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Localization of Hopfian and Cohopfian Objects in the Categories of $A - Mod$, $AGr(A - Mod)$ and $COMP(AGr(A - Mod))$

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Abstract. The aim of this paper is to study the localization of hopfian and cohopfian objects in the categories $A - Mod$ of left A-modules, $AGr(A - Mod)$ of graded left A-modules and $COMP(AGr(A - Mod))$ of complex sequences associated to graded left A-modules.
We have among others the main following results

(i) Let M be a noetherian graded left A-module, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, N a submodule of M, M_* is a noetherian quasi-injective complex sequence associated with M and N_* is an essential and completely invariant complex sub-sequence of M_*. Then, $S^{-1}(N_*)$ the complex sequence of morphisms of left $S^{-1}A$-modules is cohopfian if, and only, if $S^{-1}(M_*)$ is cohopfian ;

(ii) let M be a graded left A-module and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. If M_* is a hopfian, noetherian and quasi-injective complex sequence associated with M, then the complex sequence of morphisms of left $S^{-1}(A)$-modules $S^{-1}(M_*)$ has the following property : $\ll$ *any epimorphism of sub-complex* $S^{-1}(N_*)$ *of* $S^{-1}(M_*)$ *is an isomorphism* $\gg$;

(iii) let M be a graded left A-module, N a graded submodule of M, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. M_* the quasi-projective complex sequence associated with M and N_* a superfluous and completely invariant complex sub-sequence of M_*. Then the complex morphism sequence of left $S^{-1}(A)$-modules $S^{-1}(N_*)$ is hopfian if, and only if, $S^{-1}(M_*/N_*)$ the complex sequence associated with $S^{-1}(M/N)$ is hopfian.

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Key Words and Phrases: graded ring, a saturated multiplicative part formed by the non-zero homogeneous elements of A,Ore conditions, hopfian, cohopfian, sequence complex, chain complex, quasi-injective and quasi-projective.

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1. Introduction

In this paper, the ring A is supposed to be associative, unitary, not necessairly commutative, and any left A-modules is unifary.
In this article, we study the localization of hopfian and cohopfian objects in the categories $A-Mod$ of left A-modules, $AGr(A-Mod)$ of graded left A-modules and $COMP(AGr(A-Mod))$ of complex sequences associated to graded left A-modules. We rely on the articles, ≪*Graduation of Module of Fraction on a Graded Domain Ring not Necessarily Commutative*≫[2], ≪ *Factorization of Graded Modules of Fractions* ≫[3], ≪*Module de Fractions, Sous-modules* $S-$*saturée et Foncteur* S^{-1} ≫ [18] and ≪*Hopfian and Cohopfian Objects in the Categories of* $Gr(A-Mod)$ *and* $COMP(Gr(A-Mod))$ ≫[23], which are used as a basis for studying the notions of localization, hopficity and cohopficity. The transition to localization and the study of hopficity and cohopficity from the category of left $A-Mod$ whose objects are the left A-modules and the morphisms are the left A-module morphisms to the category $AGr(A-Mod)$ whose objects are the graded left A-module and the morphisms are the graded left A-module graded morphisms and $AGr(A-Mod)$ to the category of $COMP(AGr(A-Mod))$ whose the objects are the complex sequences of graded left A-modules and the morphisms are the chain complexes associated to the graded morphims of graded left A-modules is not easy. In our opinion, these reasons justify our work. Thus, the paper is organized as follows

In the section 2, we study the localization of hopfian and cohopfian objects in the category $A-Mod$ and we show the following results :

(i) Let A be a ring, S a saturated multiplicative part of A verifying the left Ore conditions, M a left A-module. If $S^{-1}M$ is hopfian, then M is hopfian ;

(ii) Let A be a ring, S a saturated multiplicative part of A verifying the left Ore conditions, M a left A-module. If M is a cohopfian and completely invariant submodule of left A-module $S^{-1}(M)$, then $S^{-1}(M)$ is a cohopfian left $S^{-1}(A)$(respectively A)-module ;

(iii) Let M be a noetherian quasi-injective left A-module, N be an essential and completely invariant submodule of M and S a saturated multiplicative part of A verifying the left Ore conditions. Then, the left $S^{-1}(A)$-modules $S^{-1}(N)$ is cohopfian if, and only, if $S^{-1}(M)$ is cohopfian left $S^{-1}(A)$-module ;

(iv) Let M be a quasi-projective left A-module, N a superfluous and completely invariant sub-module of M, S a saturated multiplicative part of A verifying the left Ore conditions. Then the left $S^{-1}(A)$-modules $S^{-1}(N)$ is hopfian if, and only if, $S^{-1}(M/N)$ is hopfian.

In section 3, we study the localization of hopfian and cohopfian objects in the category $AGr(A-Mod)$ and we prove the following results :

(i) Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a graded left A-module, then $S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}}(S^{-1}M)_i$ is hopfian(respectively cohopfian) if, and only, if $(S^{-1}M)_i$ is a hopfian(respectiveley cohopfien) group ;

(ii) let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M a graded left A- -module. Then, if $S^{-1}(M)$ is a hopfian left graded $S^{-1}(A)$-module, implies that M is a hopfian left graded A-module ;

(iii) if $S^{-1}(M)$ is a left graded hopfian $S^{-1}(A)$-module, then M_n for all $n \in \mathbb{Z}$ is a hopfian group ;

(iv) let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M left graded A- -module. Then, if M is a cohopfian and completely invariant submodule of left A-module $S^{-1}(M)$, implies that, $S^{-1}(M)$ is a cohopfian graded left $S^{-1}(A)$-module;

(v) if M is a cohopfian and completely invariant submodule of the left A-module $S^{-1}(M)$, then $(S^{-1}M)_i$ is a cohopfian group ;

(vi) Let M be a noetherian quasi-injective graded left A-module, N be an essential and completely invariant graded submodule of M and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. Then, the graded left $S^{-1}(A)$-modules $S^{-1}(M)$ is cohopfian if, and only, if $S^{-1}(N)$ is cohopfian graded left $S^{-1}(A)$-module ;

(vii) Let M be a graded quasi-projective left A-module, N a superfluous and completely invariant graded sub-module of M, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. Then the left $S^{-1}(A)$-modules $S^{-1}(N)$ is hopfian if, and only if, $S^{-1}(M/N)$ is hopfian.

In the section 4, we study the localization of hopfian and cohopfian objects in the category $COMP(AGr(A - Mod))$ and we show the following results :

(i) Let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M a graded left A-module and M_* the complex sequence of morphisms of graded left A-modules associated with M. Then, if the complexe $S^{-1}(M_*)$ of morphisms of graded left $S^{-1}(A)$-module is hopfian, implies that M_* is hopfian ;

(ii) Let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M a graded left A-module. If M_* the complex sequence associated with M, is cohopfian and completely invariant, then $S^{-1}(M_*)$, the complex sequence associated with $S^{-1}(M)$ is cohopfian;

(iii) let M be a noetherian graded left A-module, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, N a submodule of M, M_* is a noetherian quasi-injective complex sequence associated with M and N_* is an essential and completely invariant complex sub-sequence of M_*. Then, $S^{-1}(N_*)$ the complex sequence of morphisms of left $S^{-1}(A)$-modules is cohopfian if, and only, if $S^{-1}(M_*)$ is cohopfian ;

(iv) let M be a graded left A-module and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. If M_* is a hopfian, noetherian and quasi-injective complex sequence associated with M, then the complex sequence of morphisms of left $S^{-1}A$-modules $S^{-1}M_*$ has the following property : $\ll$*any epimorphism of sub-complex* $S^{-1}(N_*)$ *of* $S^{-1}(M_*)$ *is an isomorphism* $\gg$;

(v) let M be a graded left A-module, N a graded submodule of M, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. M_* the quasi-projective complex sequence associated with M and N_* a superfluous and completely invariant complex sub-sequence of M_*. Then the complex morphism sequence of left $S^{-1}(A)$-modules $S^{-1}(N_*)$ is hopfian if, and only if, $S^{-1}(M_*/N_*)$ the complex sequence associated with $S^{-1}(M/N)$ is hopfian.

