} = \begin{cases} d_\Gamma(v_i, v_j) \text{ if } v_i \neq v_j \\ 0 \text{ otherwise.} \end{cases}$$

Remark 1. *Circulant graphs have circulant distance matrix [9].*

The next series of graph concepts for Γ can be calculated once $\mathbf{D}(\Gamma)$ is known.

Definition 2. *The* **diameter** *of* Γ, *denoted by* $diam(\Gamma)$, *is the maximum distance between any pair of vertices in* Γ.

Remark 2. $diam(\Gamma)$ *is the maximum entry in* $\mathbf{D}(\Gamma)$.

Definition 3. *The* **transmission of** v_i **in** Γ *denoted by* $Tr_\Gamma(v_i)$, *is the sum of distances from* v_i *to all other vertices of* Γ, *that is*

$$Tr_\Gamma(v_i) = \sum_{v_j \in V(\Gamma)} d_\Gamma(v_i, v_j).$$

Remark 3. *$Tr_\Gamma(v_i)$ is the sum of the entries in the i^{th} row of $\mathbf{D}(\Gamma)$.*

Definition 4. *The Wiener index of Γ denoted by $W(\Gamma)$ is defined by*

$$W(\Gamma) = \sum_{\{v_i,v_j\}\subseteq V(\Gamma)} d_\Gamma(v_i, v_j).$$

Remark 4. *$W(\Gamma)$ is the sum of all the entries in $\mathbf{D}(\Gamma)$ divided by 2.*

Definition 5. *The* **average distance** *of Γ denoted by $\mu(\Gamma)$ is the average of all distances in Γ. In symbol*

$$\mu(\Gamma) = \frac{\sum\limits_{\{v_i,v_j\}\subseteq V(\Gamma)} d_\Gamma(v_i, v_j)}{\binom{n}{2}}.$$

Remark 5. $\mu(\Gamma) = \dfrac{W(\Gamma)}{\binom{n}{2}}$.

Definition 6. *The largest eigenvalue of the distance matrix of Γ is called the* **distance spectral radius** *of Γ and is denoted by $\rho(\Gamma)$.*

In terms of vertex transmission, a special name for a graph Γ with uniform vertex transmission is given in the next definition.

Definition 7. *A graph Γ is said to be* **s-transmission regular** *if $Tr_\Gamma(v_i) = s$ for every $v_i \in V(\Gamma)$.*

Remark 6. *Since the distance matrix of a circulant graph is circulant, it follows that circulant graphs are transmission regular graphs with transmission-regularity $Tr_\Gamma(v_0)$.*

For transmission regular graphs such as circulant graphs, the calculation of distance spectral radius is simpler.

Lemma 1 ([9])**.** *Let Γ be a circulant graph. Then $\rho(\Gamma) = Tr_\Gamma(v_0)$.*

We now define the concept of graph's vertex and edge forwarding index. To define them we need to define a series of interrelated concepts.

Definition 8. *A* **routing** *R of Γ is a set of $n(n-1)$ elementary paths (i.e. paths where no vertices appear more than once) $R(x, y)$ specified for all ordered pairs (x, y) of vertices of Γ.*

Remark 7. *The set of all possible routing in a graph Γ is denoted by $\mathcal{R}(\Gamma)$.*

For vertex-forwarding index we have

Definition 9. *Let $R \in \mathcal{R}(\Gamma)$ and $x \in V(\Gamma)$. The* **load of a vertex** *x in R of Γ denoted by $\xi_x(\Gamma, R)$ is the number of paths specified by R passing through x and admitting x as an inner vertex.*

Definition 10. *The* **vertex-forwarding index of** Γ **with respect to a routing** R*, denoted by* $\xi(\Gamma, R)$ *is the maximum number of paths of* R *going through any vertex* x *in* Γ*. Hence*

$$\xi(\Gamma, R) = max\{\xi_x(\Gamma, R) : x \in V(\Gamma)\}.$$

Definition 11. *The* **vertex-forwarding index of** Γ*, denoted by* $\xi(\Gamma)$ *is the minimum forwarding index over all possible routing of* Γ*. In symbol,*

$$\xi(\Gamma) = min\{\xi(\Gamma, R) : R \in \mathcal{R}(\Gamma)\}.$$

For edge-forwarding index we have

Definition 12. *The* **load of an edge** e *with respect to* R*, denoted by* $\pi_e(\Gamma, R)$*, is the number of the paths specified by* R *going through it.*

