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A Finite Difference Fictitious Domain Wavelet Method for Solving Dirichlet Boundary Value Problem

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Abstract. In this paper, we introduce a Finite Difference Fictitious Domain Wavelet Method (FDFDWM) for solving two dimensional (2D) linear elliptic partial differential equations (PDEs) with Dirichlet boundary conditions on regular geometric domain. The method reduces the 2D PDE into a 1D system of ordinary differential equations and applies a compactly supported wavelet to approximate the solution. The problem is embedded in a fictitious domain to aid the enforcement of the Dirichlet boundary conditions. We present numerical analysis and show that our method yields better approximation to the solution of the Dirichlet problem than traditional methods like the finite element and finite difference methods.

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Key Words and Phrases: fictitious domain, Dirichlet problem, wavelet, finite difference, finite element

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1. Introduction

The Dirichlet Problem (DP) for linear elliptic PDE in 2D is found in many physical problems that are governed by partial differential equations. These physical problems include vibration of solids, flow of fluids, diffusion of chemicals, spread of heat, structure of molecules, propagation of waves, laser beam models, financial models, etc [14]. Due to the significance of the DP in the field of engineering and the sciences, researchers have done a lot of work into improving the solution methods, both analytically and numerically

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[1, 7, 13]. However, challenges including difficulty in evaluating integrals analytically, approximation based on truncation of infinite series required, inappropriate solution space and many more, militate against analytical approaches. In particular, DP for linear elliptic PDE in 2D, in functional space may not have analytic solution and so the solution needs to be approximated numerically [15].

With the availability of powerful computers, the emphasis on solving Dirichlet Problem in 2D is gradually shifting away from the analytical method of solutions towards numerical computations and analysis [14]. Numerical methods do not involve the search for an explicit or implicit function that describes the Dirichlet Problem. They largely utilize discretization techniques to reduce the continuous linear elliptic PDE and Dirichlet boundary conditions to discrete system that are suitable for high speed computer solution. These methods are known to generate approximate solutions. The Finite Difference Method (FDM) and Finite Element Method (FEM) are some of the traditional methods used to solve this problem. Numerical methods that will produce high accurate, fast convergent and stable solution to the Dirichlet Problem in 2D than the traditional methods is paramount. Besides, the method must also reduce the complexity in obtaining the solution to this problem. In recent times, wavelet methods are known to be effective and efficient methods for solving Dirichlet Problem for linear elliptic PDEs in 2D [10, 12]. In wavelet methods, we are able to obtain information in both frequency and time domains, including boundary information of the PDE equation. Undoubtedly, the vanishing moment property of wavelet enables the wavelet series solution to converge rapidly to a point in the domain as compared to the afore mentioned traditional numerical methods for solving the Dirichlet Problem for linear elliptic PDE in 2D.

2. The FDFDWM

In this paper, we introduce the FDFDWM as a numerical approach to solving the Dirichlet Problem for linear elliptic PDE in 2D on a regular geometric domain. This method aims at providing approximate solution to the DP, that is better in terms of accuracy and rate of convergence than that of the traditional methods mentioned. It employs Daubechies scaling functions with a fictitious domain approach. Daubechies wavelet function of order N has the largest number of vanishing moments which are compactly supported on $[0, 2N-1]$. Moreover, the high number of vanishing moments lead to high compressibility of orthonormal solution in $\Omega \subset H$. The use of the Daubechies scaling function offers the FDFDWM the flexibility to obtain a more accurate and stable solution to the problem at hand in a functional space [2]. The fictitious domain approach of the FDFDWM also makes it easier to deal with the Dirichlet boundary condition. Notably, the FDFDWM reduces the linear elliptic PDE in two dimensional coordinates into a system of one dimensional ordinary differential equations. This approach reduces considerably the complexities involved in the solution process, hence reduces the computational cost.

Now, we give the outline of the FDFDWM for solving the Dirichlet problem on regular domains. We consider the case of a rectangular domain. Thus;

- The Dirichlet Problem is defined on an open domain with a rectangular boundary.

- The two dimensional problem is reduced to a one dimensional problem by discretizing along one of the variables ($y-$coordinate or $x-$coordinate) using difference quotient and leaving the other variable undiscretized.
- The original domain of the problem is embedded in a slightly larger but simple domain, in this case a rectangular domain. This extended domain is termed as a fictitious domain.
- The One dimensional problem is formulated as a variational problem in the fictitious domain.
- A compactly supported wavelet, in this case Daubechies wavelet is used for the numerical approximation in the larger domain.
- The resultant linear system in the large domain is then solved.