2. Preliminaries

Definition 1.
Let A be a ring and $\{A_n\}_{n\in\mathbb{Z}}$ a family of sub-group of A. If

(i) $A = \bigoplus_{n\in\mathbb{Z}} A_n$;

(ii) $A_n \cdot A_m \subset A_{n+m}, \forall\ n,\ m \in \mathbb{Z}$.

Then we say that A is a graded ring. Else, if $A_n = 0, \forall n < 0$. Then A is called positively graded ring.

Definition 2.
Let A a graded ring, x be a non-zero element of A, we say that x is homogeneous of degree n, if there exists n such that $x \in A_n$ and we note $deg(x) = n$.
such that $x \in A_n$ and we note that $\deg(x) = n$.

Definition 3.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring and M be a left $A-$module, then M is called a graded left $A-$module if there exists a sequence $(M_n)_{n\in\mathbb{Z}}$ of sub-group of M such that

(i) $M = \bigoplus_{n\in\mathbb{Z}} M_n$;

(ii) $A_n \cdot M_d \subset M_{n+d}, \forall\ n,\ d \in \mathbb{Z}$.

Definition 4.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *be a graded left* $A-$*module and* N *is a sub-module of* M*, then* N *is called a graded sub-module of* M*, if* $\forall x = \sum_{n\in\mathbb{Z}} x_n \in N$*, with* $x_n \in M_n$*, then* $x_n \in N$*,* $\forall n \in \mathbb{Z}$*.*

Definition 5.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$*,* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *are two graded left* $A-$*modules and* $f : M \longrightarrow N$ *is a morphism of left* $A-$*modules, then* f *is called a graded morphism if for any* $m_s \in M_s$ *then* $f(m_s) \in N_s$*.*

Theorem 1.
Let A *be a graded ring, the category of graded left* $A-$*module is the category denoted by* $AGr(A - Mod)$ *whose*

(i) The objects are the graded left $A-$*modules;*

(ii) The morphisms are the graded morphisms.

Proof.
See [23]

Theorem 2.
Let A *be a graded ring,* M *a graded left* A*-module and* S S *a saturated multiplicative part formed by the non-zero homogeneous elements of* A *verifying the left Ore conditions, then :*

(i) $S^{-1}A = \bigoplus_{i\in\mathbb{Z}} (S^{-1}A)_i$ *is a graded ring, where*
$(S^{-1}A)_i = \{\frac{a}{s} \in S^{-1}A, \exists k, a \in A_k \ and\ deg(s) = k - i\}$.

(ii) $S^{-1}M = \bigoplus_{i\in\mathbb{Z}} (S^{-1}M)_i$ *is a graded left* $S^{-1}A$*-module, where*
$(S^{-1}M)_i = \{\frac{m}{s} \in S^{-1}M, \exists p, m \in M_p \ and\ deg(s) = p - i\}$.

Proof.
See [1]

Proposition 1.
Let $A = \bigoplus_{n\in\mathbb{N}} A_n$ *be a graded ring,* $M = \bigoplus_{n\in\mathbb{Z}} M_n$ *and* $N = \bigoplus_{n\in\mathbb{Z}} N_n$ *are two graded left* A*-modules,* $f : M \longrightarrow N$ *is graded morphism and* S *the set of non-zero homogeneous elements of* A*, then we have :*

(i) the following complex sequences :

$$S^{-1}(M_*) : \cdots \longrightarrow S^{-1}(M(n+1)) \overset{S^{-1}(d_{n+1})}{\longrightarrow} S^{-1}(M(n)) \overset{S^{-1}(d_n)}{\longrightarrow} S^{-1}(M(n-1)) \longrightarrow \cdots$$

(ii) the following chain complexes :

$$\begin{array}{ccccccccc}
S^{-1}(M_*): \cdots \longrightarrow & S^{-1}(M(n+1)) & \xrightarrow{S^{-1}(d_{n+1})} & S^{-1}(M(n)) & \xrightarrow{S^{-1}(d_n)} & S^{-1}(M(n-1)) & \longrightarrow \cdots \\
\downarrow S^{-1}(f_*) & \downarrow S^{-1}(f(n+1)) & & \downarrow S^{-1}(f(n)) & & \downarrow S^{-1}(f(n-1)) & \\
S^{-1}(N_*): \cdots \longrightarrow & S^{-1}(N(n+1)) & \xrightarrow{S^{-1}(d'_{n+1})} & S^{-1}(N(n)) & \xrightarrow{S^{-1}(d'_n)} & S^{-1}(N(n-1)) & \longrightarrow \cdots
\end{array}$$

Proof.
See [3]

Theorem 3.
Let $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ be an object of $COMP(AGr(A-Mod))$. M_* is quasi-injective in $COMP(Gr(A-Mod))$ if, and only, if for all $n \in \mathbb{Z}$, $M(n)$ is a quasi-injective in $COMP(AGr(A-Mod))$.

Proof.
See [23]

Theorem 4.
Let $M_* : \ldots M(n+1) \xrightarrow{d_{n+1}} M(n) \xrightarrow{d_n} M(n-1) \longrightarrow \ldots$ be an object of $COMP(Gr(A-Mod))$. M_* is quasi-projective if, and only, if for all $n \in \mathbb{Z}$, $M(n)$ is a quasi-projective left A-modules .

Proof.
See [23]

3. Localization of hopfian and cohopfian objects in the category of $A-Mod$

Definition 6.
Let M a left A-module. Then M is said hopfian (respectively cohopfian), if any epimorphism (respectively monomorphism) of M is an automorphism of M.

Theorem 5.
Let A be a ring, S a saturated multiplicative part of A verifying the left Ore conditions, M a left A-module. If $S^{-1}(M)$ is a hopfian left $S^{-1}(A)$-module, then M is a hopfian left A-module.

Proof.
Let $f : M \longrightarrow M$ an epimorphism of left A-module. Then $S^{-1}(f) : S^{-1}(M) \longrightarrow S^{-1}(M)$ define by $[S^{-1}(f)](\frac{m}{s}) = \frac{f(m)}{s}$ is an endomorphisme of $S^{-1}(A)$-Mod. Let $\frac{m'}{s} \in S^{-1}(M)$, as f is an epimorphism, then there exists $m \in M, f(m) = m'$.
So $[S^{-1}(f)](\frac{m}{s}) = \frac{f(m)}{s} = \frac{m'}{s}$, thus $S^{-1}(f)$ is an epimorphism, since $S^{-1}(M)$ is hopfian, so $S^{-1}(f)$ is an automorphism of $S^{-1}M$.

Let m_1 and $m_2 \in M$ such that
$f(m_1) = f(m_2)$
$\Longrightarrow \frac{f(m_1)}{1} = \frac{m_2}{1} \Longrightarrow [S^{-1}(f)](m_1) = [S^{-1}(f)](m_2)$
$\Longrightarrow m_1 = m_2$
Thus f is an automorphism of M, so M is hopfian.

Theorem 6.
Let A be a ring, S a saturated multiplicative part of A verifying the left Ore conditions, M a left A-module. If M is a cohopfian and completely invariant submodule of left A-module $S^{-1}(M)$, then $S^{-1}M$ is a cohopfian left $S^{-1}(A)$(respectively A)-module.

