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General Results for General Transmuted-G Distributions

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Abstract. In this paper we have given some general results for the general transmuted families of distributions introduced by Rahman et al. (2018a,b). These results are helpful to obtain the results for any member of the general families of distributions and their special cases. We have given some examples for illustration.

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1. Introduction

The transmuted families of distributions has been in used since the work of [16]. Specifically, the transmuted family of distributions, given by [16], is defined by the cumulative distribution function (CDF)

$$F_{QT-G}(x) = G(x) + \delta G(x)\left[1 - G(x)\right], x \in \mathbb{R}, \tag{1}$$

where $G(x)$ is CDF of any baseline distribution and $|\delta| \leq 1$ is transmutation parameter. The density function corresponding to (1) is

$$f_{QT-G}(x) = g(x)\left[1 + \delta - 2\delta G(x)\right] \;,\; |\delta| \leq 1, x \in \mathbb{R}, \tag{2}$$

where $g(x)$ is density function corresponding to $G(x)$. The family (1) is referred to as the quadratic transmuted family of distributions and can be used to obtain new distributions for any baseline distribution $G(x)$. The name quadratic transmuted family of distributions emerges from the fact that the family (1) is a quadratic function of the baseline CDF $G(x)$. The density function, (2), of transmuted family of distribution can be written as the sum

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of density functions of order statistics and as sum of density functions of exponentiated distributions, as given by [4]. The density function (2) as sum of density functions of order statistics is

$$f_{QT-G}(x) = (1+\delta)\, g_{1:1}(x) - \delta g_{2:2}(x), \tag{3}$$

where $g_{1:1}(x)$ is density function of first order statistics in a sample of size 1 and $g_{2:2}(x)$ is density function of second order statistics in a sample of size 2. We can see that the representation (3) is actually weighted sum of density functions of maximum in sample of specific sizes. This representation will be generalized in this paper.

The transmuted family of distributions has been explored for various baseline distributions by many authors. The transmuted-Weibull distribution has been proposed by [6] by using Weibull distribution as baseline distribution in (1). The transmuted-Kumaraswamy distribution has been proposed by [8] by using CDF of Kumaraswamy distribution as baseline distribution in (1). Some other notable refrences on transmuted-G family of distributions are [9], [10], [11] among others.

The transmuted-G family of distributions has been extended to cubic transmutation by [7], [12] and [13]. The CDF's of cubic transmuted families of distributions, proposed by [12] and [13] are

$$F_{CT1-G}(x) = G(x) + \delta_1 G(x)\,[1-G(x)] + \delta_2 G^2(x)\,[1-G(x)] \tag{4}$$

and

$$F_{CT2-G}(x) = G(x) + \delta_1 G(x)\,[1-G(x)] + \delta_2 G(x)\,[1-G(x)]^2, \tag{5}$$

where δ_1 and δ_2 are transmutation parameters such that $-1 \leq (\delta_1, \delta_2) \leq 2$ and $-2 \leq \delta_1 + \delta_2 \leq 1$ for (4). Also for (5) the restrictions on δ_1 and δ_2 are $-2 \leq (\delta_1, \delta_2) \leq 1$ and $-1 \leq \delta_1 + \delta_2 \leq 2$. The density functions corresponding to (4) and (5) are

$$f_{CT1-G}(x) = g(x)\left[1 + \delta_1 + 2(\delta_2 - \delta_1)\, G(x) - 3\delta_2 G^2(x)\right], \tag{6}$$

and

$$f_{CT2-G}(x) = g(x)\left[(1 + \delta_1 + \delta_2) - 2(\delta_1 + 2\delta_2)\, G(x) + 3\delta_2 G^2(x)\right]. \tag{7}$$

We can see that families (4) and (5) reduces to the transmuted-G family of distributions, proposed by [16], for $\delta_2 = 0$.

The cubic transmuted families of distributions has been extended by [12] and [13] which we will discuss in the following section.