Definition 13. *The* **edge forwarding index of a graph** Γ **with respect to a routing** R*, denoted by* $\pi(\Gamma, R)$ *is the maximum number of paths specified by* R *going through any edge of* Γ*. Hence*

$$\pi(\Gamma, R) = max\{\pi_e(\Gamma, R) : e \in E(\Gamma)\}.$$

Definition 14. *The* **edge-forwarding index of a graph** Γ*, denoted by* $\pi(\Gamma)$ *is defined by*

$$\pi(\Gamma) = min\{\pi(\Gamma, R) : R \in \mathcal{R}(\Gamma)\}.$$

We end this section by giving the exact value of vertex-forwarding index and a bound for the edge-forwarding index of a graph Γ. They are given in the last two results for this section.

Lemma 2 (Lemma 4.2 [9])**.** *If* Γ *is a connected circulant graph of order* n*, then*

$$\xi(\Gamma) = \rho(\Gamma) - (n - 1).$$

Lemma 3 (Lemma 4.5 [9])**.** *If* Γ *is a connected* $r-$*regular circulant graph of order* n*, then*

$$\frac{2\rho(\Gamma)}{r} \leq \pi(\Gamma) \leq n + \rho(\Gamma) - (2r - 1).$$

3. A bfs tree construction for Γ_{m^h}

It is evident from properties (i)-(v) that the construction of $bfs_0(\Gamma_{m^h})$ will be based on $bfs_0(\Gamma_{m^{h-1}})$. Also, from property (i) and the fact that for any $x, y \in \mathbb{Z}_n$ and $s \in S$, we have if $x + y = 0$ then $(x + s) + (y - s) = 0$ and $(x - s) + (y + s) = 0$, we know that $bfs_0(\Gamma_{m^h})$ has a vertical axial-symmetry with respect to the 0-vertex. So we have a definition and a remark.

Definition 15. *The* **left part** *of* $bfs_0(\Gamma_{m^h})$ *denoted by* $L[bfs_0(\Gamma_{m^h})]$ *refers to the vertices*

$$m^0, m^1, \ldots, m^{h-1},$$

and their descendants. While the **right part** *of* $bfs_0(\Gamma_{m^h})$ *denoted by* $R[bfs_0(\Gamma_{m^h})]$ *refers to the vertices*

$$m^h - m^{h-1}, m^h - m^{h-2}, \ldots, m^h - m^{h-h},$$

and their descendants.

Remark 8. *For odd intger* m *and positive integer* h *we have*

$$L[bfs_0(\Gamma_{m^h})] = \{1, 2, \ldots, \tfrac{m^h-1}{2}\}$$

while

$$R[bfs_0(\Gamma_{m^h})] = \{\tfrac{m^h-1}{2}+1, \tfrac{m^h-1}{2}+2, \ldots, m^h-1\}.$$

Using the five properties of Γ_{m^h} presented in the introduction, a method for constructing $bfs_0(\Gamma_{m^h})$ based from $bfs_0(\Gamma_{m^{h-1}})$ is as follows:

Method on Constructing $bfs_0(\Gamma_{m^h})$

Given $bfs_0(\Gamma_{m^{h-1}})$, $bfs_0(\Gamma_{m^h})$ can be constructed as follows:

Step 1. In $bfs_0(\Gamma_{m^{h-1}})$, replace the 0-vertex by m^{h-1}.
Step 2. (Properties (ii) and (iii)) Descend the vertex m^{h-1} and the right part of $bfs_0(\Gamma_{m^{h-1}})$ by a unit and introduce the new 0-vertex.
Step 3. (Property (iv)) Reproduce the left part of $bfs_0(\Gamma_{m^{h-1}})$ with the substitution

$$0 := m^{h-1}.$$

Step 4. (Property (v)) Let $r = 2$. Introduce the vertex $r(m^{h-1})$ as a child of vertex $(r-1)(m^{h-1})$ and reproduce the genealogy of vertex $(r-1)(m^{h-1})$ with the substitution

$$(r-1)(m^{h-1}) := r(m^{h-1}).$$

Step 5. (Property (v)) Repeat Step 4 for $r = 3, 4, \ldots, \frac{m-1}{2}$
Step 6. Complete $bfs_0(\Gamma_{m^h})$ using property (i).

