EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 1, 2021, 164-172
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Ore Extension Rings Satisfy the Weak PS-Rings

Mohamed A. Farahat[1,2,*], Salha T. Al-Bogamy[1]

[1] *Department of Mathematics and Statistics, Faculty of Science, Taif University, Taif, El-Haweiah, Kingdom of Saudi Arabia (KSA)*
[2] *Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt*

Abstract. The main result of this paper is that: If R is a weak right PS-ring, then $A = R[x; \alpha, \delta]$, the Ore extension ring, is a weak right PS-ring whenever the following conditions hold on R is an (α, δ)-compatible NI-ring with $\mathrm{nil}(R)$ nilpotent, $\alpha(e) = e$ and $\delta(e) = 0$ for every idempotent $e \in R$.

2020 Mathematics Subject Classifications: 16D25, 16P60, 16W60

Key Words and Phrases: PS-ring, weak PS-ring, Ore extensions

---

1. Introduction

Throughout this article, all rings are associative with unity (R denotes such a ring) and all modules are unital R-modules unless explicitly indicated otherwise.

According to Nicholson and Watters [6], M_R is called a *PS-module* if every simple submodule is projective, equivalently if its socle, $\mathrm{Soc}(M_R)$, is projective. Examples of PS-modules include nonsingular modules and modules with zero socle. A left PS-module $_RM$ is defined analogously. A ring R is said to be a *left PS-ring* if $_RR$ is a PS-module. Equivalently, if the left annihilator of every maximal right ideal of R is a principal left ideal generated by an idempotent. Some examples of PS-rings include semiprime and p.p.-rings are PS-rings. In particular every Baer ring is a PS-ring. The notion of PS-rings is not left-right symmetric (cf. [6]). In [6], the authors proved that, if R is a PS-ring so also are $R[x]$ and $R[[x]]$. The converse of this result is false in general by the following example:

Example 1 ([6], Example 3.2)**.** If $R = \mathbb{Z}_4$, then $R[x]$ and $R[[x]]$ are PS-rings but R is not PS-ring.

Many authors investigated the behavior of PS-rings with respect to their extensions. Salem et. al., in ([9], 2015), characterized PS-modules over Ore extensions and skew generalized power series extensions. Also, Farahat and Al-Harthy, in ([3], 2017), investigated

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i1.3886

Email addresses: m_farahat79@yahoo.com,m.farahat @tu.edu.sa (M. A. Farahat), SalhaAlbogamy@hotmail.com (Salha T. Al-Bogamy)

PS-modules over generalized Mal'cev-Neumann series rings. In ([8], 2017), Paykan proved that, under suitable conditions, if R is a right PS-ring, then so the skew inverse power series rings.

Recently, Farahat and Al-Bogamy, in ([2], 2018), extend the notation of PS-rings to weak PS-rings. Recall the definition of weak PS-rings from [2]: A ring R satisfies *the right weak PS-condition* if, the weak annihilator of every maximal right ideal of R is a principal left ideal generated by an idempotent. Similarly, the left weak PS-condition was defined. A ring R satisfies the weak PS-condition if it satisfies both the right and the left weak PS-conditions. The following are some examples of rings satisfy the right weak PS-condition.

Example 2 ([2])**.** 1) Any local ring is a right weak PS-ring.
2) The ring $\mathbb{Z}_{pq}$ of integers modulo pq, where p and q are distinct prime numbers, is a reduced (weak) PS-ring.
3) Let F be a field and $R = \begin{pmatrix} F & F \\ F & F \end{pmatrix}$ is a right weak PS-ring.
4) A semisimple NI ring is a right weak PS-ring.

In this paper, we study the transfer of right weak PS-condition between a base ring R and its Ore extension $A = R[x; \alpha, \delta]$.

2. Notations

(1) $\mathrm{Id}\,(R)$ denotes idempotents of R.

(2) $\mathrm{nil}\,(R)$ denotes nilpotents of R.

(3) For a nonempty subset X of R, $\mathrm{N}_R(X)$ denotes the weak annihilator of X over R, i.e.,

$$\mathrm{N}_R(X) = \{a \in R\,|ax \in \mathrm{nil}\,(R) \text{ for all } x \in X\}.$$

It can be easily shown that

$$ab \in \mathrm{nil}\,(R) \Leftrightarrow ba \in \mathrm{nil}\,(R) \text{ for all } a, b \in R.$$

(4) R is NI if $\mathrm{nil}\,(R)$ is a two sided ideal in R.