2. General Transmuted-G Families

The general transmuted families of distributions have been proposed by [12] and [13] as an extension of (4) and (5). The CDF of general transmuted family-I of distributions (*GT1-G*), which extends (4), is

$$F_{GT1-G}(x) = G(x) + \sum_{m=1}^{k} \delta_m G^m(x)\,[1-G(x)]\ ,\ x \in \mathbb{R}, \tag{8}$$

such that $-1 \leq \delta_m \leq k$ and $-k \leq \sum_{m=1}^{k} \delta_m \leq 1$. The density function corresponding to (8) is

$$f_{GT1-G}(x) = g(x)\left[1 - \sum_{m=1}^{k} (m+1)\delta_m G^m(x) + \sum_{m=1}^{k} m\delta_m G^{m-1}(x)\right]. \tag{9}$$

Also, the CDF of general transmuted family-II of distributions ($GT2$-G), which extends (5), is

$$F_{GT2-G}(x) = G(x) + \sum_{m=1}^{k} \delta_m G(x)[1 - G(x)]^m \ , \ x \in \mathbb{R}, \tag{10}$$

such that $-k \leq \delta_m \leq 1$ and $-1 \leq \sum_{m=1}^{k} \delta_m \leq k$. The density function corresponding to (10) is

$$\begin{aligned} f_{GT2-G}(x) &= g(x)\left[1 + \sum_{m=1}^{k} \delta_m \{1 - G(x)\}^m \right. \\ &\left. - \sum_{m=1}^{k} m\delta_m G(x)\{1 - G(x)\}^{m-1}\right]. \end{aligned} \tag{11}$$

The general transmuted families of distributions (8) and (10) reduces to transmuted family of distributions (1) for $k = 1$. It is also to be noted that the family of distributions (4) and (5) appear as a special case of families (8) and (10), respectively, for $k = 2$.

In this paper we have given some general results for general transmuted families of distributions including representation as $T - X$ family of distributions, proposed by [2]. In Section 3.2 the general transmuted families of distributions are represented as distribution of order statistics and the moments of the families are obtained as moments of order statistics from parent distribution in Section 3.3. Estimation framework is discussed in Section 3.4. Section 4 contains some examples of general transmuted distributions. Conclusions and recommendations are given in Section 5.

3. General Results for General Transmuted Families

In this section we have given some general results for general transmuted families of distributions given in (8) and (10). These results are given in the following sub-sections.

3.1. Representation as Member of $T - X$ Family

The $T-X$ family of distributions have been proposed by [2] as a method of generalizing the probability distributions. The CDF of $T - X$ family of distributions is

$$F_{T-X}(x) = \int_a^{W[G(x)]} r(t)\,dt, \tag{12}$$

where $W[G(x)]$ is some function of $G(x)$ which satisfies some regularity conditions as given in [2] and $r(t)$ is density function of some random variable T defined on $[a,b]$, where a can be $-\infty$ and b can be $+\infty$. Since the emergence of $T-X$ family of distributions, (12), several authors have explored this family for various combinations of T and X. It has been noted by [1] that the transmuted family of distributions (1) can be represented as a member of $T-X$ family of distributions by using

$$r(t) = 1 + \delta - 2\delta t \ , \ 0 < t < 1$$

and $W[G(x)] = G(x)$ in (12), that is the transmuted family of distribution is obtained as

$$F_{QT-G}(x) = \int_0^{G(x)} (1 + \delta - 2\delta t\ )\,dt.$$

This fact can be extended to the case of generalized transmuted families of distributions and the results are given in the following theorems.

Theorem 1. *The general transmuted families of distributions (8) and (10) are members of the $T-X$ familiy of distributions.*

Proof. Using

$$r_1(t) = 1 + \sum_{m=1}^{k} \delta_m t^{m-1} [m - (m+1)t] \ , \ 0 < t < 1 \tag{13}$$

and

$$r_2(t) = 1 + \sum_{m=1}^{k} \delta_m (1-t)^{m-1} [1 - (m+1)t] \ , \ 0 < t < 1, \tag{14}$$

with $W[G(x)] = G(x)$ in (12) we obtain the general transmuted families (8) and (10) respectively and hence the proof is complete.

