



On Efficient Zero Ring Labeling and Restricted Zero Ring Graphs

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Abstract. In [3], Acharya et al. introduced the notion of a **zero ring labeling** of a connected graph G , where vertices are labeled by the elements of a zero ring such that the sum of the labels of adjacent vertices is not the additive identity of the ring. Archarya and Pranjali [1] also constructed a graph based on a finite zero ring called the zero ring graph. In [5], Chua et al. defined a class of zero ring labeling called efficient zero ring labeling and it was shown that a labeling scheme exists for some families of trees. In this paper, we provide an efficient zero ring labeling for some classes of graphs. We also introduce the notion of the restricted zero ring graphs and use them to show that a zero ring labeling exists for some classes of cactus graphs.

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1. Introduction

One of the fields in graph theory that has been a study of interest of many mathematical researchers is graph labeling. Most of the methods or schemes in graph labeling can be traced to or have links to a paper by Rosa [9]. At present, over 200 graph labeling techniques have been studied in over 2500 papers. For an excellent survey on different labeling schemes on graph, the readers may refer to [7].

In 2014, Acharya et al. [3] introduced zero ring labeling. In this labeling, each vertex is assigned a unique label from a zero ring such that the sum of any two adjacent vertices is not zero, i.e., the additive identity of the zero ring. It was proved that every graph admits a zero ring labeling with respect to some zero ring. The zero ring index of a graph, which is the smallest order of a zero ring in which the graph admits a zero ring labeling,

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was also studied for some well-known graphs. Acharya et al. [2] determined a necessary and sufficient condition for a finite graph of order n to attain an optimal zero ring index.

In [5], the notion of an efficient zero ring labeling was introduced. A zero ring labeling of a graph is called efficient if the cardinality of the set of distinct sums obtained from all adjacent vertices is equal to the maximum degree of the graph. It was shown that some trees have an efficient zero ring labeling. However, not all graphs have an efficient zero ring labeling.

The paper is organized as follows. Section 2 provides some background on the notion of zero ring labeling and efficient zero ring labeling. In section 3, we show that the path graphs and the complete graph on 2^k vertices admit an efficient zero ring labeling. Moreover, an explicit labeling scheme for path graphs was provided. In section 4, we construct restricted zero ring graphs. Some edge induced subgraphs of a zero ring graph were identified and shown to admit an efficient zero ring labeling.

2. Preliminaries

For completeness, we state the following definitions and concepts related to zero ring labeling and some terminologies in graph theory. The graphs considered in the paper are finite undirected simple graph.

A *tadpole graph* denoted by $T_{m,n}$ is a graph obtained from a cycle C_m and a path P_n by joining an end vertex of the path to a vertex in the cycle with an edge. A *cactus graph* is a connected graph in which any two cycles have at most one vertex in common. Equivalently, any edge of a cactus graph lies on at most one cycle. A cactus graph with minimum degree 2, maximum degree 3, and having exactly 2 cycles is a graph that can be constructed with two disjoint cycles of length n_1, n_2 and a vertex from one cycle joined to a vertex of the second cycle by a path of length m . Suppose G and H are graphs with $V(G) = \{u_0, u_1, u_2, \dots, u_{m-1}\}$ and $V(H) = \{v_0, v_1, v_2, \dots, v_{n-1}\}$. The *Cartesian product* $G \times H$ of graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and $e = (u_i, v_j)(u_k, v_l)$ is an edge of $G \times H$ if and only if either $i = k$ and $v_j v_l \in E(H)$; or $j = l$ and $u_i u_k \in E(G)$.

For other notations and concepts in graphs and groups not explicitly stated in the paper, we refer to [6, 8].

Although in some literature a zero ring is defined as the trivial ring that contains only one element, in this paper we follow the definition used by Acharya et al. [3] and use the term *zero ring* for a ring with additive identity 0 such that $ab = 0$ for any $a, b \in R$. This notion of a zero ring was previously defined by Bourbaki in [4] and called it a *pseudo ring of square zero*.

Note that a zero ring R can always be constructed from an abelian group G , by defining R as the set of the set of all 2×2 matrices of the form $R = \left\{ \begin{bmatrix} a & -a \\ a & -a \end{bmatrix}, a \in G \right\}$. We use the notation $A_a \in R$ to identify the element of the ring associated with the element $a \in G$. Throughout the paper, we denote this zero ring by $M_2^0(G)$. We often use this notation when we wish to be specific on the ring being used in the labeling scheme.