We consider a general linear second order elliptic equation of the form

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial\phi}{\partial x_i}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial\phi}{\partial x_i}+c(x)\phi=f(x),\quad x\in\Omega \tag{1}$$

where the coefficient a_{ij}, b_i, c and f satisfy the following conditions:

$$\begin{aligned} a_{ij} &\in C^1(\bar{\Omega}), \quad i,j=1,\dots,n;\\ b_i &\in C(\bar{\Omega}), \quad i=1,\dots,n;\\ c &\in C(\bar{\Omega}), \quad f(x)\in C(\bar{\Omega}), \end{aligned}$$

and

$$\sum_{i,j=1}^{n}a_{ij}(x)\eta_i\eta_j\geq\tau\sum_{i=1}^{n}\eta_i^2,\quad\forall\eta=(\eta_1,\dots,\eta_n)\in\mathbb{R}^n,\quad x\in\bar{\Omega}; \tag{2}$$

with τ being a positive constant independent of x and η. The fourth condition (2) is referred to as uniform ellipticity. In this paper, we restrict ourselves to the case where the coefficient $a_{ij}(x)$ in equation (1) reduces to a scalar multiple, aI of unit matrix, were a is a smooth function.

We consider, the Dirichlet Problem in a regular (i.e. rectangular) domain, $\Omega=[p,q]\times[r,s]\in\mathbb{R}^2$ with boundary $\partial\Omega$, given as

$$\begin{cases} -\nabla\cdot(a\nabla\phi)+b\nabla\phi+c\phi=f & \text{in } \Omega\\ \phi(x,y)=g & \text{on } \partial\Omega \end{cases} \tag{3}$$

where the coefficients $a=a(x,y), b=b(x,y)$ and $c=(x,y)$ are smooth in $\partial\bar{\Omega}$ which satisfy

$$a(x,y)\geq a_0>0,\quad c(x,y)-\frac{1}{2}\nabla\cdot b(x,y)\geq 0,\quad\text{for all } x,y\in\Omega \tag{4}$$

and where f is a given function and g is the boundary data.

The first stage of the FDFDWM is to reduce the two dimensional DP in equation (3) to a system of ordinary differential equations. To achieve this, equation (3) is discretized along one of the spatial variables (say y), with equally spaced sample, $y^i = i\Delta y$. Thus

$$p = y_0 < y_1 < \cdots < y_{N_y} = q$$

and

$$\Delta y = \frac{q-p}{N_y} = \frac{q-p}{2^m}$$

where $N_y = 2^m$ and m is the resolution.

We use central difference approximation:

$$\frac{d^2\phi}{dy^2} \approx \frac{\phi(x, y+\Delta y) - 2\phi(x,y) + \phi(x, y-\Delta y)}{\Delta y^2} \tag{5}$$

and

$$\frac{d\phi}{dy} \approx \frac{\phi(x, y+\Delta y) - \phi(x, y-\Delta y)}{2\Delta y} \tag{6}$$

with an error term $O(y^2)$ for each approximation.

Substituting equations (5) and (6) into (3), we have

$$-a\left[\frac{d^2\phi}{dx^2} + \frac{\phi^{i+1}(x) - 2\phi^i(x) + \phi^{i-1}(x)}{\Delta y^2}\right] + b\left[\frac{d\phi}{dx} + \frac{\phi^{i+1}(x) - \phi^{i-1}(x)}{2\Delta y}\right] + c\phi^i(x) = f(x,y) \tag{7}$$

Expanding and simplifying (7), we obtain

$$-\beta_1 \frac{d^2\phi^i}{dx^2} + \beta_2 \frac{d\phi^i}{dx} + \beta_3\phi^{i+1} + \beta_4\phi^i + \beta_5\phi^{i-1} = \beta_6 f^i \tag{8}$$

where

$$\beta_1 = a\Delta y^2,\ \ \beta_2 = b\Delta y^2,\ \ \beta_3 = b\frac{\Delta y}{2} - a,\ \ \beta_4 = 2a + c\Delta y^2,$$

$$\beta_5 = -(a + b\frac{\Delta y}{2}) \ \text{ and } \ \beta_6 = \Delta y^2$$

Now, we let

$$\phi^i_\Sigma(x) = \beta_3\phi^{i+1} + \beta_4\phi^i + \beta_5\phi^{i-1}. \tag{9}$$

Then equation (8) is

$$-\beta_1 \frac{d^2\phi^i}{dx^2} + \beta_2 \frac{d\phi^i}{dx} + \phi^i_\Sigma(x) = \beta_6 f^i, \tag{10}$$

which is represented in a vector form as

$$-\nabla\cdot(\beta_1\nabla\phi^i) + \beta_2\nabla\phi^i + \phi^i_\Sigma = \beta_6 f^i \quad \text{for } \ i = 1, 2, 3, \ldots. \tag{11}$$

Equation (11) is a one dimensional system of ordinary differential equations obtained as a result of the reduction of (1).