3. Ore extension rings satisfy the weak PS-condition

Ore extensions, named after Øystein Ore (1899–1968), are special types of ring extensions whose properties are relatively well understood. These extensions cover a large class of noncommutative polynomial extensions. are special types of ring extensions whose properties are relatively well understood. The definition of noncommutative polynomial rings with identity was first introduced by Øystein Ore [7]. Ever since the appearance of Ore's fundamental paper [7], Ore extensions have played an important role in noncommutative. Ore extensions have wide applications. Not only do they provide interesting examples in noncommutative algebra, they have also been a valuable tool used first by David Hilbert (1862–1943) in the study of the independence of geometry axioms.

Let R be a ring with identity 1 and α an endomorphism of R. Then a map $\delta : R \longrightarrow R$ is called an α-derivation of R if

$$\delta(a+b) = \delta(a) + \delta(b) \text{ and } \delta(ab) = \delta(a)\, b + \alpha(a)\, \delta(b),$$

for all $a, b \in R$.

We denote by $A = R[x; \alpha, \delta]$, the *Ore extension* of R whose elements are polynomials over R, the addition is defined as usual and the multiplication is subject to the relation (Ore commutation rule)

$$xa = \alpha(a)\, x + \delta(a), \text{ for each } a \in R.$$

We assume that 1 is the identity element of $A = R[x; \alpha, \delta]$. This means that $\alpha(1) = 1$ and $\delta(1) = 0$, since

$$x = x1 = \alpha(1)\, x + \delta(1) \Rightarrow \alpha(1) = 1 \text{ and } \delta(1) = 0.$$

Notation ([5]). *For integers i, j with $j \geq i \geq 0$, $\lambda_i^j \in \mathrm{End}(R,+)$ denotes the map which is the sum of all possible "words" in α and δ built with i letters of α and $j-i$ letters of δ. For instance*

$$\begin{aligned} \lambda_0^0 &= \mathrm{Id}_R,\ \lambda_j^j = \alpha^j,\ \lambda_0^j = \delta^j \text{ and} \\ \lambda_{j-1}^j &= \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \ldots + \delta\alpha^{j-1}. \end{aligned}$$

Lemma 1 ([5]). *For any positive integer n and $r \in R$, we have*

$$x^n r = \sum_{i=0}^{n} \lambda_i^n(r) x^i.$$

This formula uniquely determines a general product of (left) polynomials in $R[x; \alpha, \delta]$ and will be used freely in what follows.

The ring-theoretical properties of Ore extension have been investigated by many authors (see [9], [8], [5], [1], [4], for instance). There are many other papers addressed $\delta = 0$ and α an automorphism or the case where α is the identity. However the recent surge of interest in quantum groups and quantized algebras has brought renewed interest in general Ore extensions, due to the fact that many of these quantized algebras and their representations can be expressed in terms of Ore extension rings. When we move from these "unmixed" polynomials to the general case with an endomorphism α and an α-derivation δ, we face a much greater challenge.

Annin [1] introduced the notion of (α, δ)-compatibility as follows:

Definition 1 ([1]). Given a module M_R, an endomorphism $\alpha : R \longrightarrow R$ and an α-derivation $\delta : R \longrightarrow R$. We say that M_R is α-compatible if for each $m \in M$ and $r \in R$, we have $mr = 0 \Leftrightarrow m\alpha(r) = 0$. Moreover, we say that M_R is δ-compatible if for each $m \in M$ and $r \in R$, we have $mr = 0 \Longrightarrow m\delta(r) = 0$. If M_R is both α-compatible and δ-compatible, we say that M_R is (α, δ)-compatible.

A ring R is called (α,δ)-compatible if R_R is an (α,δ)-compatible module. The (α,δ)-compatible condition on the module M_R is a natural, independently interesting condition from which we can derive a number of interesting properties, and it will be of invaluable service in the proof of our main results.

Remark ([1])**.** (1) If M_R is α-compatible, then M_R is α^i-compatible for all $i \geq 1$,
(2) If M_R is δ-compatible, then M_R is δ^i-compatible for all $i \geq 1$,
(3) If M_R is (α,δ)-compatible, then for each $m \in M$ and $r \in R$, we have $mr = 0 \Longrightarrow m\lambda_i^j(r) = 0$ for all $j \geq i \geq 0$.