Let $\Gamma = (V, E)$ be a graph with vertex set $V =: V(\Gamma)$ and edge set $E =: E(\Gamma)$, and let R be a finite zero ring. An injective function $f : V \rightarrow R$ is called a *zero ring labeling* of Γ if $f(u) + f(v) \neq 0$ for every edge $uv \in E$

In [5], the authors defined the notion about k -zero ring labelings and efficient zero ring labeling of a graph. A zero ring labeling f of a graph $\Gamma = (V, E)$ is called a *k -zero ring labeling* if $|K| = |\{f(u) + f(v) : uv \in E\}| = k$. If $|K| = \Delta(\Gamma)$ where $\Delta(\Gamma)$ denotes the maximum degree of a vertex in Γ , then the zero ring labeling is *efficient*. It was shown that some families of trees have an efficient zero ring labeling. Although it was shown in [3] that every finite graph admits a zero ring labeling, not all graphs have an efficient zero ring labeling. In particular it was shown in [5] that some cycles do not have an efficient zero ring labeling.

Theorem 1. [5] *Cycles of odd length do not have an efficient zero ring labeling.*

3. Families of Graphs with an Efficient Zero ring labeling

In this section, we show that several common families of graphs admit an efficient zero ring labeling by explicitly providing a labeling scheme. In [5], it was shown that several families of graphs admit an efficient zero ring labeling; however, some labeling schemes were not explicitly shown, particularly for paths and complete graphs.

One useful result in [5] that can be used to show that a graph Γ admits an efficient zero ring labeling is shown in the following theorem.

Theorem 2. [5]. *If Γ is a graph that admits an efficient zero ring labeling, then any edge induced subgraph Γ' of Γ such that $\Delta(\Gamma') = \Delta(\Gamma)$ has an efficient zero ring labeling.*

In [5], it was shown that the path P_n admits an efficient zero ring labeling, which is just a corollary to the labeling scheme used for caterpillars. We now give a different and explicit labeling scheme for paths as stated in the following theorem.

Theorem 3. *The path $P_n, n \geq 1$ is an integer, admits an efficient zero ring labeling.*

Proof. Let v_0, v_1, \dots, v_{n-1} be the vertices of a path P_n whose edges are of the form (v_i, v_{i+1}) where $i = 0, 1, \dots, n-2$. We consider the case when n is even or odd. Let A_i denote the matrix $\begin{bmatrix} i & -i \\ i & -i \end{bmatrix} \in M_2^0(\mathbb{Z}_n)$.

Case 1. Let $n > 1$ be even. Define a function $f : V(P_n) \rightarrow M_2^0(\mathbb{Z}_n)$ such that

$$f(v_i) = \begin{cases} A_i & \text{when } i = 1 \text{ or } i \text{ is even} \\ A_{n+2-i} & \text{if } i \text{ is odd and } i \geq 3 \end{cases}.$$

Then f is an injective function and $f(v_0) + f(v_1)$ and $f(v_1) + f(v_2)$ are equal to A_1 and A_3 , respectively. Furthermore, we have

$$f(v_i) + f(v_{i+1}) = \begin{cases} A_1 & \text{when } i \text{ is even, } i \geq 2 \\ A_3 & \text{if } i \text{ is odd, } i \geq 3 \end{cases}$$

which shows that $f(v_i) + f(v_{i+1}) \neq A_0$ for all $i = 0, 1, \dots, n - 2$.

Since $K = \{f(v_i) + f(v_{i+1}) \mid i = 0, 1, \dots, n - 2\} = \{A_1, A_3\}$, f is a 2-zero ring labeling of P_n .

Case 2. Let n be odd. Define a function $f : V(P_n) \rightarrow M_2^0(\mathbb{Z}_n)$ such that when n is of the form $4k + 1$, $k \in \mathbb{Z}^+$

$$f(v_i) = \begin{cases} A_{\lfloor \frac{n}{2} \rfloor - i + 1} & \text{for } i = 0, 2, \dots, \lfloor \frac{n}{2} \rfloor - 2, \lfloor \frac{n}{2} \rfloor \\ A_{n+i-\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = 1, 3, \dots, \lfloor \frac{n}{2} \rfloor - 5, \lfloor \frac{n}{2} \rfloor - 3 \\ A_{i-\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 3 \dots, n - 2 \\ A_{n-i+\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 4, \dots, n - 1 \end{cases}$$