Proof.
Let $g : S^{-1}(M) \longrightarrow S^{-1}(M)$ be a $S^{-1}A$-morphism, we remark also that g is a A-morphism.
As M is a completely invariant submodule of left A-module $S^{-1}(M)$, so $g(M) \subset M$.
Suppose that g is a monomorphism.
Thus the induce morphism $g_{ind} : M \longrightarrow M$ is a monomorphisme of M, since M is cohopfian, then g_{ind} is an automorphism of M.
Let's consider
$S^{-1}(g_{ind}) : S^{-1}(M) \longrightarrow S^{-1}(M)$
$\frac{m}{s} \longmapsto \frac{g_{ind}(m)}{s}$
So $\frac{g_{ind}(m)}{s} = \frac{1}{s}.\frac{g(m)}{1}$.
Or g is a $S^{-1}(A)$-morphisme, so $\frac{1}{s}.\frac{g_{ind}(m)}{1} = g_{ind}(\frac{1}{s}\frac{m}{1}) = g_{ind}(\frac{m}{s}) \Longrightarrow S^{-1}(g_{ind}) = g$
g is a monomorphism, then g_{ind} is a monomorphism.
As M is cohopfian, then g_{ind} is an automorphism of M.
Let $\frac{m'}{s} \in S^{-1}(M) \Longrightarrow m' \in M$, so there exists $m \in M : g_{ind}(m) = m'$
thus $\frac{m}{s} \in S^{-1}M$, $[S^{-1}(g_{ind})](\frac{m}{s}) = \frac{g(m)}{s} = \frac{m'}{s}$
hence $(g(\frac{m}{s})) = \frac{m'}{s}$.
Thus $S^{-1}M$ is a cohopfian left $S^{-1}A$-module.
Let's prove now that $S^{-1}(M)$ is a cohopfian left A-module

Let $f : S^{-1}(M) \longrightarrow S^{-1}(M)$ be a monomorphism of left A-module, then $f : S^{-1}(M) \longrightarrow S^{-1}(M)$ is a monomorphism, indeed,
let $\frac{m}{s}$ and $\frac{m'}{s'} \in S^{-1}M : [S^{-1}(f)](\frac{m}{s}) = [S^{-1}(f)](\frac{m'}{s'}) \Longrightarrow \frac{f(m)}{s} = \frac{f(m')}{s'} \Longrightarrow \exists x, y \in S$ such that :
$\begin{cases} x.f(m) = y.f(m') \\ xs = ys' \end{cases} \Longrightarrow \begin{cases} f(x.m) = f(y.m') \\ xs = ys' \end{cases} \Longrightarrow \begin{cases} x.m = y.m' \\ xs = ys' \end{cases} \Longrightarrow \frac{m}{s} = \frac{m'}{s} \Longrightarrow$
$S^{-1}(f)$ is a monomorphisme, or $S^{-1}(M)$ is cohopfian, then $S^{-1}(f)$ is an automorphism.

Let $\frac{m'}{s'} \in S^{-1}(M)$, since $S^{-1}(f)$ is an automorphism, then there exists $\frac{m}{s} \in S^{-1}M$ such that
$[S^{-1}(f)](\frac{m}{s}) = \frac{m'}{s'} \Longrightarrow \frac{f(m)}{s} = \frac{m'}{s'} \Longrightarrow \frac{\frac{s}{s}.m}{s} = \frac{m'}{s'} \Longrightarrow \frac{s.f(\frac{m}{s})}{s} = \frac{m'}{s'} \Longrightarrow \frac{f(m)}{s} = \frac{m'}{s'}$,
thus there exists $\frac{m}{s} \in S^{-1}M$ such that $f(\frac{m}{s}) = \frac{m'}{s'} \Longrightarrow f$ is an epimorphism, hence $S^{-1}(M)$ is cohopfian.

Theorem 7.
Let M be a noetherian quasi-injective left A-module, N be an essential and completely invariant submodule of M and S a saturated multiplicative of A verifying the left Ore conditions. Then, the left $S^{-1}(A)$-modules $S^{-1}(N)$ is cohopfian if, and only, if $S^{-1}(M)$ is cohopfian left $S^{-1}(A)$-module.

Proof.
Suppose that $S^{-1}(M)$ is cohopfian and left $S^{-1}(f) : S^{-1}(N) \longrightarrow S^{-1}(N)$ be a monomorphism. As $S^{-1}(M)$ is quasi-injective because M is noetherian. Then, there exists $S^{-1}(g) \in End(S^{-1}(M))$ such that $S^{-1}(g_{|S^{-1}N}) = S^{-1}(f)$.
$S^{-1}(g)$ is injective since $S^{-1}(N)$ is essential in $S^{-1}(M)$, and as $S^{-1}(M)$ is cohopfian, $S^{-1}(g)$ is invertible. Let $\frac{x}{s} \in S^{-1}(N)$, there exists $\frac{y}{t} \in S^{-1}(M)$ such that $\frac{x}{s} = [S^{-1}(g)](\frac{y}{t})$. Or $S^{-1}(g^{-1}) \in End(S^{-1}()N))$ and $S^{-1}(N)$ is completely invariant, so $\frac{y}{t} = [S^{-1}(g^{-1})](\frac{x}{s}) \in S^{-1}(N)$, thus $S^{-1}(f)$ is an automorphisme, consequentely $S^{-1}(N)$ is cohopfian.
Reciprocally, suppose that $S^{-1}(N)$ is cohopfian and let $S^{-1}(f) : S^{-1}(M) \longrightarrow S^{-1}(M)$ a monomorphism. Then $S^{-1}(f_{|S^{-1}(N)})$ is a monomorphism of $S^{-1}(N)$. Thus, $S^{-1}(f) \in Aut(S^{-1}(N))$, hence $[S^{-1}(f)](S^{-1}(N)) = S^{-1}(N)$. As $S^{-1}(M)$ is quasi-injective, then there exists $S^{-1}(L)$ a submodule of $S^{-1}(M)$ such that $S^{-1}(M) = [S^{-1}(f)](S^{-1}(M)) \oplus S^{-1}(L)$. Thus, we have $0 = [S^{-1}(f)](S^{-1}N) \cap S^{-1}(L) = S^{-1}(N) \cap S^{-1}(L)$, since $S^{-1}(N)$ is essential, then $S^{-1}(L) = 0$, hence $S^{-1}(M) = [S^{-1}(f)](S^{-1}M)$, thus $S^{-1}(f)$ is an epimorphism, so $S^{-1}(M)$ is cohopfian.

Theorem 8.
Let M be a quasi-projective left A-module, N an superfluous and completely invariant submodule of M, S a saturated multiplicative part of A verifying the left Ore conditions. Then the left $S^{-1}(A)$-modules $S^{-1}(N)$ is hopfian if, and only if, $S^{-1}(M/N)$ is hopfian.

Proof.
Suppose that $S^{-1}(M/N)$ is hopfian and let $S^{-1}(f) : S^{-1}(M) \longrightarrow S^{-1}(M)$ an epimorphism.
As $S^{-1}(N)$ is completely invariant, then $[S^{-1}(f)](S^{-1}(N)) \subset S^{-1}(N)$, implies $S^{-1}(f)$ induces an epimorphism $S^{-1}(\overline{f}) : S^{-1}(M/N) \longrightarrow S^{-1}(M/N)$, since $S^{-1}(M/N)$ is hopfian, then $S^{-1}(\overline{f})$ is a graded automorphism. Put $S^{-1}(K) = ker(S^{-1}(f))$ and $S^{-1}(\pi) : S^{-1}(M) \longrightarrow S^{-1}(M/N)$ the canonical projection, we have :
$S^{-1}(\overline{f}) \circ [S^{-1}(\pi)](S^{-1}(K)) = S^{-1}(\pi \circ f(K)) = 0$
Indeed, $\forall \frac{x}{s} \in S^{-1}(K)$, we have :
$S^{-1}[(\overline{f} \circ \pi)](\frac{x}{s}) = [S^{-1}(\pi \circ f)](\frac{x}{s})$, so
$[S^{-1}(\pi \circ f)](\frac{x}{s}) = [S^{-1}(\pi)](\frac{f(x)}{s}) = [S^{-1}(\pi)](\frac{0}{s}) = 0$, thus $[S^{-1}(\overline{f} \circ \pi)](K) = 0$.
we have :
$[S^{-1}(\overline{f} \circ \pi)](K) = [S^{-1}(\pi \circ f)](K) = 0 \Longrightarrow [S^{-1}(\overline{f})](\pi(K)) = 0 \Longrightarrow [S^{-1}(\pi)](K) \subset S^{-1}(N) \Longrightarrow S^{-1}(K) \subset S^{-1}(N)$
Since M is quasi- projective $\Longrightarrow S^{-1}(M)$ is quasi-projective, there exists an endomorphism $S^{-1}(s) : S^{-1}(M) \longrightarrow S^{-1}(M)$ such that $S^{-1}(f \circ s) = S^{-1}(id_{S^{-1}(M)})$, this implies $S^{-1}(M) = S^{-1}(K \oplus Im(s))$, or $K = N$ and $S^{-1}(N)$ is superfluous in $S^{-1}(M)$, then