Remark 9. *The method just presented is an extension of a method presented in [3] for constructing* $bfs_0(\Gamma_{3^h})$.

Example 1. *We illustrate the method by constructing a bfs tree rooted at 0-vertex for the graph* Γ_{5^2} *using* $bfs_0(\Gamma_{5^1})$ *in Figure 1 as an input. Using the propose method, we have a bfs tree rooted at 0-vertex for* Γ_{5^2} *as shown in Figure 3.*

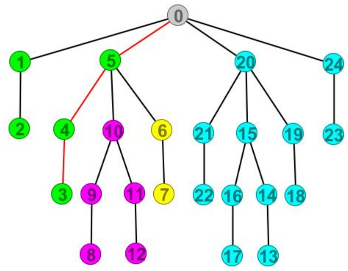

Figure 3: A bfs tree of the graph Γ_{5^2} with root 0. The green-colored vertices refer to the vertices that originally appeared in $bfs_0(\Gamma_{5^1})$. While the green-colored vertices with red edges refer to the descended vertices in $bfs_0(\Gamma_{5^1})$. The yellow-colored vertices refer to the resulting vertices as a result of reproducing the left part of $bfs_0(\Gamma_{5^1})$ with the substitution $0 := 5$. The violet-colored vertices refer to the resulting vertices as a result of introducing the vertex $5+5$ as a child of vertex 5 and reproducing the genealogy of vertex 5 with the substitution $5 =: 5+5$. Finally, the blue-colored vertices are the vertices obtained using property (i).

Example 2. *In this example, we illustrate the method by constructing a bfs tree for the graph Γ_{7^2} with root 0 using $bfs_0(\Gamma_{7^1})$ shown in Figure 4 as an input. Using the propose method, we have a bfs tree for Γ_{7^2} with root 0 as shown in Figure 5.*

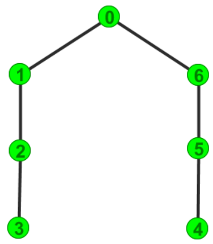

Figure 4: The bfs tree of Γ_{7^1} with 0-vertex as the root.

Based on the bfs tree construction for Γ_{m^h} with 0 as the root vertex, we have

Theorem 1. *Let h be a positive integer. Then*

$$d_{\Gamma_{m^h}}(0,j) = \begin{cases} d_{\Gamma_{m^{h-1}}}(0,j) & \text{if } j = 0,1,2,\ldots,\frac{m^{h-1}-1}{2} \\ d_{\Gamma_{m^{h-1}}}(0,j)+1 & \text{if } j =, \frac{m^{h-1}-1}{2}+1\ldots,m^{h-1}-1. \end{cases} \tag{1}$$

Moreover, if k_j, l_{ij^+}, and $l_{ij^-} \in V(\Gamma_{m^h})$ such that $k_j = m^{h-1}+j$, $l_{ij^+} = (i+1)(m^{h-1})+j$ and $l_{ij^-} = (i+1)(m^{h-1})-j$ where $i = 1,2,\ldots,\frac{m-1}{2}-1$ and $j = 0,1,\ldots,\frac{m^{h-1}-1}{2}$ then

$$d_{\Gamma_{m^h}}(0,k_j) = d_{\Gamma_{m^{h-1}}}(0,j)+1, \tag{2}$$

and

$$d_{\Gamma_{m^h}}(0,l_{ij^+}) = d_{\Gamma_{m^h}}(0,l_{ij^-}) = d_{\Gamma_{m^{h-1}}}(0,j)+(i+1) \tag{3}$$