In what follows, we characterize Ore extension rings that satisfy the weak PS-condition. We need first the following Lemmas which will help us in our target.

Lemma 2. *Let R be an (α,δ)-compatible NI ring. If $a \in \mathrm{nil}(R)$, then $\lambda_i^j(a) \in \mathrm{nil}(R)$ for all $j \geq i \geq 0$.*

Proof. Clearly for any R-endomorphism α, we have $\alpha^k(\mathrm{nil}(R)) \subseteq \mathrm{nil}(R)$, for any positive integer k. Since R is an (α,δ)-compatible ring, we have also that $\delta^k(\mathrm{nil}(R)) \subseteq \mathrm{nil}(R)$. Since R is an NI ring, we conclude that $\lambda_i^j(a) \in \mathrm{nil}(R)$ for any $a \in \mathrm{nil}(R)$ and for all $j \geq i \geq 0$.

Lemma 3. *Let R be an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent and $f(x) = \sum_{i=0}^{n} a_i x^i \in A = R[x;\alpha,\delta]$. Then $f(x) \in \mathrm{nil}(A)$ if and only if $a_i \in \mathrm{nil}(R)$ for all integers $0 \leq i \leq n$.*

Proof. $(\Longrightarrow)$ Suppose that $f(x) \in \mathrm{nil}(A)$. Then there exists some positive integer k such that

$$0 = (f(x))^k = \left(a_0 + a_1x + a_2x^2 + ... + a_nx^n\right)^k.$$

Then

$$0 = (f(x))^k = \text{"lower terms"} + a_n\alpha^n(a_n)\alpha^{2n}(a_n)...\alpha^{(k-1)n}(a_n)x^{kn}.$$

Hence $a_n\alpha^n(a_n)\alpha^{2n}(a_n)...\alpha^{(k-1)n}(a_n) = 0$, and α-compatiblility of R, gives $a_n \in \mathrm{nil}(R)$. So $\lambda_i^j(a_n) \in \mathrm{nil}(R)$ for all $j \geq i \geq 0$. Let $Q = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1}$. Then we have

$$\begin{aligned} 0 &= (Q + a_nx^n)^k = \underbrace{(Q + a_nx^n)(Q + a_nx^n)...(Q + a_nx^n)}_{k\text{-}factor} \\ &= \left(Q^2 + Qa_nx^n + a_nx^nQ + a_nx^na_nx^n\right)...(Q + a_nx^n) \\ &= Q^k + \Delta, \end{aligned}$$

where $\Delta \in A$. Note that the coefficients of Δ can be written as sums of monomials in a_i and $\lambda_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, a_2, ..., a_n\}$ and $v \geq u \geq 0$, and each monomial has a_n and $\lambda_u^v(a_n)$ as a factor. Since $\mathrm{nil}(R)$ is an ideal, we obtain that each monomial of Δ is in $\mathrm{nil}(R)$ and so $\Delta \in \mathrm{nil}(R)[x;\alpha,\delta]$. Thus we obtain

$$Q^k = \left(a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1}\right)^k$$

$$\begin{aligned} &= \text{“lower terms”} + a_{n-1}\alpha^{n-1}(a_{n-1})...\alpha^{(k-1)(n-1)}(a_{n-1})x^{k(n-1)} \\ &\in \operatorname{nil}(R)[x;\alpha,\delta]. \end{aligned}$$

Therefore $a_{n-1}\alpha^{n-1}(a_{n-1})...\alpha^{(k-1)(n-1)}(a_{n-1}) \in \operatorname{nil}(R)$ and so $a_{n-1} \in \operatorname{nil}(R)$. By using induction on n we obtain $a_i \in \operatorname{nil}(R)$ for all $0 \leq i \leq n$.
($\Longleftarrow$) Consider the finite subset $S = \{a_0, a_1, a_2, ..., a_n\} \subseteq \operatorname{nil}(R)$. Since R is an NI ring with $\operatorname{nil}(R)$ nilpotent, there exist integers k_i such that

$$(a_iR)^{k_i} = 0, \ 0 \leq i \leq n.$$

Let $k = k_0 + k_1 + ... + k_n + 1$. Then we have $(a_iR)^k = 0$, $0 \leq i \leq n$. We have