It is to be noted that the, for $k = 1$, density function $r_1(t)$, given in (13) reduces to the density function $r(t)$ which is used in representing transmuted family of distributions as member of $T-X$ family of distributions. Also for $k = 2$ the density function $r_1(t)$ reduces to the density function $r(t)$ which is used by [12] to represent cubic transmuted family 1, (4). The same is true for density $r_2(t)$.

3.2. Representation as Distribution of Order Statistics

We have seen in (3) that the transmuted family of distributions can be written as weighted sum of maximum for various sample sizes from the baseline distribution $G(x)$, that is the transmuted family of distribution is a weighted sum of distributions of order statistics from baseline distribution $G(x)$. This result can be extended to the density function of general transmuted families of distributions, given in (9) and (11). In the following we will represent the general transmuted families of distributions as distribution or order statistics from the baseline distribution $G(x)$.

3.3. Representation as Weighted Sum of Maximums

We have seen, in (3), that the transmuted family of distribution is written as weighted sum of distributions of maximums in samples of sizes 1 and 2 from the baseline distribution $G(x)$. This result can be extended to the case of general transmuted families of distributions and are given in the following theorems.

Theorem 2. *The density function of general transmuted family-I (9) can be written as weighted sum of maximums as*

$$\begin{aligned} f_{GT1-G}(x) &= (1+\delta_1)\,g_{1:1}(x) + \sum_{m=2}^{k}(\delta_m-\delta_{m-1})\,g_{m:m}(x) \\ &\quad -\delta_k g_{k+1:k+1}(x), \end{aligned} \tag{15}$$

where $g_{n:n}(x)$ is distribution of the maximum in a sample of size n from $G(x)$.

Proof. The density function of general transmuted family of distributions is given in (9). It can be easily seen that the density function can be written as

$$\begin{aligned} f_{GT1-G}(x) &= g(x)\Bigg[(1+\delta_1) + \sum_{m=2}^{k} m\,(\delta_m-\delta_{m-1})\,G^{m-1}(x) \\ &\quad -(k+1)\,\delta_k G^k(x)\Bigg] \end{aligned}$$

or

$$\begin{aligned} f_{GT1-G}(x) &= (1+\delta_1)\,g(x) + \sum_{m=2}^{k}(\delta_m-\delta_{m-1})\,mg(x)\,G^{m-1}(x) \\ &\quad -(k+1)\,\delta_k g(x)\,G^k(x). \end{aligned}$$

Now using the distribution of order statistics, as given in [5] and [15], the above density can be written as given in (15) and the proof is complete.

The representation of general transmuted-II family of distributions (GT2-G) as weighted sum of distribution of the maximum is given in the following theorem.

Theorem 3. *The density function of general transmuted family-II (11) can be written as weighted sum of maximums as*

$$f_{GT2-G}(x) = \sum_{m=0}^{k}\sum_{j=0}^{m}(-1)^j\,\delta_m\binom{m}{j}g_{j+1:j+1}(x), \tag{16}$$

where $g_{n:n}(x)$ is distribution of the maximum in a sample of size n from $G(x)$.

Proof. The CDF of general transmuted family-II of distributions is given in (10) as

$$F_{GT2-G}(x) = G(x) + \sum_{m=1}^{k} \delta_m G(x) [1 - G(x)]^m \ , \ x \in \mathbb{R}.$$

This can be written as

$$F_{GT2-G}(x) = G(x) \sum_{m=0}^{k} \delta_m \{1 - G(x)\}^m ,$$

where $\delta_0 = 1$. Expanding binomially, above CDF can be written as

$$F_{GT2-G}(x) = \sum_{m=0}^{k} \sum_{j=0}^{m} (-1)^j \delta_m \binom{m}{j} G^{j+1}(x).$$

The density function corresponding to above CDF is

$$f_{GT2-G}(x) = \sum_{m=0}^{k} \sum_{j=0}^{m} (-1)^j \delta_m \binom{m}{j} (j+1) g(x) G^j(x).$$

Now using the distribution of order statistics, as given by [5] and [15], the above density can be written as given in (16) and the proof is complete.