and when n is of the form $4k + 3$, $k \in \mathbb{Z}^+$

$$f(v_i) = \begin{cases} A_{n+i-\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = 0, 2, \dots, \lfloor \frac{n}{2} \rfloor - 5, \lfloor \frac{n}{2} \rfloor - 3 \\ A_{\lfloor \frac{n}{2} \rfloor - i + 1} & \text{for } i = 1, 3 \dots, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - 4, \lfloor \frac{n}{2} \rfloor - 2 \\ A_{i-\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = \lfloor \frac{n}{2} \rfloor - 1, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 3 \dots, n - 1 \\ A_{n-i+\lfloor \frac{n}{2} \rfloor + 1} & \text{for } i = \lfloor \frac{n}{2} \rfloor + 2, \lfloor \frac{n}{2} \rfloor + 4, \dots, n - 2 \end{cases}$$

Clearly, f is injective for both subcases. Now we verify that f is an efficient zero ring labeling. We show that the sums $f(v_i) + f(v_{i+1}) \neq A_0$ for $0 \leq i \leq n - 2$, that is, within the intervals $0 \leq i < \lfloor \frac{n}{2} \rfloor - 2$, $\lfloor \frac{n}{2} \rfloor - 2 \leq i < \lfloor \frac{n}{2} \rfloor + 1$ and $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 2$.

For the interval $0 \leq i < \lfloor \frac{n}{2} \rfloor - 2$, the sums $f(v_i) + f(v_{i+1})$ are

$$A_{\lfloor \frac{n}{2} \rfloor - i + 1} + A_{n+(i+1)-\lfloor \frac{n}{2} \rfloor + 1} = A_3$$

when i is odd and $n + 3$ taken modulo n , and

$$A_{n+i-\lfloor \frac{n}{2} \rfloor + 1} + A_{\lfloor \frac{n}{2} \rfloor - (i+1) + 1} = A_1$$

when i is even and $n + 1$ is taken modulo n .

For the interval $\lfloor \frac{n}{2} \rfloor - 2 \leq i < \lfloor \frac{n}{2} \rfloor + 1$, we have the sums

$$A_{\lfloor \frac{n}{2} \rfloor - i + 1} + A_{i+1-\lfloor \frac{n}{2} \rfloor + 1} = A_3$$

when $i = \lfloor \frac{n}{2} \rfloor - 2$ and $i = \lfloor \frac{n}{2} \rfloor$, and when $i = \lfloor \frac{n}{2} \rfloor - 1$, $A_{i-\lfloor \frac{n}{2} \rfloor + 1} + A_{\lfloor \frac{n}{2} \rfloor - (i+1) + 1} = A_1$.

For the interval $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 2$, we have the sums

$$A_{i-\lfloor \frac{n}{2} \rfloor + 1} + A_{n-(i+1)+\lfloor \frac{n}{2} \rfloor + 1} = A_1$$

when i is odd and $n + 1$ is taken modulo n , and

$$A_{n-i+\lfloor \frac{n}{2} \rfloor + 1} + A_{i+1-\lfloor \frac{n}{2} \rfloor + 1} = A_3$$

when i is even and $n + 3$ is taken modulo n .

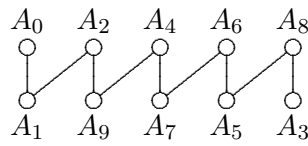


Figure 1: An efficient zero ring labeling for P_{10} using the zero ring $R = M_2^0(\mathbb{Z}_{10})$.

Similar computations can be done for the case when n is odd and of the form $4k + 3$ to yield the same result. Since $K = \{f(v_i) + f(v_{i+1}) | i = 0, 1, \dots, n - 2\} = \{A_1, A_3\}$, f is a 2-zero ring labeling of P_n . Therefore, P_n admits an efficient zero ring labeling. \square

The zero ring labeling for P_{10} shown in Figure 1 using the zero ring $M_2^0(\mathbb{Z}_{10})$ is an efficient zero ring labeling where $K = \{A_1, A_3\}$.

It is a well-known result in group theory that if in a finite group G the order of the non-identity elements is 2, then the order of the group is a power of 2. We state this as the following lemma.