$S^{-1}(M) = S^{-1}(Im(s))$, so $S^{-1}(K) = ker[S^{-1}(f)] = 0$, thus $S^{-1}(f)$ is a monomorphism, finally, $S^{-1}(f)$ is a monomorphism, so $S^{-1}(M)$ is a hopfian left $S^{-1}(A)$-module.
Reciprocally,
let $S^{-1}(M)$ be hopfian, show that $S^{-1}(M/N)$ is hopfian.
Let $S^{-1}(\varphi) : S^{-1}(M/N) \longrightarrow S^{-1}(M/N)$ be an epimorphism of left $S^{-1}A$-module, as M is quasi-projective, then $S^{-1}(M)$ is quasi-projective. Consider $S^{-1}(\pi) : S^{-1}(M) \longrightarrow S^{-1}(M/N)$, then there exists $S^{-1}(f) \in End(S^{-1}(M))$ such that $S^{-1}(\pi \circ f) = S^{-1}(\varphi \circ \pi)$. Since $S^{-1}(\varphi)$ is an epimorphism, $\forall \overline{\frac{x}{s}} \in S^{-1}(M/N), \exists(\overline{\frac{y}{t}}) \in S^{-1}(M/N)$ such that
$[S^{-1}(\varphi)](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}} = [S^{-1}(\varphi)](\frac{\pi(y)}{t})$
$[S^{-1}(\pi \circ f)](\frac{y}{t}) = [S^{-1}(\varphi \circ \pi)](\frac{y}{t})$
$[S^{-1}(\pi)](\frac{f(x)}{s}) = [S^{-1}(\varphi)](\overline{\frac{y}{t}})$
$\Longrightarrow \underline{[S^{-1}(\varphi)](\overline{\frac{y}{t}})} = [S^{-1}(\overline{f})](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}}$
$\Longrightarrow \frac{f(y)}{t} - \frac{x}{s} = \overline{0}$
$\Longrightarrow \frac{f(y)}{t} - \frac{x}{s} \in S^{-1}(N)$,
then $S^{-1}(M) = Im(S^{-1}(f)) + S^{-1}(N)$, as $S^{-1}(N)$ is superfluous, then $Im(S^{-1}(f)) = S^{-1}(M) \Longrightarrow S^{-}(f)$ is an epimorphism. So, $S^{-1}f$ is an automorphism, because $S^{-1}(M)$ is hopfian.
So the restriction of $S^{-1}(f)$ over $S^{-1}(N)$ is a automorphism of $S^{-1}(N)$.
If $[S^{-1}(\varphi)](\overline{\frac{x}{s}}) = [S^{-1}(\overline{f})](\frac{x}{s}) = \overline{0}$, then $[S^{-1}(f)](\frac{x}{s}) \in S^{-1}(N)$, or $S^{-1}(N)$ is completely invariant, then $\frac{x}{s} \in S^{-1}(N)$, so $\overline{\frac{x}{s}} = \overline{0} \Longrightarrow ker(S^{-1}(\varphi)) = S^{-1}(N) = \overline{0} \Longrightarrow S^{-1}(\varphi)$ is a monomorphism $\Longrightarrow \varphi$ is an automorphism, lastly $S^{-1}(M/N)$ is hopfian, hence $S^{-1}(M/N)$ is a hopfian left $S^{-1}(A)$-module.

4. Localization of hopfian and cohopfian objects in the category of $AGr(A - Mod)$

Definition 7.
Let M a graded left A-module. Then M is said hopfian (respectively cohopfian), if any epimorphism (respectively monomorphism) of M is an automorphism of M.

Lemma 1.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a graded left A-module, then M is hopfian(respectively cohopfian) if, and only if, M_n is a hopfian(respectiveley cohopfien) groupe.

Proof.
Suppose that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is hopfian, show that M_n is a hopfian group.

Let $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded morphism, so we have the induce morphism of groups : $f_n : M_n \longrightarrow M_n$, for all $x \in M_n$, we have $f_n(x) = f(x_n)$. We see that f_n is well defined because f is graded, moreover, for all x_n and $y_n \in M_n$, we have $f_n(x_n + y_n) = f(x_n + y_n) = f(x_n) + f(y_n) = f_n(x_n) + f_n(y_n) \Longrightarrow f_n$ is a morphism of

groups.
let $h : M_n \longrightarrow M_n$ be an epimorphism of groups.
Put for all $x_i, i \neq n$, $f(x_i) = x_i$ and for all $x_n \in M_n, f(x_n) = h(x_n)$, it is easy to prove that f is a graded epimorphism of graded left A-modules.
Since f is an automorphism because M hopfian by hypothesis, so h is an automorphism, thus M_n is hopfien.
Suppose that M_n is hopfian, show that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is hopfian.
Let $f : M = \bigoplus_{n\in\mathbb{Z}} M_n \longrightarrow M = \bigoplus_{n\in\mathbb{Z}} M_n$ be an epimorphism of left A-module. Show that f is an automorphism. Prove that $f_n : M_n \longrightarrow M_n$ is epimorphism of groups for all $n \in \mathbb{Z}$
Let $y_n \in M_n$, then there exists $x \in M = \bigoplus_{n\in\mathbb{Z}} A_n$.
Suppose that $x = \sum_{finie} x_t$, or for all t, $f(x_t) \in M_t$
if $t \neq n, f(x_t) = 0$, since $f(x) = y_n \in M_n$, so for all $y_n \in M_n$, there exists $x_n \in M_n$ such that $f(x_n) = y_n$, or $f(x_n) = f_n(x_n) \Longrightarrow f_n(x_n) = y_n$, so f_n is an epimorphism, as M_n is hopfian, so f_n is an automorphism $\Longrightarrow f$ is also an automorphism, consequentely, M is hopfian.
For the hopfian case, the proof is similary.

Theorem 9.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a graded left A-module, then $S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}} (S^{-1}M)_i$ is hopfian(respectively cohopfian) if, and only, if $(S^{-1}M)_i$ is a hopfian(respectiveley cohopfien) group.

Proof.
It suffices to prove that $(S^{-1}f) : S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}} (S^{-1}M)_i \longrightarrow S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}} (S^{-1}M)_i$ is graded.
Since $(S^{-1}M)_i = \{\frac{m}{s} \in S^{-1}M, \exists p, m \in M_p \text{ and } deg(s) = p - i\}$, we have $[S^{-1}(f)](\frac{m}{s}) = \frac{f(m)}{s}$ or f is graded $\Longrightarrow f(m) \in M_p \Longrightarrow \frac{f(m)}{s} \in (S^{-1}M)_i$, so $S^{-1}(f)$ is a graded morphism.
Then, by Lemma(1), we obtain the result.