3.4. Representation as Weighted Sum of Minimums

It is sometime easy to study the distribution of minimum in a random sample from some baseline distribution, for example the distribution of minimum in a random sample from exponential distribution is also exponential. It is, therefore, useful to represent the density functions of general transmuted families of distributions as weighted sum of minimums in a random sample from baseline distribution $G(x)$. In the following theorems we have given representations of the density functions of general transmuted families of distributions as weighted sum of distributions of minimums.

Theorem 4. *The density function of general transmuted family-I (9) can be written as weighted sum of minimums as*

$$\begin{aligned} f_{GT1-G}(x) &= g_{1:1}(x) + \sum_{m=1}^{k} \delta_m \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \\ &\times \left[\frac{m+1}{j+2} g_{1:j+2}(x) - \frac{1}{j+1} g_{1:j+1}(x) \right]. \end{aligned} \tag{17}$$

where $g_{1:n}(x)$ is distribution of the minimum in a sample of size n from $G(x)$.

Proof. The density function of general transmuted family of distributions is given in (9). It can be easily seen that the density function can be written as

$$f_{GT1-G}(x) = g(x)\left[1 + \sum_{m=1}^{k} \delta_m \left[1 - \{1 - G(x)\}\right]^{m-1} \{m[1 - G(x)] - G(x)\}\right].$$

Now expansing $\left[1 - \{1 - G(x)\}\right]^{m-1}$ binomially and after some re-arrangements we have

$$\begin{aligned} f_{GT1-G}(x) &= g(x) + \sum_{m=1}^{k} \delta_m \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \\ &\quad \times \left[\frac{m}{j+2} g_{1:j+2}(x) - \frac{1}{(j+1)(j+2)} g_{2:j+2}(x)\right], \end{aligned} \tag{18}$$

where $g_{1:n}(x)$ is density function of minimum in a sample of size n from $G(x)$ and $g_{2:n}(x)$ is distribution of second order statistics in a sample of size n from $G(x)$. Now using following relation between density functions of order statistics, given in [5] and [15],

$$r g_{r+1:n}(x) = n g_{r:n-1}(x) - (n-r) g_{r:n}(x),$$

with $r = 1$ and $n = j + 2$ we have

$$g_{2:j+2}(x) = (j+2) g_{1:j+1}(x) - (j+1) g_{1:j+2}(x).$$

Using this in (18) and after some re-arrangements we have (17) and hence the proof is complete.

The representation of general transmuted-II family of distributions (GT2-G) as weighted sum of distribution of minimums is given in the following theorem.

Theorem 5. *The density function of general transmuted family-II (11) can be written as weighted sum of minimums as*

$$f_{GT2-G}(x) = g_{1:1}(x) + \sum_{m=1}^{k} \delta_m \left[g_{1:m+1}(x) - g_{1:m}(x)\right] \tag{19}$$

where $g_{1:n}(x)$ is distribution of the minimum in a sample of size n from $G(x)$.

Proof. The density function of general transmuted family-II of distributions, given in (11), can be written as

$$f_{GT2-G}(x) = g(x)\left[1 - \sum_{m=1}^{k} \delta_m \{1 - G(x)\}^{m-1} \{(m+1) G(x) - 1\}\right]$$

Now using the fact that

$$g_{1:m}(x) = m g(x) \{1 - G(x)\}^{m-1}$$

and

$$g_{2:m+1}(x) = m(m+1)\,g(x)\,G(x)\left\{1-G(x)\right\}^{m-1},$$

the above density function can be written as

$$f_{GT2-G}(x) = g_{1:1}(x) + \sum_{m=1}^{k}\frac{\delta_m}{m}\left[g_{1:m}(x) - g_{2:m+1}(x)\right]. \tag{20}$$

Again using the relation

$$rg_{r+1:n}(x) = ng_{r:n-1}(x) - (n-r)\,g_{r:n}(x),$$

with $r = 1$ and $n = m+1$ we have

$$g_{2:m+1}(x) = (m+1)\,g_{1:m}(x) - mg_{1:m+1}(x).$$

Using this in (20) and after some re-arrangements we have (19) and the proof is complete.