Lemma 1. *If G is a finite group such that every non-identity element is of order 2, then $|G| = 2^k$ for some positive integer k .*

Now consider a finite group G with identity e such that $x^2 = e$ for any $x \in G$. Note that by Lemma 1, $|G| = 2^n$ for $n \geq 0$. This group is always abelian. Thus, a zero ring R can be constructed from G .

Consider a zero ring $R = M_2^0(G)$ where G is a finite group whose non-identity elements have order 2. Consider the complete graph $\Gamma = (V, E)$ with 2^n vertices and any zero ring labeling $f : V \rightarrow R$ on Γ . Since f is injective, the sum of any two distinct elements of R is not the identity and $\{f(u) + f(v) : uv \in E\} = R - \{A_0\}$, hence f is an efficient zero ring labeling of the complete graph. From this observation, the next theorem follows.

Theorem 4. *The complete graph on 2^n vertices, $n \geq 0$, denoted by K_{2^n} admits an efficient zero ring labeling.*

4. Restricted Zero Ring Graphs

In [1], Acharya and Pranjali defined the notion of a zero ring graph as follows. Let R be a finite zero ring. The zero ring graph $\Gamma(R)$ is a simple undirected graph whose vertices are the elements of R and two distinct vertices x and y are adjacent if and only if $x + y \neq 0$ where 0 is the additive identity of R . In this study, we define a variation of this graph called restricted zero ring graph. Let $S \subseteq R - \{0\}$ for any zero ring R . We define the restricted zero ring graph of R in S denoted by $\Gamma_S(R) = (V, E)$ to be the graph with $V = R$ and edge set given by $E = \{(u, v) : u + v \in S\}$. In view of the above definition, if $S = R - \{0\}$, then the graph $\Gamma_S(R)$ is isomorphic to the zero ring graph $\Gamma(R)$.

Consider the zero ring $R = M_2^0(\mathbb{Z}_4)$, the following are examples of restricted zero ring graphs on R where $|S| = 2$.

Note that if $S_1, S_2 \subset R - \{0\}$, the graphs $\Gamma_{S_1}(R), \Gamma_{S_2}(R)$ are not necessarily isomorphic even if $|S_1| = |S_2|$.

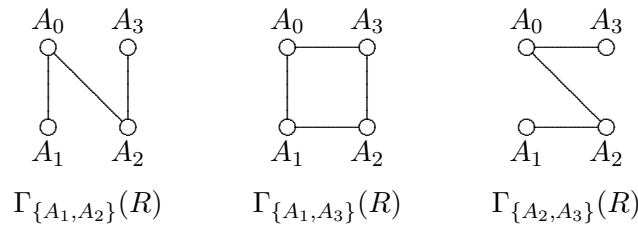


Figure 2: Some restricted zero ring graphs in $M_2^0(\mathbb{Z}_4)$.

The following lemma follows directly from the definition of the restricted zero ring graph and the fact that R together with the addition operation on the ring is an abelian group.

Lemma 2. Consider a zero ring R with additive identity 0 and a non-empty $S \subseteq R - \{0\}$ such that $|R| = n$ and $|S| = m \geq 1$. Then the graph $\Gamma_S(R) = (V, E)$ satisfies the following:

- (i) $\Delta(\Gamma_S(R)) = m$
- (ii) $\Gamma_S(R)$ is a disconnected graph for $n > 2$ and $m = 1$.

Proof. For (i) : Note that for any $S = \{s_1, \dots, s_m\} \subseteq R - \{0\}$, the edge $(0, s_i) \in E(\Gamma_S(R))$ by the definition of $\Gamma_S(R)$. Suppose there is a vertex v in the graph with neighbor set $N(v) = \{u_1, \dots, u_k\}$ where $k > m$. Since $\{v + u_i : i = 1, 2, \dots, k\} \subset S$, there is at least one pair of distinct vertices u_1, u_2 where $v + u_1 = s_i = v + u_2$ for some $s_i \in S$. Then $u_1 = u_2$, which is a contradiction. Hence, the maximum degree of a vertex is m . Statement (ii) follows directly from (i) for any zero ring R with $|R| > 2$. \square

The following are some examples of restricted zero ring graphs.

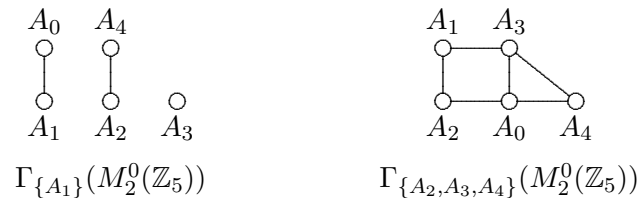


Figure 3: Some restricted zero ring graphs in $M_2^0(\mathbb{Z}_n)$.