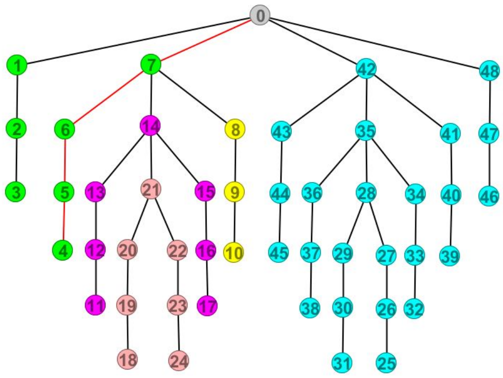

Figure 5: A bfs tree of the graph Γ_{7^2}. The green-colored vertices refer to the vertices that originally appeared in $bfs_0(\Gamma_{7^1})$. While the green-colored vertices with red edges refer to the descended vertices in $bfs_0(\Gamma_{7^1})$. The yellow-colored vertices refer to the resulting vertices as a result of reproducing the left part of $bfs_0(\Gamma_{7^1})$ with the substitution $0 := 7$. The violet-colored vertices refer to the resulting vertices as a result of introducing the vertex $7+7$ as a child of vertex 7 and reproducing the genealogy of vertex 7 with the substitution $7 =: 7+7$. The beige-colored vertices refer to the resulting vertices as a result of introducing the vertex $7+7+7$ as a child of vertex $7+7$ and reproducing the genealogy of vertex $7+7$ with the substitution $7+7 =: 7+7+7$. The blue-colored vertices are the vertices obtained using property (i).

Proof. Steps 1 and 2 imply that if $j = m^{h-1}$, then $d_{\Gamma_{m^h}}(0,j) = 1 = d_{\Gamma_{m^{h-1}}}(0,0)+1$. And that

$$d_{\Gamma_{m^h}}(0,j) = \begin{cases} d_{\Gamma_{m^{h-1}}}(0,j) & \text{if } j \in L[bfs_0(\Gamma_{m^{h-1}})] \\ d_{\Gamma_{m^{h-1}}}(0,j)+1 & \text{if } j \in R[bfs_0(\Gamma_{m^{h-1}})]. \end{cases}$$

By referring to Remark 8 we verified equation (1).

Next, we consider the implication of Step 3. Step 3 implies that if $k_j = m^{h-1}+j$ where $j = 1,2,\ldots,\frac{m^{h-1}-1}{2}$, we have $d_{\Gamma_{m^h}}(0,k_j) = d_{\Gamma_{m^{h-1}}}(0,j)+1$. Combining this with the fact that for $j = m^{h-1}$, we have $d_{\Gamma_{m^h}}(0,j) = 1 = d_{\Gamma_{m^{h-1}}}(0,0)+1$ proves equation (2).

The substitution part of Step 4 implies that for $i = 1$ and $j = 0$, we have $d_{\Gamma_{m^h}}(0,l_{ij}) = d_{\Gamma_{m^h}}(0,l_{0j})+1$ where $l_{0j} = m^{h-1}$. While the part involving reproduction of genealogy implies that for $i = 1$ and $j = 1,2,\ldots,\frac{m^{h-1}-1}{2}$ we have $d_{\Gamma_{m^h}}(0,l_{ij^+}) = d_{\Gamma_{m^h}}(0,l_{ij^-}) = d_{\Gamma_{m^h}}(0,k_j)+1$. Using equation (2) we get $d_{\Gamma_{m^h}}(0,l_{ij^+}) = d_{\Gamma_{m^h}}(0,l_{ij^-}) = d_{\Gamma_{m^{h-1}}}(0,j)+1+1$. This proves the $i = 1$ case of equation (3).

Finally, Step 5 implies the validity of equation (3) for $i = 2,3,\ldots,\frac{m-1}{2}-1$. This completes the proof of the theorem.

Example 3. *Using the constructed bfs tree for Γ_{7^2} in Figure 5 we have*

$$\{0,1,2,3,4,3,2,1,2,3,4,5,4,3,2,3,4,5,6,5,4,3,4,5,6,\\ 6,5,4,3,4,5,6,5,4,3,2,3,4,5,4,3,2,1,2,3,4,3,2,1\}$$

as the first row entries of $\mathbf{D}(\Gamma_{7^2})$.

Using Theorem 1, given the first row of the distance matrix of the graph Γ_{7^2}, we can determine the first row of the distance matrix of the graph Γ_{7^3}. The first row of the distance matrix of the graph Γ_{7^3} is given by