The results given in Theorems (3.2)-(3.5) are very useful in studying the properties of any members of general transmuted families of distributions. The choice between (15) or (17) and between (16) or (19) depends on the form of $G(x)$. The relations of density functions given above are also useful in obtaining moments for members of general transmuted families of distributions which are given in the following sections.

4. Moments of General Transmuted Families of Distributions

The moments are useful in studying the properties of any distribution. The moments of any member of general transmuted family of distributions can be obtained from the momenst of order statistics from the baseline distribuion $G(x)$ and are given in the following theorems.

Theorem 6. *The moments of any member of general transmuted families of distributions can be written as weighted sum of moments of maximums in a sample of specific size from baseline distribution $G(x)$ as*

$$\mu^{/q}_{GT1-G} = (1+\delta_1)\,\mu^{/q}_{G(1:1)} + \sum_{m=2}^{k}(\delta_m - \delta_{m-1})\,\mu^{/q}_{G(m:m)} - \delta_k\mu^{/q}_{G(k+1:k+1)} \tag{21}$$

and

$$\mu^{/q}_{GT2-G} = \sum_{m=0}^{k}\sum_{j=0}^{m}(-1)^j\,\delta_m\binom{m}{j}\mu^{/q}_{G(j+1:j+1)}, \tag{22}$$

where $\mu^{/q}_{G(n:n)}$ is qth raw moment of maximum in a sample of size n from baseline distribution $G(x)$.

Proof. The theorem can be readily proved by using the representations of density functions of general transmuted families of distributions as weighted sum of maximums, given in (15) and (16).

Again the moments of general transmuted families of distributions can be obtained from moments of minimum in a sample of specific sizes from baseline distribution $G(x)$ and is given in the following theorem.

Theorem 7. *The moments of any member of general transmuted families of distributions can be written as weighted sum of moments of minimums in a sample of specific size from baseline distribution $G(x)$ as*

$$\begin{aligned} \mu_{GT1-G}^{/q} &= \mu_{G(1:1)}^{/q} + \sum_{m=1}^{k} \delta_m \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \\ &\quad \times \left[\frac{m+1}{j+2} \mu_{G(1:j+2)}^{/q} - \frac{1}{j+1} \mu_{G(1:j+1)}^{/q} \right], \end{aligned} \tag{23}$$

and

$$\mu_{GT2-G}^{/q} = \mu_{G(1:1)}^{/q} + \sum_{m=1}^{k} \delta_m \left\{ \mu_{G(1:m+1)}^{/q} - \mu_{G(1:m)}^{/q} \right\}, \tag{24}$$

where $\mu_{G(1:n)}^{/q}$ is qth raw moment of minimum in a sample of size n from baseline distribution $G(x)$.

Proof. The theorem can be readily proved by using the representations of density functions of general transmuted families of distributions as weighted sum of minimums, given in (17) and (19).

In the following section we will discuss an example of general transmuted distribution, the general transmuted Weibull distribution.

5. General Transmuted Weibull Distribution

The Weibull distribution, proposed by [17], is a useful distribution and has widespread applications in many area of life. The density and distribution functions of Weibull distribution are

$$f(x) = \frac{\beta}{\theta^\beta} x^{\beta-1} \mathrm{e}^{-(x/\theta)^\beta} \ ; \ x, \beta, \theta > 0 \tag{25}$$

and

$$F(x) = 1 - \mathrm{e}^{-(x/\theta)^\beta} \ ; \ x, \beta, \theta > 0. \tag{26}$$

The Weibull distribution has been used by [3] with the quadratic transmutation to propose the transmuted Weibull distribution. The distribution has been used by [14] to propose a cubic transmuted Weibull distribution. In the following we will use the Weibull distribution with the general transmuted families of distributions to propose the general transmuted