Theorem 10.
Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, $M = \bigoplus_{n\in\mathbb{Z}} M_n$ a graded left A-module. Then, if $S^{-1}(M)$ is a hopfian left graded $S^{-1}(A)$-module, implies that M is a hopfian left graded A-module.

Proof.
Let $f : M \longrightarrow M$ be a graded epimorphism of M. Then $S^{-1}f : S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}}(S^{-1}M)_i \longrightarrow S^{-1}(M) = \bigoplus_{i\in\mathbb{Z}}(S^{-1}M)_i$ defined by $[S^{-1}(f)](\frac{m}{s}) = \frac{f(m)}{s}$ is a graded endomorphism of graded left $S^{-1}(A)$-module, because $[S^{-1}(f)](\frac{m_i}{s}) = \frac{f(m_i)}{s}$, since f is graded, then $f(m_i) \in M_i$, thus $\frac{f(x_i)}{s} \in (S^{-1}M)_i$. Let $\frac{m'}{s} \in S^{-1}(M)$, as f is a graded epimorphism, then there exists $m \in M, f(m) = m'$.
So $[S^{-1}(f)](\frac{m}{s}) = \frac{f(m)}{s} = \frac{m'}{s}$, hence $S^{-1}(f)$ is a graded epimorphism, since $S^{-1}(M)$ is hopfian, then $S^{-1}(f)$ is a graded automorphism of $S^{-1}(M)$.
Let m_1 and $m_2 \in M$ such that
$f(m_1) = f(m_2)$
$\Longrightarrow \frac{f(m_1)}{1} = \frac{f(m_2)}{1} \Longrightarrow [S^{-1}(f)](m_1) = [S^{-1}(f)](m_2)$
$\Longrightarrow m_1 = m_2$
Thus f is a graded automorphism of M, so M is a hopfian graded left A-module.

Corollary 1.
Under the same conditions of the previous theorem. If $S^{-1}(M)$ is a hopfian graded left $S^{-1}(A)$-module, then M_n for all $n \in \mathbb{Z}$ is a hopfian group.

Proof.
It's obvious by Theorem(10)

Theorem 11.
Let A graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M left graded A-module. Then, if M is a cohopfian and completely invariant submodule of the left A-module $S^{-1}(M)$, implies that, $S^{-1}(M)$ is a left cohopfian graded $S^{-1}(A)$-module.

Proof.
Let $g : S^{-1}(M) \longrightarrow S^{-1}(M)$ a graded $S^{-1}(A)$-morphisme, we remark also that g is a graded A-morphisme. As M is completely invariant as graded submodule of graded left A-module $S^{-1}M$, so $g(M) \subset M$.
Suppose that g is a graded monomorphism.
So the induce graded morphism $g_{ind} : M \longrightarrow M$ is a graded monomorphism of M and, as M is cohopfian, then g_{ind} is a graded automorphism of M.
Consider
$S^{-1}(g_{ind}) : S^{-1}(M) \longrightarrow S^{-1}(M)$
$\frac{m}{s} \longmapsto \frac{g(m)}{s}$
Thus $\frac{g(m)}{s} = \frac{1}{s}\frac{g(m)}{1}$.
Or g is a graded $S^{-1}(A)$-morphism, so $\frac{1}{s} = \frac{g(m)}{1} = g(\frac{1}{s}\frac{m}{1}) = g(\frac{m}{s}) \Longrightarrow S^{-1}(g_{ind}) = g$
g is a graded monomorphism, then g_{ind} is a graded monomorphism.
Since M is cohopfian, then g_{ind} is a graded automorphism of M.

Let $\frac{m'}{s} \in S^{-1}(M) \Longrightarrow m' \in M$, so there exists $m \in M : g_{ind}(m) = m'$
hence $\frac{m}{s} \in S^{-1}(A)$, $[S^{-1}(g_{ind})](\frac{m}{s}) = \frac{g(m)}{s} = \frac{m'}{s}$
so $(g(\frac{m}{s})) = \frac{m'}{s}$.
Thus $S^{-1}(M)$ is a cohopfian left $S^{-1}A$-module.

Corollary 2.
Under the same conditions of the previous theorem If M is a cohopfian and completely invariant submodule of the left A-module $S^{-1}M$, then $(S^{-1}M)_i$ is a cohopfian group.

Proof.
By Theorem(11)

Theorem 12.
Let M be a noetherian quasi-injective graded left A-module, N be an essential and completely invariant graded submodule of M and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. Then, the garded left $S^{-1}A$-modules $S^{-1}(M)$ is cohopfian if, and only, if $S^{-1}(N)$ is cohopfian garded left $S^{-1}(A)$-module.

Proof.
Suppose that $S^{-1}(M)$ is cohopfian and let $S^{-1}(f) : S^{-1}(N) \longrightarrow S^{-1}(N)$ be a graded monomorphism. As $S^{-1}(M)$ is quasi-injective because M is noetherian. Then, there exists a graded morphism $S^{-1}(g) \in End(S^{-1}(M))$ such that $S^{-1}(g_{|S^{-1}(N)}) = S^{-1}(f)$.
$S^{-1}(g)$ is injective since $S^{-1}(N)$ is essential in $S^{-1}(M)$, and as $S^{-1}(M)$ is cohopfian, $S^{-1}(g)$ is invertible. Let $\frac{x}{s} \in S^{-1}(N)$, there exists $\frac{y}{t} \in S^{-1}(M)$ such that $\frac{x}{s} = [S^{-1}(g)](\frac{y}{t})$.
Or $S^{-1}(g^{-1}) \in End(S^{-1}(N))$ and $S^{-1}(N)$ is completely invariant, so $\frac{y}{t} = [S^{-1}(g^{-1})](\frac{x}{s}) \in S^{-1}(N)$, thus $S^{-1}(f)$ is an automorphisme, consequentely $S^{-1}(N)$ is cohopfian.
Reciprocally, suppose that $S^{-1}(N)$ is cohopfian and let $S^{-1}(f) : S^{-1}(M) \longrightarrow S^{-1}(M)$ a graded monomorphism. Then $S^{-1}(f_{|S^{-1}(N)})$ is a graded monomorphism of $S^{-1}(N)$. Thus, $S^{-1}(f) \in Aut(S^{-1}(N))$, hence $[S^{-1}(f)](S^{-1}(N)) = S^{-1}(N)$. As $S^{-1}(M)$ is quasi-injective, then there exists $S^{-1}(L)$ a submodule of $S^{-1}(M)$ such that $S^{-1}(M) = [S^{-1}(f)](S^{-1}(M)) \oplus S^{-1}(L)$. Thus, we have $0 = [S^{-1}(f)](S^{-1}(N)) \cap S^{-1}(L) = S^{-1}(N) \cap S^{-1}(L)$, since $S^{-1}(N)$ is essential, then $S^{-1}(L) = 0$, hence $S^{-1}(M) = S^{-1}[(f)](S^{-1}(M))$, thus $S^{-1}(f)$ is an epimorphism, so $S^{-1}(M)$ is cohopfian left $S^{-1}A$-module.

Theorem 13.
Let M be a graded quasi-projective left A-module, N a superfluous and completely invariant graded sub-module of M and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. Then the left $S^{-1}(A)$-modules $S^{-1}(N)$ is hopfian if, and only if, $S^{-1}(M/N)$ is hopfian.