Now we consider the restricted zero ring graph $\Gamma_S(R)$ where $R = M_2^0(\mathbb{Z}_n)$. The following lemma can be easily verified from the definition of $\Gamma_S(R)$ and the fact that R together with the additive operation in the ring is an abelian group.

Lemma 3. If $|S| = 1$, then $\Gamma_S(M_2^0(\mathbb{Z}_n))$ is a disconnected graph. In particular, the graph $\Gamma_S(M_2^0(\mathbb{Z}_n))$ consists of at most $\lfloor \frac{n}{2} \rfloor$ disjoint copies of K_2 .

Proof. By Lemma 2, the maximum degree of a vertex in $\Gamma_S(M_2^0(\mathbb{Z}_n))$ is 1. Thus, the graph should consist only of disjoint edges. For an integer $n \geq 2$, the maximum number of disjoint edges is $\lfloor \frac{n}{2} \rfloor$.

Theorem 5. Consider the zero ring $R = M_2^0(\mathbb{Z}_n)$ and $S \subset R - \{0\}$, where $|S| = m > 1$. Let $n_e = |\{A_i \in S : i \text{ even}\}|$ and $n_o = |\{A_i \in S : i \text{ odd}\}|$. Then

- (i) If n is odd, then the number of edges in $\Gamma_S(R)$ is $\frac{m(n-1)}{2}$.
- (ii) If n is even, then the number of edges in $\Gamma_S(R)$ is $\frac{n_e(n-2)}{2} + \frac{n_o(n)}{2}$.

Proof. Consider the additive group table of \mathbb{Z}_n . If n is odd, then every element of the group occurs exactly $\frac{(n-1)}{2}$ times above the main diagonal entries of the table. Thus, if $|S| = m$ then there are $\frac{m(n-1)}{2}$ distinct pairs of indices i, j such that $A_i + A_j \in S$. Now, if n is even, the number of occurrences of an odd index above the main diagonal entries of the table is $\frac{n}{2}$ while there are $\frac{(n-2)}{2}$ occurrences of an even index above the main diagonal entries. Hence, if there are n_e, n_o even and odd indices respectively in S , there is a total of $\frac{n_e(n-2)}{2} + \frac{n_o(n)}{2}$ distinct pairs of indices i, j whose sum is an index in S . \square

In Theorem 3, it was shown that P_n admits an efficient zero ring labeling by providing an explicit labeling of its vertices. Moreover, given the zero ring $M_2^0(\mathbb{Z}_n)$, the restricted zero ring graph $\Gamma_S(M_2^0(\mathbb{Z}_n))$ where $S = \{A_1, A_2\}$ is isomorphic to P_n as shown in Figure 4.

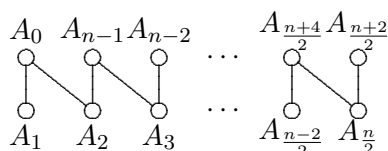


Figure 4: The graphs $\Gamma_{\{A_1, A_2\}}(M_2^0(\mathbb{Z}_n))$

Lemma 4. Let $R = M_2^0(\mathbb{Z}_n)$ and $S_i = \{A_i, A_{i+1}\}$ where $0 \notin S_i$. Then $\Gamma_{S_i}(R) \cong \Gamma_{S_j}(R)$ for $i, j \in \{1, 2, \dots, n-2\}$.

Proof. Since $\Gamma_{\{1,2\}}(R) \cong P_n$, we need to show that each of $\Gamma_{S_i}(R) \cong P_n$. Let n be even. For the set S_i , consider the arrangement of the elements of R in a matrix of size $2 \times \frac{n}{2}$ in the order A_0, A_1, \dots, A_{n-1} in a counterclockwise manner such that the entry of the first row column $\lfloor \frac{i}{2} \rfloor + 1$ is A_0 . In this arrangement, the sum of the entries for each column taken under addition modulo n is A_i if i is odd and A_{i+1} if i is even. The sum of the $(1, j)$ -entry and $(2, j+1)$ entry is A_{i+1} for $j = 1, 2, \dots, \frac{n}{2} - 1$ if i is odd, and the sum of the $(2, j)$ -entry and $(1, j+1)$ entry is A_i for $j = 1, 2, \dots, \frac{n}{2} - 1$ if i is even. With this arrangement of the elements of R , it is easy to see that the graph $\Gamma_{S_i}(R)$ is a path on n vertices.