Weibull distributions. The general transmuted Weibull distributions will be proposed by using (8) and (10). Now, using the distribution function of Weibull random variable in (8) the CDF of *general transmuted Weibull-I* (GTW-I) distribution is

$$F_{GT1-W}(x) = \left[1 - \mathrm{e}^{-(x/\theta)^\beta}\right] + \sum_{m=1}^{k} \delta_m \mathrm{e}^{-(x/\theta)^\beta} \left[1 - \mathrm{e}^{-(x/\theta)^\beta}\right]^m. \tag{27}$$

Again, using the CDF of Weibull distribution in (10), the CDF of *general transmuted Weibull-II* (GTW-II) distribution is

$$F_{GT2-W}(x) = \left[1 - \mathrm{e}^{-(x/\theta)^\beta}\right] + \sum_{m=1}^{k} \delta_m \mathrm{e}^{-m(x/\theta)^\beta} \left[1 - \mathrm{e}^{-(x/\theta)^\beta}\right]. \tag{28}$$

The density functions corresponding to (27) and (28) are

$$\begin{aligned} f_{GT1-W}(x) &= \frac{\beta}{\theta^\beta} x^{\beta-1} \mathrm{e}^{-(x/\theta)^\beta} \left[1 - \sum_{m=1}^{k} (m+1)\,\delta_m \left\{1 - \mathrm{e}^{-(x/\theta)^\beta}\right\}^m \right. \\ &\quad \left. + \sum_{m=1}^{k} m\delta_m \left\{1 - \mathrm{e}^{-(x/\theta)^\beta}\right\}^{m-1}\right], \end{aligned} \tag{29}$$

for $x, \beta, \theta > 0$ and

$$\begin{aligned} f_{GT2-G}(x) &= \frac{\beta}{\theta^\beta} x^{\beta-1} \mathrm{e}^{-(x/\theta)^\beta} \left[1 + \sum_{m=1}^{k} \delta_m \mathrm{e}^{-m(x/\theta)^\beta} - \sum_{m=1}^{k} m\delta_m \right. \\ &\quad \left. \times \mathrm{e}^{-(m-1)(x/\theta)^\beta} \left\{1 - \mathrm{e}^{-(x/\theta)^\beta}\right\}\right], \end{aligned} \tag{30}$$

for $x, \beta, \theta > 0$. The density functions of GTW-I and GTW-II can be represented as linear combinations of the distributions of order statistics from the Weibull distribution. We have presented the same in the following subsection.

5.1. General Transmuted Weibull Distributions as Distributions of Order Statistics

In the following we will obtain representation of the density functions of GTW-I and GTW-II distributions, given in (29) and (30), as linear combinations of distributions of maximum and minimum for a random sample from Weibull distribution. These representations are obtained by first seeing that the distribution of maximum and minimum for a random sample of size n from Weibull distribution are

$$f_{1:n}(x) = \frac{n\beta}{\theta^\beta} x^{\beta-1} \mathrm{e}^{-n(x/\theta)^\beta} \;;\; x, \beta, \theta > 0, n \geq 1 \tag{31}$$

and

$$f_{n:n}(x) = \frac{n\beta}{\theta^\beta} x^{\beta-1} \mathrm{e}^{-(x/\theta)^\beta} \left\{1 - \mathrm{e}^{-(x/\theta)^\beta}\right\}^{n-1}. \tag{32}$$

Now using (32) in (15) and (16) the density functions of GTW-I and GTW-II can be written as linear combinations of distributions of maximum from the Weibull distribution and are

$$\begin{aligned} f_{GT1-W}(x) &= \frac{(1+\delta_1)\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta} + \sum_{m=2}^{k}(\delta_m-\delta_{m-1})\frac{m\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta} \\ &\quad \times\left\{1-\mathrm{e}^{-(x/\theta)^\beta}\right\}^{m-1} - \frac{\delta_k(k+1)\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta}\left\{1-\mathrm{e}^{-(x/\theta)^\beta}\right\}^{k} \end{aligned} \tag{33}$$