$$\begin{aligned} (f(x))^k &= \left(a_0 + a_1x + a_2x^2 + ... + a_nx^n\right)^k \\ &= \sum_{i=0}^{n} a_i\lambda_0^i(a_0) + \left(\sum_{i=0}^{n} a_0\lambda_0^i(a_1) + \sum_{i=1}^{n} a_i\lambda_1^i(a_0)\right)x \\ &\quad + \left(\sum_{i=0}^{n} a_i\lambda_0^i(a_2) + \sum_{i=1}^{n} a_i\lambda_1^i(a_1) + \sum_{i=2}^{n} a_i\lambda_2^i(a_0)\right)x^2 \\ &\quad + ... + \left(\sum_{s=0}^{k}\left(\sum_{i=s}^{n} a_i\lambda_s^i(a_{k-s})\right)\right)x^k + ... + a_n\alpha^n(a_n)x^n. \end{aligned}$$

We show that the coefficients of $(f(x))^k$ can be written as sums of monomials of length k in a_i and $\lambda_u^v(a_j)$, where $a_i, a_j \in \{a_0, a_1, a_2, ..., a_n\}$ and $v \geq u \geq 0$ are integers. By using (α, δ)-compatiblility of R and $(a_iR)^k = 0$, $0 \leq i \leq n$, we have

$$a_{i_1}\lambda_{u_{i_2}}^{v_{i_2}}(a_{i_2})\lambda_{u_{i_3}}^{v_{i_3}}(a_{i_3})...\lambda_{u_{i_k}}^{v_{i_k}}(a_{i_k}) = 0,$$

where $\{a_{i_1}, a_{i_2}, ..., a_{i_k}\} \subseteq S$. Thus $(f(x))^k = 0$. Hence $f(x)$ is a nilpotent of $A = R[x;\alpha,\delta]$.

Corollary 1. *If $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n \in A = R[x;\alpha,\delta]$, R is an (α,δ)-compatible ring and satisfies any one of the following conditions:*
1) R is a Noetherian ring,
2) R has either the ACC or DCC on left annihilators,
then $f(x) \in \operatorname{nil}(A)$ if and only if $a_i \in \operatorname{nil}(R)$ for all $0 \leq i \leq n$.

Proof. If R satisfies any one of the conditions (1) and (2), then R is an NI ring with $\operatorname{nil}(R)$ nilpotent. Hence the result follows directly from Lemma 3.

Lemma 4. *Let R be an (α,δ)-compatible NI ring with $\operatorname{nil}(R)$ nilpotent and $a, b \in R$. Then $ab \in \operatorname{nil}(R)$ if and only if $a\lambda_u^v(b) \in \operatorname{nil}(R)$, where $v \geq u \geq 0$ are integers.*

Proof. ($\Longrightarrow$) Suppose that $ab \in \operatorname{nil}(R)$, so $ba \in \operatorname{nil}(R)$. Assume that $f(x) = b$ and $g(x) = ax \in A = R[x;\alpha,\delta]$. Then $f(x)g(x) \in \operatorname{nil}(A)$, so $g(x)f(x) = a\delta(b) + a\alpha(b)x \in \operatorname{nil}(R)[x;\alpha,\delta]$. Thus $a\delta(b), a\alpha(b) \in \operatorname{nil}(R)$. Now suppose that $h(x) = \alpha(b)$ and $k(x) =$

$ax \in A = R[x;\alpha,\delta]$. Then $h(x)k(x) \in \mathrm{nil}(A)$, so $k(x)h(x) = a\delta(\alpha(b)) + a\alpha^2(b)x \in \mathrm{nil}(R)[x;\alpha,\delta]$. Thus $a\delta(\alpha(b)), a\alpha^2(b) \in \mathrm{nil}(R)$. Since $a\delta(b) \in \mathrm{nil}(R)$, for $p(x) = \delta(b)$ and $q(x) = ax \in A = R[x;\alpha,\delta]$, we have $p(x)q(x) \in \mathrm{nil}(A)$, so $q(x)p(x) = a\delta^2(b)+a\alpha(\delta(b))x \in \mathrm{nil}(R)[x;\alpha,\delta]$. Thus $a\delta^2(b), a\alpha(\delta(b)) \in \mathrm{nil}(R)$. Continuing in this process we get

$$a\alpha^{n_1}(\delta^{m_1}(\alpha^{n_2}(\delta^{m_2}...\alpha^{n_i}(\delta^{m_j}(b))))) \in \mathrm{nil}(R),$$

where n_i, m_j are nonnegative integers. Thus $a\lambda_u^v(b) \in \mathrm{nil}(R)$, where $v \geq u \geq 0$ are integers.
($\Longleftarrow$) Suppose that $a\lambda_u^v(b) \in \mathrm{nil}(R)$, where $v \geq u \geq 0$ are integers. By using (α,δ)-compatiblility of R, we can conclude that $ab \in \mathrm{nil}(R)$.