$$\{0, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 3, 4, 5, 6, 5, 4, 3, 4, 5, 6,$$
$$7, 6, 5, 4, 5, 6, 7, 6, 5, 4, 3, 4, 5, 6, 5, 4, 3, 2, 3, 4, 5, 4, 3, 2,$$
$$1, 2, 3, 4, 5, 4, 3, 2, 3, 4, 5, 6, 5, 4, 3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7,$$
$$8, 7, 6, 5, 6, 7, 8, 7, 6, 5, 4, 5, 6, 7, 6, 5, 4, 3, 4, 5, 6, 5, 4, 3,$$
$$2, 3, 4, 5, 6, 5, 4, 3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7, 8, 7, 6, 5, 6, 7, 8,$$
$$9, 8, 7, 6, 7, 8, 9, 8, 7, 6, 5, 6, 7, 8, 7, 6, 5, 4, 5, 6, 7, 6, 5, 4,$$
$$3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7, 8, 7, 6, 5, 6, 7, 8, 9, 8, 7, 6, 7, 8, 9,$$
$$9, 8, 7, 6, 7, 8, 9, 8, 7, 6, 5, 6, 7, 8, 7, 6, 5, 4, 5, 6, 7, 6, 5, 4,$$
$$3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7, 8, 7, 6, 5, 6, 7, 8, 9, 8, 7, 6, 7, 8, 9,$$
$$8, 7, 6, 5, 6, 7, 8, 7, 6, 5, 4, 5, 6, 7, 6, 5, 4, 3, 4, 5, 6, 5, 4, 3,$$
$$2, 3, 4, 5, 6, 5, 4, 3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7, 8, 7, 6, 5, 6, 7, 8,$$
$$7, 6, 5, 4, 5, 6, 7, 6, 5, 4, 3, 4, 5, 6, 5, 4, 3, 2, 3, 4, 5, 4, 3, 2,$$
$$1, 2, 3, 4, 5, 4, 3, 2, 3, 4, 5, 6, 5, 4, 3, 4, 5, 6, 7, 6, 5, 4, 5, 6, 7,$$
$$6, 5, 4, 3, 4, 5, 6, 5, 4, 3, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, 3, 2, 1\}$$

In the above set, we use six colors to represent the distances of each vertices per group. We use color green for the group of vertices covered by the first part of equation (1) *while red for the group of vertices covered by second part. Color yellow were used for the group of vertices covered by equation* (2)*, color violet were used for the group of vertices covered by equation* (3) *in the first implementation while color orange were used for the group of vertices covered by equation* (3) *in the second/final implementation. Finally, we used color blue for the group of vertices covered by property (i).*

Remark 10. *The first row of the distance matrix of* Γ_{7^2} *and* Γ_{7^3} *are verified to be correct using Wolfram Mathematica [7] with the inputs*

d = GraphDistanceMatrix[CirculantGraph[49, {1,7}]]; d[[1]]

and

d = GraphDistanceMatrix[CirculantGraph[343, {1,7,49}]]; d[[1]].

Once the distance of all the vertices in $V(\Gamma_{m^h})$ from the 0-vertex is known, the distance matrix of Γ_{m^h} can be easily determined using Remark 1. In the next section, we discuss some of the many graph properties of Γ_{m^h} that can be determined using its distance matrix.

4. Some consequences of the bfs tree construction for Γ_{m^h}

In this section, we use our proposed construction to reprove some known results involving the diameter, average distance and distance spectral radius of Γ_{m^h}. We also determine the following graph-related properties for Γ_{m^h}: Wiener index, vertex-forwarding index, and bounds for its edge-forwarding index. Except for the diameter and average distance, the results in this section is a generalization of the results presented in [3] for Γ_{3^h}.

On the diameter, average distance and distance spectral radius of Γ_{m^h}

In 1974, Wong and Coppersmith [15] introduced a combinatorial problem related to multimodule memory organizations which involves "memory circulator", a bank of interconnected registers and control circuitry. One model of a memory circulator that was considered in [15] is actually the graph Γ_{m^h}. They determined its diameter as well as its average distance by calculating the points (with integral coordinate) which can be reached from 0 in a given number of steps displayed in the Cartesian coordinate plane showing a uniform filled pattern.

Wong and Coppersmith found out that the diameter of Γ_{m^h} for odd base m is given by $h\left(\frac{m-1}{2}\right)$. They also found out that the average distance of Γ_{m^h} where the "average distance" refers to the sum of all entries in $\mathbf{D}\left(\Gamma_{m^h}\right)$ divided by the number of entries is given by $\frac{h}{m}\left(\frac{m^2-1}{4}\right)$. As a consequence, since the distance matrix of Γ_{m^h} is circulant, the distance spectral radius of Γ_{m^h} is then given by $\left(\frac{m^2-1}{4}\right)h(m^{h-1})$.

We reprove the results involving Γ_{m^h}'s diameter and distance spectral radius using our proposed bfs tree construction in this subsection. This subsection is motivated by the work of Liu et al. [9] where they determined the distance spectral radius of certain class of circulant graphs.