and

$$\begin{aligned} f_{GT2-W}(x) &= \sum_{m=0}^{k}\sum_{j=0}^{m}(-1)^j\delta_m\binom{m}{j}\frac{(j+1)\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta} \\ &\quad \times\left\{1-\mathrm{e}^{-(x/\theta)^\beta}\right\}^{j}, \end{aligned} \tag{34}$$

for $x,\beta,\theta>0$. Again, using (31) in (17) and (19) the density functions of GTW-I and GTW-II can be written as linear combinations of distributions of minimum from the Weibull distribution and are

$$\begin{aligned} f_{GT1-W}(x) &= \frac{\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta} + \sum_{m=1}^{k}\delta_m\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\left[\frac{m+1}{j+2}\right. \\ &\quad \left.\times\frac{(j+2)\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(j+2)(x/\theta)^\beta} - \frac{(j+2)\beta}{(j+1)\theta^\beta}x^{\beta-1}\mathrm{e}^{-(j+1)(x/\theta)^\beta}\right], \end{aligned} \tag{35}$$

and

$$\begin{aligned} f_{GT2-W}(x) &= \frac{\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(x/\theta)^\beta} + \sum_{m=1}^{k}\delta_m\left[\frac{(m+1)\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-(m+1)(x/\theta)^\beta}\right. \\ &\quad \left.-\frac{m\beta}{\theta^\beta}x^{\beta-1}\mathrm{e}^{-m(x/\theta)^\beta}\right], \end{aligned} \tag{36}$$

for $x,\beta,\theta>0$. The representation of transmuted Weibull and cubic transmuted Weibull distributions as linear combinations of maximum can be obtained from (33) and (34) by using $k=1$ and $k=2$. Similarly, representation of transmuted Weibull and cubic transmuted Weibull distributions as linear combinations of minimum can be obtained from (35) and (36) by using $k=1$ and $k=2$.

The representations (35) and (36) are useful to obtain the moments of GTW-I and GTW-II distributions by using moments of minimum from the Weibull distribution which are obtained in the following.

5.2. Moments of General Transmuted Weibull Distributions

The moments of GTW-I distribution can be obtained by using either (21) or (23) and the moments of GTW-II distribution can be obtained by using either (22) or (24). We will obtain the moments of GTW-I and GTW-II distributions by using (23) and (24) as these depends upon the moments of minimum from Weibull distribution which are in compact form. Now, the qth moment of minimum in a random sample of size n from the Weibull distribution, given in (31), is

$$\mu_{1:n}^{q} = E\left(X_{1:n}^{q}\right) = \frac{\theta^{q}}{n^{q/\beta}} \Gamma\left(\frac{\beta}{q}+1\right). \tag{37}$$

Now, using (37) in (23) the qth moment of GTW-I distribution is

$$\begin{aligned} \mu_{GT1-W}^{p} &= \theta^{q} \Gamma\left(\frac{\beta}{q}+1\right) + \sum_{m=1}^{k} \delta_{m} \sum_{j=0}^{m-1} (-1)^{j} \binom{m-1}{j} \left[\frac{m+1}{j+2} \frac{\theta^{q}}{(j+2)^{q/\beta}} \right. \\ & \left. \times \Gamma\left(\frac{\beta}{q}+1\right) - \frac{1}{j+1} \frac{\theta^{q}}{(j+1)^{q/\beta}} \Gamma\left(\frac{\beta}{q}+1\right)\right]. \end{aligned} \tag{38}$$

Similarly, using (37) in (24) the qth moment of GTW-II distribution is

$$\mu_{GT2-W}^{p} = \theta^{q} \Gamma\left(\frac{\beta}{q}+1\right) + \sum_{m=1}^{k} \delta_{m} \left\{\frac{\theta^{q}}{(m+1)^{q/\beta}} \Gamma\left(\frac{\beta}{q}+1\right) - \frac{\theta^{q}}{m^{q/\beta}} \Gamma\left(\frac{\beta}{q}+1\right)\right\}. \tag{39}$$

The means and variances of GTW-I and GTW-II distributions can be obtained from (38) and (39) respectively.

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