Proposition 1. *Let R be an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in A = R[x;\alpha,\delta]$. Then $f(x)g(x) \in \mathrm{nil}(A)$ if and only if $a_i b_j \in \mathrm{nil}(R)$ for all integers $0 \leq i \leq n$ and $0 \leq j \leq m$.*

Proof. Suppose that $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in A$ such that $f(x)g(x) \in \mathrm{nil}(A)$. Since R is an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, we get, from Lemma 3, the following:

$$\Delta_{n+m} = a_n\alpha^n(b_m) \in \mathrm{nil}(R), \tag{1}$$

$$\Delta_{n+m-1} = a_n\alpha^n(b_{m-1}) + a_{n-1}\alpha^{n-1}(b_m) + a_n\lambda_{n-1}^n(b_m) \in \mathrm{nil}(R), \tag{2}$$

$$\Delta_{n+m-2} = a_n\alpha^n(b_{m-2}) + \sum_{i=n-1}^{n} a_i\lambda_{n-1}^i(b_{m-1}) + \sum_{i=n-2}^{n} a_i\lambda_{n-2}^i(b_m) \in \mathrm{nil}(R). \tag{3}$$

From Eq.(1) and Lemma 4, we obtain $a_n b_m \in \mathrm{nil}(R)$. So, $b_m a_n \in \mathrm{nil}(R)$. If we multiply Eq.(2) on the left side by b_m, then we get $b_m a_{n-1}\alpha^{n-1}(b_m) \in \mathrm{nil}(R)$. Thus, by Lemma 4, we obtain $a_{n-1}b_m \in \mathrm{nil}(R)$. Again from Eq.(2) and Lemma 4, we obtain $a_n b_{m-1} \in \mathrm{nil}(R)$. Applying the preceding method repeatedly, we deduce that $a_i b_j \in \mathrm{nil}(R)$ for all integers $0 \leq i \leq n$ and $0 \leq j \leq m$.
Conversely, suppose that $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j \in A$ such that $a_i b_j \in \mathrm{nil}(R)$ for all integers $0 \leq i \leq n$ and $0 \leq j \leq m$. We show that $f(x)g(x) \in \mathrm{nil}(A)$. From Lemma 3 and Lemma 4, we get the following:

$$a\lambda_u^v(b) \in \mathrm{nil}(R), \text{ where } v \geq u \geq 0 \text{ are integers.}$$

Hence $f(x)g(x) \in \mathrm{nil}(A)$.

Now we can turn to our main Theorems in the paper.

Theorem 1. *Let R be an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, such that $\alpha(e) = e$ and $\delta(e) = 0$ for every $e \in \mathrm{Id}(R)$. If R is a weak right PS-ring, then $A = R[x;\alpha,\delta]$ is a weak right PS-ring.*

Proof. Let L be a maximal right ideal of $A = R[x;\alpha,\delta]$. We will show that either $\mathrm{N}_A(L) \subseteq \mathrm{nil}(A)$ or $\mathrm{N}_A(L) = Aq$, where $q \in \mathrm{Id}(A)$. Let I be the set of all coefficients of all polynomials in L and let J be the right ideal of R generated by I, i.e.,

$$J = \langle I \rangle_r = IR.$$

If $J = R$, then there exist $a_1, a_2, ..., a_n \in I$ and $r_1, r_2, ..., r_n \in R$, such that

$$1 = a_1 r_1 + a_2 r_2 + ... + a_n r_n.$$

Suppose that $\varphi(x) = \sum_{i=0}^{k} b_i x^i \in \mathrm{N}_A(L)$, then for every $f(x) = \sum_{j=0}^{n} a_j x^j \in L$, we have

$$\varphi(x) f(x) = \left(\sum_{i=0}^{k} b_i x^i \right) \left(\sum_{j=0}^{n} a_j x^j \right) \in \mathrm{nil}(A).$$