Proof.
Suppose that $S^{-1}(M/N)$ is hopfian and let $S^{-1}(f) : S^{-1}(M) \longrightarrow S^{-1}(M)$ a graded epimorphism.
As $S^{-1}(N)$ is completely invariant, then $[S^{-1}(f)](N) \subset S^{-1}(S^{-1}(N))$, implies $S^{-1}(f)$

induces a graded epimorphism $S^{-1}(\overline{(f)}) : S^{-1}(M/N) \longrightarrow S^{-1}(M/N)$, since $S^{-1}(M/N)$ is hopfian, then $S^{-1}(\overline{(f)})$ is a graded automorphism. Put $S^{-1}(K) = ker(S^{-1}(f))$ and $S^{-1}(\pi) : S^{-1}(M) \longrightarrow S^{-1}(M/N)$ the canonical projection, we have :
$S^{-1}(\overline{(f)}) \circ [S^{-1}(\pi)](S^{-1}(K)) = S^{-1}(\pi \circ f(K)) = 0$
Indeed, $\forall \frac{x}{s} \in S^{-1}(K)$, we have :
$[S^{-1}(\overline{f} \circ \pi)](\frac{x}{s}) = [S^{-1}(\pi \circ f)](\frac{x}{s})$, so
$[S^{-1}(\pi \circ f)](\frac{x}{s}) = [S^{-1}(\pi)](\frac{f(x)}{s}) = [S^{-1}(\pi)](\frac{0}{s}) = 0$, thus $[S^{-1}(\overline{f} \circ \pi)](K) = 0$.
we have :
$[S^{-1}(\overline{f} \circ \pi)](K) = S^{-1}[\pi \circ f](K) = 0 \Longrightarrow [S^{-1}(\overline{f})](\pi(K)) = 0 \Longrightarrow [S^{-1}(\pi)](K) \subset S^{-1}(N) \Longrightarrow S^{-1}(K) \subset S^{-1}(N)$
Since M is graded quasi- projective $\Longrightarrow S^{-1}(M)$ is graded quasi-projective, there exists a graded endomorphism $S^{-1}(s) : S^{-1}(M) \longrightarrow S^{-1}(M)$ such that $S^{-1}(f \circ s) = S^{-1}(id_{S^{-1}(M)})$, this implies $S^{-1}(M) = S^{-1}(K \oplus Im(s))$, or $K = N$ and $S^{-1}(N)$ is superfluous in $S^{-1}(M)$, then $S^{-1}(M) = S^{-1}(Im(s))$, so $S^{-1}(K) = ker(S^{-1}(f)) = 0$, thus $S^{-1}(f)$ is a monomorphism,
finally, $S^{-1}(f)$ is a monomorphism, so $S^{-1}(M)$ is a graded hopfian left $S^{-1}(A)$-module.
Reciprocally, suppose that $S^{-1}(M)$ is hopfian, show that $S^{-1}(M/N)$ is hopfian.
Let $S^{-1}(\varphi) : S^{-1}(M/N) \longrightarrow S^{-1}(M/N)$ a graded epimorphism of left $S^{-1}(A)$-module, as M is quasi-projective, then , $S^{-1}(M)$ is quasi-projective. Consider $S^{-1}(\pi) : S^{-1}(M) \longrightarrow S^{-1}(M/N)$, then there exists $S^{-1}(f) \in End(S^{-1}(M))$ such that $S^{-1}(\pi \circ f) = S^{-1}(\varphi \circ \pi)$.
Since $S^{-1}(\varphi)$ is an epimorphism, $\forall\ (\overline{\frac{x}{s}}) \in S^{-1}(M/N), \exists\ (\overline{\frac{y}{t}}) \in S^{-1}(M/N)$ such that
$[S^{-1}(\varphi)](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}} = [S^{-1}(\varphi)](\frac{\pi(y)}{t})$
$[S^{-1}(\pi \circ f)](\frac{y}{t}) = [S^{-1}(\varphi \circ \pi)](\frac{y}{t})$
$[S^{-1}(\pi)](\frac{(f(x))}{s}) = [S^{-1}(\varphi)](\overline{\frac{y}{t}})$
$\Longrightarrow [S^{-1}(\varphi)](\overline{\frac{y}{t}}) = [S^{-1}(\overline{f})](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}}$
$\Longrightarrow (\overline{\frac{f(y)}{t} - \frac{x}{s}} = \overline{0}$
$\Longrightarrow \frac{f(y)}{t} - \frac{x}{s} \in S^{-1}(N)$,
then $S^{-1}(M) = Im(S^{-1}(f)) + S^{-1}(N)$, as $S^{-1}(N)$ is superfluous, then $Im(S^{-1}(f)) = S^{-1}(M) \Longrightarrow S^{-1}(f)$ is a graded epimorphism. So, $S^{-1}(f)$ is a graded automorphism, because $S^{-1}(M)$ is hopfian.
So the restriction of $S^{-1}(f)$ over $S^{-1}(N)$ is a graded automorphism of $S^{-1}(N)$.
If $[S^{-1}(\varphi)](\overline{\frac{x}{s}}) = [S^{-1}(\overline{f})](\frac{x}{s}) = \overline{0}$, then $[S^{-1}(f)](\frac{x}{s}) \in S^{-1}(N)$, or $S^{-1}(N)$ is completely invariant, then $\frac{x}{s} \in S^{-1}(N)$, so $\overline{\frac{x}{s}} = \overline{0} \Longrightarrow ker(S^{-1}(\varphi)) = S^{-1}(N) = \overline{0} \Longrightarrow S^{-1}(\varphi)$ is a monomorphism $\Longrightarrow S^{-1}(\varphi)$ is an automorphism, lastly $S^{-1}(M/N)$ is hopfian, hence $S^{-1}(M/N)$ is a hopfian left $S^{-1}A$-module.

5. Localization of hopfian and cohopfian objects in the $COMP(AGr(A - Mod))$ category

Definition 8.
Let M_* an object of $COMP(AGr(A{-}Mod))$. Then M_* is said to be hopfian (resp. cohopfian)

if any epimorphism (resp. monomorphism) f_* of M_* is an automorphism.

Lemma 2.
Let M be a graded left A-module, $f : M \longrightarrow M$ a graded morphism and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. If f is an epimorphism (respectively a monomorphism), then $S^{-1}(f(n))$ is an epimorphism (respectively monomorphism).

Proof.
Since $f(n)$ is the induce of f, then $S^{-1}(f(n))$ is an epimorphism (respectively a monomorphisme).

Lemma 3.
Let M be a graded left A-module and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. Then $S^{-1}(M)$ is a hopfian (respectively cohopfian) garded left A-module if any $n \in \mathbb{Z}$, $S^{-1}(M(n))$ is a hopfian (respectively cohopfian) graded left A-module.

Proof.
Let $S^{-1}(g) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n))$ be a graded epimorphism (respectively a monomorphism), since $S^{-1}(M) = S^{-1}(M(n)\bigoplus_{k>n} M_{n+k}) \cong S^{-1}(M(n)\bigoplus_{k>n} M_{n+k})$. Put $S^{-1}(f) = S^{-1}(g) + S^{-1}(id_{M_{n+k}})$, where $S^{-1}(f)$ is an epimorphism (respectively a monomorphism) of $S^{-1}(M)$.
As M is hopfian (respectively cohopfian) this implies that f is an isomorphism, i.e $S^{-1}(f)$ is an isomorphism of $S^{-1}(M)$, thus $S^{-1}(g)$ is an isomorphism of $S^{-1}M(n))$, so $S^{-1}(M(n))$ is hopfian (respectivement cohopfian).

Theorem 14.
Let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M a graded left A-module and M_* the complex sequence of morphisms of graded left A-modules associated with M. Then, if the complexe complex $S^{-1}(M_*)$ of morphisms of graded left $S^{-1}(A)$-modules is hopfian, implies that M_* is hopfian.