Let n be an odd integer, we now consider the arrangement of the elements of R in array with 2 rows such that the first row contains $\lfloor \frac{n}{2} \rfloor$ columns when i is odd and $\lceil \frac{n}{2} \rceil$ columns when i is even. The second row always contains $\lceil \frac{n}{2} \rceil$ columns. The elements of R are arranged in the following order A_0, A_1, \dots, A_{n-1} in a clockwise manner such that the entry of the first row column $\lfloor \frac{i}{2} \rfloor + 1$ is A_0 . In a similar manner, the sum of the entries for

each column taken under addition modulo n is A_i if i is odd and A_{i+1} if i is even. Using the same arguments in the case for an even n , the sum of the $(1, j)$ -entry and $(2, j + 1)$ -entry is A_{i+1} for $j = 1, 2, \dots, \frac{n}{2} - 1$ if i is odd, and the sum of the $(2, j)$ -entry and $(1, j + 1)$ -entry is A_i for $j = 1, 2, \dots, \frac{n}{2} - 1$ if i is even. \square

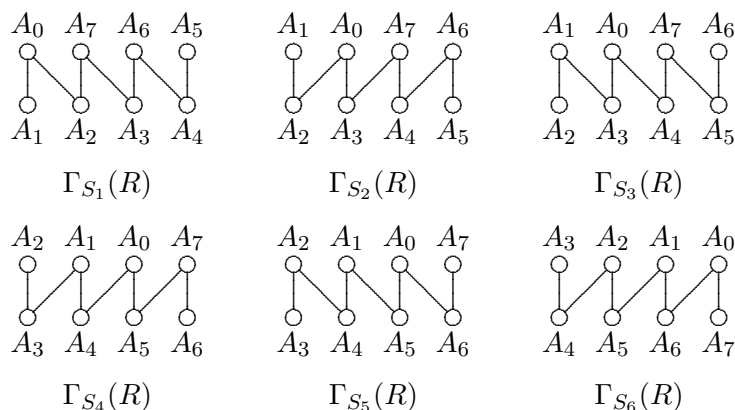


Figure 5: The graphs $\Gamma_{S_i}(R)$ for $i = 1, \dots, 6$ respectively where $S_i = \{i, i + 1\}$, and $R = M_2^0(\mathbb{Z}_8)$.

The graph $\Gamma_S(M_2^0(\mathbb{Z}_n))$ where $S = \{A_1, A_2, A_3\}$ is shown in Figure 6 when n is even.

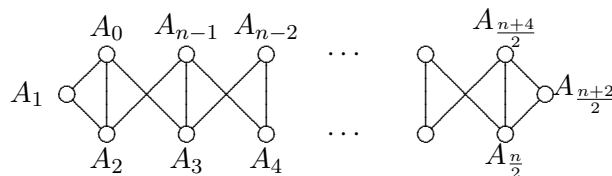


Figure 6: The graph $\Gamma_S(R)$ where $S = \{A_1, A_2, A_3\}$, and $R = M_2^0(\mathbb{Z}_n)$.

The graph $\Gamma_S(M_2^0(\mathbb{Z}_n))$ where $S = \{A_1, A_2, A_3\}$ is shown in Figure 7 when n is odd.

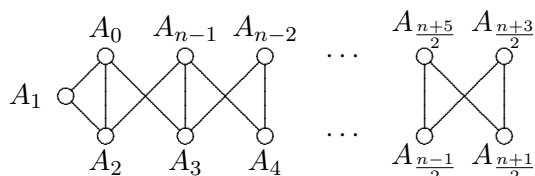


Figure 7: The graph $\Gamma_S(R)$ where $S = \{A_1, A_2, A_3\}$, and $R = M_2^0(\mathbb{Z}_n)$.