Since R is an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, from Lemma 1, we get that $b_i a_j \in \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$ and $0 \leq j \leq n$. Consequently, for every $a \in I$, $b_i a \in \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$. Hence $b_i \in \mathrm{N}_R(J) = \mathrm{N}_R(R) = \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$. Therefore $\varphi(x) \in \mathrm{nil}(A)$. Hence $\mathrm{N}_A(L) \subseteq \mathrm{nil}(A)$.
If $J \neq R$, we show that J is a maximal right ideal of R. Let $r \in R - J$. If $r \in L$, then $r \in I$ and so $r \in J$, which is a contradiction. Thus $r \notin L$. Since L is a maximal right ideal of A, we have

$$A = L + rA.$$

It follows that there exist $f(x) = \sum_{i=0}^{n} a_i x^i \in L$ and $h(x) = \sum_{j=0}^{m} b_j x^j \in A$, such that

$$1 = a_0 + r b_0.$$

If $a_0 = 0$, then $1 = r b_0 \in rR$ and so $R = J + rR$.
If $a_0 \neq 0$, then $a_0 \in I \subset J$ which implies that $R = J + rR$. Hence J is a maximal right ideal of R.
Since R is a weak right PS-ring, it follows that either $\mathrm{N}_R(J) \subseteq \mathrm{nil}(R)$ or $\mathrm{N}_R(J) = Re$, where $e \in \mathrm{Id}(R)$.
Case (1): Assume that $\mathrm{N}_R(J) \subseteq \mathrm{nil}(R)$. We will show that $\mathrm{N}_A(L) \subseteq \mathrm{nil}(A)$. Let $\varphi(x) = \sum_{i=0}^{k} m_i x^i \in \mathrm{N}_A(L)$. Then for every $g(x) = \sum_{j=0}^{n} a_j x^j \in L$, we have

$$\varphi(x) g(x) = \left(\sum_{i=0}^{k} m_i x^i \right) \left(\sum_{j=0}^{n} a_j x^j \right) \in \mathrm{nil}(A).$$

Since R is an (α,δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, from Lemma 1, we get that $b_i a_j \in \mathrm{nil}(R)$, for all integers $0 \leq i$ and $0 \leq j$. Consequently, for every $a \in I$, $b_i a \in \mathrm{nil}(R)$, for all integers $0 \leq i$. Hence $b_i \in \mathrm{N}_R(J) = \mathrm{N}_R(R) = \mathrm{nil}(R)$, for all integers $0 \leq i$. Therefore $\varphi(x) \in \mathrm{nil}(A)$ and we have $\mathrm{N}_A(L) \subseteq \mathrm{nil}(A)$.

Case (2): Assume that $\mathrm{N}_R(J) = Re$, where $e \in \mathrm{Id}(R)$. We will show that $\mathrm{N}_A(L) = Ah$, where $h \in \mathrm{Id}(A)$. Let $\varphi(x) = \sum\limits_{i=0}^{k} b_i x^i \in \mathrm{N}_A(L)$ and $\varphi(x) \notin \mathrm{nil}(A)$, then for every $f(x) = \sum\limits_{j=0}^{n} a_j x^j \in L$, we have

$$\varphi(x)f(x) = \left(\sum_{i=0}^{k} b_i x^i\right)\left(\sum_{j=0}^{n} a_j x^j\right) \in \mathrm{nil}(A).$$

Since R is an (α, δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent, from Lemma 1, we get that $b_i a_j \in \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$ and $0 \leq j \leq n$. Consequently, for every $a \in I$, $b_i a \in \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$. For any $m \in J$, there exist $a_1, a_2, ..., a_n \in I$ and $r_1, r_2, ..., r_n \in R$, such that

$$\begin{aligned} q &= a_1 r_1 + a_2 r_2 + ... + a_n r_n, \\ b_i q &= (b_i a_1) r_1 + (b_i a_2) r_2 + ... + (b_i a_n) r_n, \end{aligned}$$

hence $b_i q \in \mathrm{nil}(R)$, for all integers $0 \leq i \leq k$, so $b_i \in \mathrm{N}_R(J) = Re$, for all integers $0 \leq i \leq k$. Therefore there exist $t_i \in R$ such that $b_i = t_i e$, for all integers $0 \leq i \leq k$. Since for any idempotent $e \in R$ we have $\alpha(e) = e$ and $\delta(e) = 0$, we can conclude that

$$\varphi(x) = \sum_{i=0}^{k} b_i x^i = \sum_{i=0}^{k} t_i e x^i = \left(\sum_{i=0}^{k} t_i x^i\right) e \in Ah,$$
$$\text{where } h = e = e^2 = h^2 \in A.$$

Therefore $\mathrm{N}_A(L) = Ah$, where $h \in \mathrm{Id}(A)$ and the result is proved.