Proof.
Let $f_* : M_* \longrightarrow M_*$ be an epimorphism of M_*. Then $S^{-1}(f(n)) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n))$ is a graded epimorphism of graded left $S^{-1}(A)$-module for all $n \in \mathbb{Z}$. as $S^{-1}M(n)$ is hopfian for all $n \in \mathbb{Z}$, so $S^{-1}(f(n))$ is a graded automorphism of $S^{-1}(M(n))$.
Let m_1 and $m_2 \in M(n)$ such that
$f(n)(m_1) = f(n)(m_2)$
$\Longrightarrow \frac{f(n)(m_1)}{1} = \frac{f(n)(m_2)}{1} \Longrightarrow [S^{-1}(f(n))](m_1) = [S^{-1}(f(n))](m_2)$
$\Longrightarrow m_1 = m_2$
Thus $f(n)$ is a graded automorphism of $M(n)$ for any $n \in \mathbb{Z}$, so M is hopfian.

Theorem 15.
Let A be a graded ring, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, M a graded left A-module. If M_* the complex sequence associated with M, is cohopfian and completely invariant, then $S^{-1}(M_*)$, the complex sequence associated with $S^{-1}(M)$ is cohopfian.

Proof.
Let $g : S^{-1}(M_*) \longrightarrow S^{-1}(M_*)$ be a graded $S^{-1}(A)$-morphism, we remark also that g is a graded A-morphism. Since M_* is completely invariant complex sequence of left $S^{-1}A$-modules , so $g(M_*) \subset M_*$.
Suppose that g is an monomorphism.
Thus the induce morphism $g_{ind} : M_* \longrightarrow M_*$ is a monomorphism of M_* and since M_* is cohopfian, then g_{ind} is an automorphism graded of M_*.
Consider
$S^{-1}(g_{ind}(n)) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n))$
$\frac{m}{s} \longmapsto \frac{g(n)(m)}{s}$
So $\frac{g(n)(m)}{s} = \frac{1}{s}\frac{g(n)(m)}{1}$.
Or $g(n)$ is a graded $S^{-1}(A)$-morphism , thus $\frac{1}{s} = \frac{g(n)(m)}{1} = g(n)(\frac{1}{s}\frac{m}{1}) = g(n)(\frac{m}{s}) \Longrightarrow S^{-1}(g_{ind}(n)) = g(n)$
$g(n)$ is a graded monomorphism, then $g_{ind}(n)$ is a graded monomorphism.
As $M(n)$ is cohopfian for all $n \in \mathbb{Z}$, then $g_{ind}(n)$ is a graded automorphism of $M(n)$.
Let $\frac{m'}{s} \in S^{-1}(M(n)) \Longrightarrow m' \in M(n)$, so there exists $m \in M(n) : g_{ind}(n)(m) = m'$
hence $\frac{m}{s} \in S^{-1}(M(n))$, $[S^{-1}(g_{ind}(n))](\frac{m}{s}) = \frac{g(n)(m)}{s} = \frac{m'}{s}$
thus $(g(n)(\frac{m}{s})) = \frac{m'}{s}$.
hence $S^{-1}(M(n))$ for all $n \in \mathbb{Z}$ is cohopfian, consequentely $S^{-1}(M_*)$ is cohopfian.

Theorem 16.
Let M be a noetherian graded left A-module, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions, N a submodule of M, M_* is a noetherian quasi-injective complex sequence associated with M and N_* is an essential and completely invariant complex sub-sequence of M_*. Then, $S^{-1}(N_*)$ the complex sequence of morphisms of left $S^{-1}(A)$-modules is cohopfian if, and only, if $S^{-1}(M_*)$ is cohopfian.

Proof.
Suppose that $S^{-1}(M_*)$ is cohopfian and let $S^{-1}(f_*) : S^{-1}(N_*) \longrightarrow S^{-1}(N_*)$ be a monomorphism. As M_* is noetherian, then $S^{-1}(M_*)$ is noetherian quasi-injective, so for all $n \in \mathbb{Z}, S^{-1}(M(n))$ is noetherian and quasi-injective. Then, there exists $S^{-1}(g(n)) \in End(S^{-1}(M(n)))$ $\forall n \in \mathbb{Z}$ such that $S^{-1}(g_{|S^{-1}(N(n))}) = S^{-1}(f(n))$.
$S^{-1}(g(n))$ is injective since $S^{-1}(N_*)$ is essential in $S^{-1}(M_*)$, hence $\forall n \in \mathbb{Z}$, $S^{-1}(N(n))$ is essential, and as $S^{-1}(M_*)$ is cohopfian, $S^{-1}(g)$ is invertible. Let $\frac{x}{s} \in S^{-1}(N(n))$, there exists $\frac{y}{t} \in S^{-1}(M(n))$ such that $\frac{x}{s} = [S^{-1}(g(n))](\frac{y}{t})$. Or $S^{-1}(g^{-1}(n)) \in End(S^{-1}(N(n)))$ and $S^{-1}(N(n))$ is completely invariant, so $\frac{y}{t} = S^{-1}(g^{-1})(\frac{x}{s}) \in S^{-1}(N)$, thus $S^{-1}(f(n))$ is an epimorphism for all $n \in \mathbb{Z}$. Thus $S^{-1}(f_*)$ is an automorphisme, consequentely $S^{-1}(N_*)$

is cohopfian.
Reciprocally, suppose that $S^{-1}(N_*)$ is cohopfian and let $S^{-1}(f_*) : S^{-1}(M_*) \longrightarrow S^{-1}(M_*)$ a monomorphism. Then $S^{-1}(f_{*|S^{-1}N_*})$ is a monomorphism of $S^{-1}(N_*)$, so for all $n \in \mathbb{Z}, S^{-1}(f(n)_{|S^{-1}N(n)})$ is a monomorphism of $S^{-1}(N(n))$. Thus, $S^{-1}(f(n)) \in Aut(S^{-1}(N(n)))$, hence $[S^{-1}(f(n))](S^{-1}(N(n))) = S^{-1}(N(n))$. As $S^{-1}(M(n))$ is quasi-injective, then there exists $S^{-1}(L(n))$ a submodule of $S^{-1}(M(n))$ such that $S^{-1}(M(n)) = [S^{-1}(f(n))](S^{-1}M(n)) \oplus S^{-1}(L(n))$. Thus, we have $0 = [S^{-1}(f(n))](S^{-1}N) \cap S^{-1}(L(n)) = S^{-1}(N(n)) \cap S^{-1}(L(n))$, since $S^{-1}(N(n))$ is essential, then $S^{-1}(L(n)) = 0$, hence $S^{-1}(M(n)) = [S^{-1}(f(n))](S^{-1}M(n))$, thus $S^{-1}(f(n))$ is an epimorphism for all $n \in \mathbb{Z} \Longrightarrow S^{-1}(f_*)$ is an epimorphism of chain complex, so $S^{-1}(M_*)$ is cohopfian complex sequence.

Proposition 2.
let M be a graded left A-module and S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. If M_* is a hopfian, noetherian and quasi-injective complex sequence associated with M, then the complex sequence of morphisms of left $S^{-1}(A)$-modules $S^{-1}(M_*)$ has the following property : $\ll$any epimorphism of sub-complex $S^{-1}(N_*)$ of $S^{-1}(M_*)$ is an isomorphism $\gg$.

Proof.
Let N_* be a subcomplex of M_* and $S^{-1}(f_*) : S^{-1}N_* \longrightarrow S^{-1}M_*$ an epimorphism of chain complex. Since M_* is quasi-injective, then for all $n \in \mathbb{Z}$, $M(n)$ is quasi-injective $\Longrightarrow$ $S^{-1}(M(n))$ is quasi-injective for all $n \in \mathbb{Z}$, so, there exists $S^{-1}(\tilde{f}(n)) \in End(S^{-1}(M(n)))$ such that $S^{-1}(\tilde{f}(n))_{|S^{-1}(N(n))} = S^{-1}(f(n))$. Or $S^{-1}(f(n))$ is surjective, then for all $\frac{x}{s} \in S^{-1}(M(n))$, there exists $\frac{y}{t} \in S^{-1}(N(n))$ such that $\frac{x}{s} = [S^{-1}(f(n))](\frac{y}{t})$, hence $S^{-1}(\tilde{f}(n))$ is surjective, as $S^{-1}(M(n))$ is hopfian, thus $\widetilde{S^{-1}}(f)(n) \in Aut(S^{-1}(N(n)))$ for all $n \in \mathbb{Z}$. We deduce that $S^{-1}(f(n))$ is a monomorphism of $S^{-1}(N(n))$ into $S^{-1}(M(n))$, for all $n \in \mathbb{Z}$. Consequentely, $S^{-1}(f_*)$ is an isomorphism of chain complex.