Note that Theorem 2 can be used to determine if a graph admits an efficient zero ring labeling. If a graph Γ can be shown to be an edge induced subgraph of $\Gamma_S(R)$ for some subset S of $R - \{0\}$, such that the maximum degree of a vertex in $\Gamma = |S|$, then Γ admits an efficient zero ring labeling. In particular, the following graphs can be embedded in the restricted zero ring graph $\Gamma_S(R)$ where $S = \{A_1, A_2, A_3\}$ and $R = M_2^0(\mathbb{Z}_m)$: tadpole graphs, cactus graphs with exactly 2 cycles and maximum degree 3, minimum degree 2 and the cartesian product of $P_2 \times P_n$ for $n \geq 3$. Thus, these graphs admit an efficient zero ring labeling for some appropriate zero ring R .

Theorem 6. *The following graphs admit an efficient zero ring labeling:*

- (i) *Tadpole graphs $T_{n,m}$*
- (ii) *Cactus graph with exactly two cycles having a minimum and maximum degree of a vertex 2 and 3, respectively.*
- (iii) *$P_2 \times P_n$ for any $n \geq 3$*

Proof. The proof follows from the fact that these graphs are edge induced subgraphs of the graph $\Gamma_S(M_2^0(\mathbb{Z}_n))$ for some positive integer n and $S = \{A_1, A_2, A_3\}$. Thus, by Theorem 2, these graphs admit an efficient zero ring labeling. In particular, the graph $T_{n,m}$ is an edge induced subgraph of $\Gamma_S(M_2^0(\mathbb{Z}_k))$ where $k \geq n + m$ and $S = \{A_1, A_2, A_3\}$. Suppose Γ is a cactus graph with exactly two cycles, and has minimum and maximum degree of a vertex equal to 2 and 3, respectively. If the cycles in Γ have lengths n_1 and n_2 and are connected by a path of length m , then the graph Γ is an edge induced subgraph of $\Gamma_S(M_2^0(\mathbb{Z}_k))$ where $S = \{A_1, A_2, A_3\}$ and for $k \geq n_1 + n_2 + m - 1$. The graph $P_2 \times P_n$ can be embedded in the restricted zero ring graph $\Gamma_A(M_2^0(\mathbb{Z}_k))$ where $S = \{A_1, A_2, A_3\}$ and for $k = 2n + 2$. □

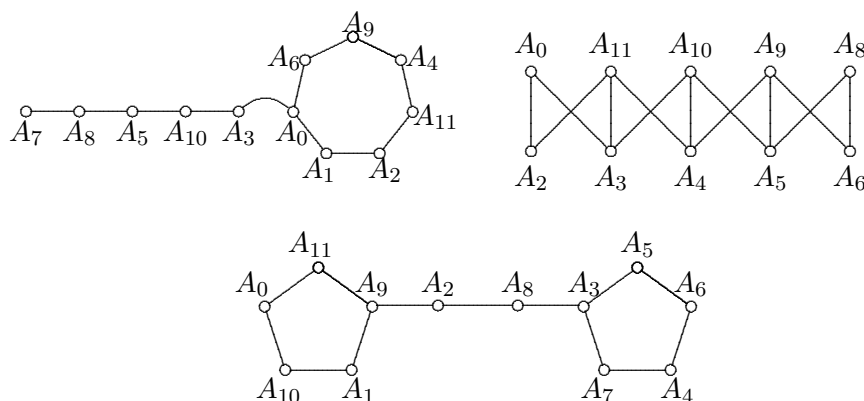


Figure 8: Efficient zero ring labelings of Tadpole graph $T_{7,5}$, Cartesian product $P_2 \times P_5$ and Cactus graph by $R = M_2^0(\mathbb{Z}_{12})$

Figure 8 shows an efficient zero ring labeling of $T_{7,5}$, $P_2 \times P_5$ and a cactus graph with exactly two cycles as edge induced subgraphs of $\Gamma_S(R)$, where $S = \{A_1, A_2, A_3\}$ and $R = M_2^0(\mathbb{Z}_{12})$.

So far, the only known families of graphs that do not admit an efficient zero ring labeling are the odd cycles. In [5], several families of trees have been shown to admit an efficient zero ring labeling. The next logical question on this is to ask whether all trees admit an efficient zero ring labeling, and in [5] it was conjectured that all trees admit an efficient zero ring labeling.

If the conjecture that all trees admit an efficient zero ring labeling is proved to be true, then the cycle graph on an odd number of vertices is the smallest graph, in terms of the number of edges, that does not admit an efficient zero ring labeling.

Problem: Let $S, T \subseteq R - \{0\}$ such that $|S| = |T|$. Determine necessary and sufficient conditions such that $\Gamma_S(R) \cong \Gamma_T(R)$.

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