To determine the diameter of Γ_{m^h}, we begin by proving a relationship between the diameters of Γ_{m^h} and $\Gamma_{m^{h-1}}$.

Theorem 2. *The two diameters $diam(\Gamma_{m^h})$ and $diam(\Gamma_{m^{h-1}})$ are related by*

$$diam(\Gamma_{m^h}) = diam(\Gamma_{m^{h-1}}) + \frac{m-1}{2}. \tag{4}$$

Proof. We start by initially assuming that $diam(\Gamma_{m^h})$=$diam(\Gamma_{m^{h-1}})$. Performing the steps necessary to construct $bfs_0(\Gamma_{m^h})$ from $bfs_0(\Gamma_{m^{h-1}})$ gives the following update in the initial diameter of Γ_{m^h}

Step 1: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})$
Step 2: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})+1$
Step 3: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})+1$
Step 4: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})+1+1$
Step 5: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})+1+1+\underbrace{1+1+\ldots+1}_{\frac{m-1}{2}-2}.$
Step 6: $diam(\Gamma_{m^h})=diam(\Gamma_{m^{h-1}})+1+1+\underbrace{1+1+\ldots+1}_{\frac{m-1}{2}-2}.$

Hence $diam(\Gamma_{m^h}) = diam(\Gamma_{m^{h-1}}) + \frac{m-1}{2}$.

Corollary 1. *The diameter of Γ_{m^h} is $h\left(\frac{m-1}{2}\right)$.*

Proof. Note that for all odd integer $m > 1$, we have $diam(\Gamma_{m^1}) = \frac{m-1}{2}$. Using Theorem 2, we have

$$\begin{aligned} diam(\Gamma_{m^2}) &= diam(\Gamma_{m^1}) + \frac{(m-1)}{2} \\ &= 2\left(\frac{m-1}{2}\right). \end{aligned}$$

$$\begin{aligned} diam(\Gamma_{m^3}) &= diam(\Gamma_{m^2}) + \frac{(m-1)}{2} \\ &= 3\left(\frac{m-1}{2}\right). \\ &\vdots \end{aligned}$$

$$\begin{aligned} diam(\Gamma_{m^h}) &= diam(\Gamma_{m^{h-1}}) + \frac{(m-1)}{2} \\ &= (h-1)\left(\frac{m-1}{2}\right) + \frac{(m-1)}{2}. \\ &= h\left(\frac{m-1}{2}\right). \end{aligned}$$

The next result gives the relationship between the two distance spectral radii $\rho(\Gamma_{m^h})$ and $\rho(\Gamma_{m^{h-1}})$.

Theorem 3. *The two distance spectral radii $\rho(\Gamma_{m^h})$ and $\rho(\Gamma_{m^{h-1}})$ are related by*

$$\rho(\Gamma_{m^h}) = m\rho(\Gamma_{m^{h-1}}) + \frac{(m-1)(m+1)}{2}m^{h-1}. \tag{5}$$

Proof. Note that the distance spectral radius of Γ_{m^h} corresponds to the sum of all distances in $bfs_0(\Gamma_{m^h})$. Initially, we have $\rho(\Gamma_{m^h}) = \rho(\Gamma_{m^{h-1}})$. As we go over the steps of constructing $bfs_0(\Gamma_{m^h})$ from $bfs_0(\Gamma_{m^{h-1}})$, the value of $\rho(\Gamma_{m^h})$ will be updated.

After performing Step 2, the initial value of $\rho(\Gamma_{m^h})$ will be added by the number of distance created as a result of descending the vertices m^{h-1} and the right part of $bfs_0(\Gamma_{m^{h-1}})$ by a unit. The number of created distance of the just stated action is exacty $|R[bfs_0(\Gamma_{m^{h-1}})]| + 1$. So we have $\rho(\Gamma_{m^h}) = \rho(\Gamma_{m^{h-1}}) + |R[bfs_0(\Gamma_{m^{h-1}})]| + 1$ after Step 2.

For Step 3, reproducing the left part of $bfs_0(\Gamma_{m^{h-1}})$ will create a distance of $\frac{\rho(\Gamma_{m^{h-1}})}{2}$. Since the reproduction starts at vertex m^{h-1} which is of distance 1 to the $0-$vertex, we need to add another $|L[bfs_0(\Gamma_{m^{h-1}})]|$. So, after Step 3, we have $\rho(\Gamma_{m^h}) = \rho(\Gamma_{m^{h-1}}) + |R[bfs_0(\Gamma_{m^{h-1}})]| + 1 + \frac{\rho(\Gamma_{m^{h-1}})}{2} + |L[bfs_0(\Gamma_{m^{h-1}})]|$.