Theorem 2. *Let R be an (α, δ)-compatible NI ring with $\mathrm{nil}(R)$ nilpotent. If R is a weak left PS-ring, then $A = R[x; \alpha, \delta]$ is a weak left PS-ring.*

Proof. The proof is similar to the previous proof of Theorem 1. The only thing we need to note here is that, If L is a maximal left ideal of $A = R[x; \alpha, \delta]$, then, by analogue manner as above, we get in case (2) that $b_i \in \mathrm{N}_R(J) = Re$, for all integers $0 \leq i \leq k$. Therefore there exist $t_i \in R$ such that $b_i = e t_i$, for all integers $0 \leq i \leq k$. So

$$\varphi(x) = \sum_{i=0}^{k} b_i x^i = \sum_{i=0}^{k} e t_i x^i = e\left(\sum_{i=0}^{k} t_i x^i\right) \in hA,$$
$$\text{where } h = e = e^2 = h^2 \in A.$$

Therefore $\mathrm{N}_A(L) = hA$, where $h \in \mathrm{Id}(A)$ and the result is proved.

Assume that δ is the zero map, then $A = R[x;\alpha]$, the usual skew polynomial ring over R, and we get the following corollaries:

Corollary 2. *Let R be an α-compatible NI ring with* $\mathrm{nil}(R)$ *nilpotent, such that $\alpha(e) = e$ for every $e \in$* $\mathrm{Id}(R)$. *If R is a weak right PS-ring, then $A = R[x;\alpha]$ is a weak right PS-ring.*

Corollary 3. *Let R be an α-compatible NI ring with* $\mathrm{nil}(R)$ *nilpotent. If R is a weak left PS-ring, then $A = R[x;\alpha]$ is a weak left PS-ring.*

Assume that α is the identity map, then $A = R[x;\delta]$, the differential polynomial ring over R, and we get the following corollaries:

Corollary 4. *Let R be a δ-compatible NI ring with* $\mathrm{nil}(R)$ *nilpotent, such that $\delta(e) = 0$ for every $e \in$* $\mathrm{Id}(R)$. *If R is a weak right PS-ring, then $A = R[x;\delta]$ is a weak right PS-ring.*

Corollary 5. *Let R be a δ-compatible NI ring with* $\mathrm{nil}(R)$ *nilpotent. If R is a weak left PS-ring, then $A = R[x;\delta]$ is a weak left PS-ring.*

Assume that α is the identity map and δ is the zero map, then $A = R[x]$, the usual polynomial ring over R, and we get the following corollary:

Corollary 6. *Let R be an NI ring with* $\mathrm{nil}(R)$ *nilpotent. If R is a weak right (left) PS-ring, then $A = R[x]$ is a weak right (left) PS-ring.*

References

[1] S. Annin. Cassociated primes over ore extension rings. *J. Alg. Appl.*, 3:193–205, 2004.

[2] M. Farahat and S. Al-Bogamy. PS-ringss over skew Hurwitz series. *Eur. J. Pure Appl. Math.*, 11(1):244–259, 2018.

[3] M. Farahat and N. Al-Harthy. PS-modules of generalized Mal'cev-Neumann series rings. *Hacet. J. Math. Stat.*, 46(5):1–6, 2017.

[4] C. Hong, N. Kim, and T. Kwak. Ore extensions of Baer and p.p.-rings. *J. Pure and Appl. Alg.*, 151:215–226, 2000.

[5] T. Lam, A. Leory, and J. Matczuk. Primeness, semiprimeness and the prime radical of Ore extensions. *Comm. Alg.*, 25(8):2459–2516, 1997.

[6] W. Nicholson and J. Watters. Rings with projective socle. *Proc. Amer. Math. Soc.*, 102:443–450, 1988.

[7] Ø . Ore. Theory of noncommutative polynomials. *Ann. Math.*, 34(3):480–508, 1933.

[8] K. Paykan. Skew inverse power series rings over a ring with projective socle. *Czec. Math. J.*, 67(142):389–398, 2017.

[9] R. Salem, M. Farahat, and H. Abd-ELmalk. PS-modules over Ore extensions and skew generalized power series rings. *Int. J. Math. Sci.*, 2015.