Theorem 17.
Let M be a graded left A-module, N a graded submodule of M, S a saturated multiplicative part formed by the non-zero homogeneous elements of A verifying the left Ore conditions. M_* the quasi-projective complex sequence associated with M and N_* a superfluous and completely invariant complex sub-sequence of M_*. Then the complex morphism sequence of left $S^{-1}(A)$--modules $S^{-1}(N_*)$ is hopfian if, and only if, $S^{-1}(M_*/N_*)$ the complex sequence associated with $S^{-1}(M/N)$ is hopfian.

Proof.
Suppose that $S^{-1}(M_*/N_*)$ is hopfian and let $f_* : M_* \longrightarrow M_*$ an epimorphism.
As $S^{-1}(N_*)$ is completely invariant, then $\forall n \in \mathbb{Z}, [S^{-1}(f(n))](S^{-1}(N(n))) \subset S^{-1}(N(n))$, implies that $S^{-1}(f(n))$ induces an epimorphism $S^{-1}(\overline{f})(n) : S^{-1}(M(n)/N(n)) \longrightarrow S^{-1}(M(n)/N(n))$, since $S^{-1}(M_*/N_*)$ is hopfian, then $S^{-1}(\overline{f})(n)$ is an automorphism. Put $S^{-1}(K(n)) = ker(S^{-1}f(n))$ and $S^{-1}(\pi(n)) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n)/N(n))$ the canonical projection, we have :
$S^{-1}(\overline{f})(n) \circ [S^{-1}(\pi(n))](S^{-1}(K(n))) = [S^{-1}(\pi(n) \circ f(n))](K) = 0$

Indeed, $\forall \frac{x}{s} \in S^{-1}(K(n))$, we have :
$[S^{-1}(\overline{f}(n) \circ \pi(n))](\frac{x}{s}) = [S^{-1}(\pi(n) \circ f(n))](\frac{x}{s})$, so
$[S^{-1}(\pi(n) \circ f(n))](\frac{x}{s}) = [S^{-1}(\pi(n))](\frac{f(n)(x)}{s}) = [S^{-1}(\pi(n))](\frac{0}{s}) = 0$, thus $[S^{-1}(S^{-1}(\overline{f})(n) \circ \pi(n))](K) = 0$.
we have :
$[S^{-1}(\overline{f}(n) \circ \pi(n))](K(n)) = [S^{-1}(\pi(n) \circ f(n))](K(n)) = 0 \Longrightarrow [S^{-1}(\overline{f}(n))](\pi(n)(K(n))) = 0 \Longrightarrow [S^{-1}(\pi(n))](K(n)) \subset S^{-1}(N(n)) \Longrightarrow S^{-1}(K(n)) \subset S^{-1}(N(n))$
Since M_* is quasi- projective $\Longrightarrow S^{-1}M_*$ quasi-projective, there exists an endomorphism $S^{-}(s(n)) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n))$ such that $S^{-1}(f(n) \circ s(n)) = S^{-1}(id(n)_{S^{-1}(M(n))})$, this implies $S^{-1}(M(n)) = S^{-1}(K(n) \oplus Ims(n))$, or $K(n) = N(n)$ and $S^{-1}(N(n))$ is superfluous in $S^{-1}(M(n)), \forall n \in \mathbb{Z}$, then $S^{-1}(M(n)) = S^{-1}(Ims(n))$, so $S^{-1}(K(n)) = ker(S^{-1}(f)(n)) = 0, \forall n \in \mathbb{Z}$, thus $S^{-1}(f(n))$ is a monomorphism, $\forall n \in \mathbb{Z}$
finally, $S^{-1}(f_*)$ is a monomorphism, so $S^{-1}(M_*)$ is hopfian.
Reciprocally,
If $S^{-1}(M_*)$ is hopfian, show that $S^{-1}(M_*/N_*)$ is hopfian.
Let $S^{-1}(\varphi(n)) : S^{-1}(M_*/N_*) \longrightarrow S^{-1}(M_*/N_*)$ an epimorphism of chain complex, as $S^{-1}(M_*)$ is quasi-projective, then $\forall n \in \mathbb{Z}, S^{-}(M(n))$ is quasi-projective. Consider $S^{-1}(\pi(n)) : S^{-1}(M(n)) \longrightarrow S^{-1}(M(n)/N(n))$, then there exists $S^{-1}(f(n)) \in End(S^{-1}(M(n)))$ such that $S^{-1}(\pi(n) \circ f(n)) = S^{-1}(\varphi(n) \circ \pi(n)$.
Since $S^{-1}(\varphi(n))$ is an epimorphism, $\forall \overline{\frac{x}{s}} \in S^{-1}(M(n)/N(n)), \exists (\overline{\frac{y}{t}}) \in S^{-1}(M(n)/N(n))$ such that
$[S^{-1}(\varphi(n))](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}} = [S^{-1}(\varphi(n))](\frac{\pi(n)(y)}{t})$
$[S^{-1}(\pi(n) \circ f(n))](\frac{y}{t}) = [S^{-1}(\varphi(n) \circ \pi(n))](\frac{y}{t})$
$[S^{-1}(\pi(n))](\frac{(f(n)(x))}{s}) = [S^{-1}(\varphi)](\overline{\frac{y}{t}})$
$\Longrightarrow [S^{-1}(\varphi(n))](\overline{\frac{y}{t}}) = [S^{-1}(\overline{f}(n))](\overline{\frac{y}{t}}) = \overline{\frac{x}{s}}$
$\Longrightarrow (\overline{\frac{f(n))(y)}{t} - \frac{x}{s}} = \overline{0}$
$\Longrightarrow \frac{f(n)(y)}{t} - \frac{x}{s} \in S^{-1}(N(n))$,
then $S^{-1}(M) = S^{-1}(Im(f)(n)) + S^{-1}(N(n)$, as $S^{-1}(N(n))$ is superfluous, then $S^{-1}(Im(f)(n)) = S^{-1}(M(n)) \Longrightarrow S^{-}(f(n))$ is an epimorphism. So, $S^{-1}f(n)$ is an automorphism, because $S^{-1}(M(n))$ is hopfian for all $n \in \mathbb{Z}$.
So the restriction of $S^{-1}(f(n))$ over $S^{-1}(N(n))$ is an automorphisme of $S^{-1}(N(n))$.
If $[S^{-1}(\varphi)(n)](\overline{\frac{x}{s}}) = [S^{-1}(\overline{f}(n))](\frac{x}{s}) = \overline{0}$, then $S^{-1}[(f(n))](\frac{x}{s}) \in S^{-1}(N(n))$, or $S^{-1}(N(n))$ is completely invariant, then $\frac{x}{s} \in S^{-1}(N(n))$, so $\overline{\frac{x}{s}} = \overline{0} \Longrightarrow ker(S^{-1}(\varphi(n))) = S^{-1}(N(n)) = \overline{0} \Longrightarrow S^{-1}(\varphi(n))$ is a monomorphism $\Longrightarrow \varphi(n)$ is an automorphism for all $n \in \mathbb{Z}$, lastly $S^{-1}(M(n)/N(n))$ is hopfian for all $n \in \mathbb{Z}$, hence $S^{-1}(M_*/N_*)$ is a hopfian complex sequence.

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