The action reproduce the genealogy of m^{h-1} in Step 4 will create a distance of $\rho(\Gamma_{m^{h-1}})$. Moreover, since the reproduction starts at vertex $2m^{h-1}$ which is of distance 2 to the 0−vertex, we need to add another $2(m^{h-1})$. As a result, we have $\rho(\Gamma_{m^h}) = \rho(\Gamma_{m^{h-1}}) + |R[bfs_0(\Gamma_{m^{h-1}})]| + 1 + \frac{\rho(\Gamma_{m^{h-1}})}{2} + |L[bfs_0(\Gamma_{m^{h-1}})]| + \rho(\Gamma_{m^{h-1}}) + 2(m^{h-1})$.

The principle that holds in Step 4 is the same principle that holds for Step 5. In general, for $r \in \{3, 4, \ldots, \frac{m-1}{2}\}$ we have an additional distance $\rho(\Gamma_{m^{h-1}}) + r(m^{h-1})$. So after Step 5, we have

$$\begin{aligned}\rho(\Gamma_{m^h}) &= \rho(\Gamma_{m^{h-1}}) + |R[bfs_0(\Gamma_{m^{h-1}})]| + 1 + \frac{\rho(\Gamma_{m^{h-1}})}{2} + |L[bfs_0(\Gamma_{m^{h-1}})]| \\ &\quad + \sum_{r=2}^{\frac{m-1}{2}} \left[\rho(\Gamma_{m^{h-1}}) + r(m^{h-1})\right] \\ &= \rho(\Gamma_{m^{h-1}}) + \frac{\rho(\Gamma_{m^{h-1}})}{2} + m^{h-1} + \left(\frac{m-1}{2} - 1\right)\rho(\Gamma_{m^{h-1}}) + \sum_{r=2}^{\frac{m-1}{2}} r(m^{h-1}) \\ &= \left(1 + \frac{1}{2} + \frac{m-1}{2} - 1\right)\rho(\Gamma_{m^{h-1}}) + \sum_{r=1}^{\frac{m-1}{2}} r(m^{h-1}) \\ &= \frac{m}{2}\rho(\Gamma_{m^{h-1}}) + \frac{(m-1)(m+1)}{4}(m^{h-1}).\end{aligned}$$

Finally, performing Step 6 doubles the current value of $\rho(\Gamma_{m^h})$. As a result, we have the final value of

$$\begin{aligned}\rho(\Gamma_{m^h}) &= 2\left[\frac{m}{2}\rho(\Gamma_{m^{h-1}}) + \frac{(m-1)(m+1)}{4}(m^{h-1})\right] \\ &= m\rho(\Gamma_{m^{h-1}}) + \frac{(m-1)(m+1)}{2}m^{h-1}.\end{aligned}$$

An explicit formula for the distance spectral radius of the graph Γ_{m^h} for any positive integer h is given in the next result.

Corollary 2. *For all positive integer h, we have*

$$\rho(\Gamma_{m^h}) = \left(\frac{m^2-1}{4}\right) h(m^{h-1}).$$

Proof. For $h = 1$, we have $\rho(\Gamma_{m^1}) = \sum_{i=1}^{\frac{m-1}{2}} 2i = \left(\frac{m^2-1}{4}\right)(1)(m^{1-1})$. Now, let $h > 1$ be an integer and suppose that for all $k < h$ we have $\rho(\Gamma_{m^k}) = \left(\frac{m^2-1}{4}\right) k(m^{k-1})$. We show that for h we have $\rho(\Gamma_{m^h}) = \left(\frac{m^2-1}{4}\right) h(m^{h-1})$.

By Theorem 3 we have

$$\rho(\Gamma_{m^h}) = m\rho(\Gamma_{m^{h-1}}) + \frac{(m-1)(m+1)}{2}m^{h-1}.$$

Now since $h-1 < h$, using our induction hypothesis yields

$$\begin{aligned}\rho(\Gamma_{m^h}) &= m\left[\left(\frac{m^2-1}{4}\right)(h-1)(m^{h-2})\right] + \left(\frac{m^2-1}{2}\right)m^{h-1} \\ &= \left(\frac{m^2-1}{4}\right)(h-1)(m^{h-1}) + \left(\frac{m^2-1}{2}\right)m^{h-1} \\ &= \left(\frac{m^2-1}{4}\right)(h-1+1)(m^{h-1}) \\ &= \left(\frac{m^2-1}{4}\right)h(m^{h-1}).\end{aligned}$$

Remark 11. *For* $h = 1, 2, \ldots$, *the sequence* $\left(\frac{m^2-1}{4}\right)h(m^{h-1})$ *denotes the distance spectral radius of* Γ_{m^h}. *For* $m = 3$, *the sequence generated is the sequence A212697 [13] in The On-line Encyclopedia of Integer Sequence (OEIS). For* $m = 5$, *the sequence generated is the sequence A269760 [6] in the OEIS.*

The Wiener Index and Average Distance of Γ_{m^h}

This subsection is motivated by the works of Ali et al. [1, 2], where they determined some distance-based topological indices for certain class of circulant graphs. The computation of the Wiener index of the graph Γ_{m^h} follows immediately from Corollary 2, Remark 4, and Remark 1.

Theorem 4. *The Wiener index of* Γ_{m^h} *is* $\frac{h}{8}(m^{2h-1})(m^2-1)$.

The next result about the average distance of $MC(m^h)$ follows immediately from Theorem 4 and Remark 5.

Theorem 5. *The average distance of* Γ_{m^h} *is* $\frac{\frac{h}{4}(m^2-1)(m^{h-1})}{m^h-1}$.

Exact Value of $\Gamma'_{m^h}s$ Vertex-Forwarding Index

This subsection and the last is motivated by the work of Liu et al. [9] where they determined the exact values of vertex-forwarding index and bounds for the edge-forwarding index of some class of circulant graphs. The exact value of the vertex-forwarding index of Γ_{m^h} is given in the next result.

Theorem 6. *Let $m > 1$ be odd and h be a positive integer. Then*

$$\xi(\Gamma_{m^h}) = \left(\frac{m^2-1}{4}\right) h(m^{h-1}) - (m^h - 1).$$

Proof. Follows from Theorem 2 and Lemma 2.

Bounds for $MC(m^h)'s$ Edge-Forwarding Index

Our final result in this section gives an upper and lower bounds for $\Gamma'_{m^h}s$ edge-forwarding index. The result follows from Theorem 2, Lemma 3 and the fact that Γ_{m^h} is a $2h$-regular graph.

Theorem 7. *Let $m > 1$ be odd and h be a positive integer. Then*

$$\left(\frac{m^2-1}{4}\right)(m^{h-1}) \leq \pi(\Gamma_{m^h}) \leq m^{h-1}\left(m + \frac{m^2-1}{4}h\right) - 4h + 1.$$

5. Future applications of the bfs tree construction for Γ_{m^h}

In this short section, we state some particular research works in which the proposed bfs tree construction can be applied.

As stated earlier, Ali et al. [1, 2]computed some distance-based topological indices for some class of circulant graphs. In particular, they determined the Wiener index, hyper-Wiener index, and Schultz molecular topological index of circulant graph class $Cay(\mathbb{Z}_n, \{1, a\})$ where $a = 2, 3, 4, 5$.

Since the proposed construction presented in this paper determines the distance of 0-vertex to all the other vertices of the graph Γ_{m^h}, and the distance matrix of Γ_{m^h} is circulant, we can use the proposed construction to obtain Γ_{m^h}'s distance matrix. Once the distance matrix of Γ_{m^h} is known, the computation for some distance-based topological indices can be performed.

The distance matrix of Γ_{m^h} can also be used to aid in the study of various distance-based coloring problem related to multiplicative circulant graphs. For instance, the $L(h, k)$-coloring problem.

6. Conclusion

In this paper, we successfully presented a method in constructing a breadth-first search tree for multiplicative circulant graphs of order power of odd with 0-vertex as the root. As a consequence, we were able to reprove some known results about multiplicative circulant graph's diameter, average distance, and distance spectral radius. We also determined the Wiener index, vertex-forwarding index, and bounds for the edge-forwarding index of the studied multiplicative circulant graph. New integer sequences were also generated. Finally, we stated some particular research works in which the proposed bfs tree construction can be applied.

In our next paper, we wish to determine some distance-based topological indices for multiplicative circulant graphs that utilizes our bfs tree construction.

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