2.1. Variational Formulation of the DP

The next stage of the FDFWDM is to write equation (11) in a weak form and seek a solution in a Sobolev space H^1. We achieve this by multiplying (11) by a test function, $v \in H_0^1$ and integrating over the domain Ω, that is

$$\int_\Omega \left(-\nabla \cdot (\beta_1 \nabla \phi^i) + \beta_2 \nabla \phi^i + \phi^i_\Sigma\right) v dx = \int_\Omega \beta_6 f^i v \, dx \tag{12}$$

Using the first Green's identity and noting that $v = 0$ on the $\partial\Omega$, we arrive at the following weak form,

$$\begin{cases} \text{find } \phi^i \in H^1(\Omega_F) \text{ such that} \\ \int_\Omega \left(\beta_1 \nabla \phi^i \nabla v + \beta_2 \nabla \phi^i v + \phi^i_\Sigma v\right) dx = \int_\Omega \beta_6 f^i v \, dx \quad \forall v \in H_0^1. \end{cases} \tag{13}$$

We define $\alpha : V \times V \to \mathbb{R}$ and a linear functional $L : V \to \mathbb{R}$, where $\alpha(\cdot, \cdot)$ over space V is continuous. Then we express equation (13) in a bilinear form as

$$\alpha(v, \eta) = \int_\Omega (\beta_1 \nabla v \nabla \eta + \beta_2 \nabla v \eta + v\eta) \, dx \quad \forall v, \eta \in V \tag{14}$$

and

$$L(v) = \beta_6 \int_\Omega f v dx \tag{15}$$

We note from (1) that when $b = 0$, the bilinear becomes symmetrical, that is $\alpha(v, \eta) = \alpha(\eta, v)$.

2.2. Fictitious Domain Formulation of the DP

The FDFDWM uses the fictitious domain approach to handle the boundary conditions. The idea behind the fictitious domain approach is to embed the original domain, Ω of the DP (1) in a slightly larger but simple (rectangular) domain, Ω_F . This is done in order to handle the difficulties usually associated with taking care of boundary conditions.

Now, we let V be a closed subspace $H_p^1(\Omega_F)$ given as

$$\{v : v = \tilde{v}|_\Omega, \tilde{v} \in V\} = H_p^1(\Omega_F)$$

The choice for V, considering the Dirichlet boundary condition is $H_0^1(\Omega)$. We also define $V_p(\Omega_F)$ by

$$V_p(\Omega_F) = \{v \in H_0^1(\Omega_F) : v \text{ on } \partial\Omega \text{ is periodic on.} \tag{16}$$

Given some $s > 0$, suppose that $\Omega_F = (0, s)^2$ then the periodicity property in (16) implies that $v(0, y) = v(s, y)$ and $v(x, 0) = v(x, s)$. Following from equations (14) and (15), the DP (3) can be formulated as a variational problem. That is,

$$\begin{cases} \text{find } \phi^i \in H^1(\Omega_F), \ \forall v \in H_0^1 \text{ so that} \\ \alpha(\phi^i, v) = L(v). \end{cases} \tag{17}$$

2.3. Wavelet Approximation of the DP

One cardinal aspect of the FDFDWM is the approximation of the DP using wavelet series solution. Generally, wavelet approximations give a more stable and accurate solutions and also provide information on the boundary data. We use the Daubechies wavelet for approximating the solution of the DP due to the fact that it has the highest number of vanishing moments among the compactly supported wavelets [4, 5]. This makes the wavelet series approximation converges rapidly to the desired solution.

We begin formulating the FDFDWM solution by letting, V_j be a finite dimensional subspace of V. The problem now becomes;

$$\begin{cases} \text{find } \phi_j^i \text{ in } V_j \text{ such that} \\ \alpha(\phi_j^i, v_j) = L(v_j) \quad \forall v_j \in V_j. \end{cases} \tag{18}$$

We express equation (18) in an expanded form as

$$\beta_1 \int_{\Omega_F} \nabla\phi_j^i \nabla v_j dx + \beta_2 \int_{\Omega_F} \nabla\phi_j^i v_j dx + \int_{\Omega_F} \phi_{\Sigma,j}^i v_j dx = \beta_6 \int_{\Omega_F} f^i v_j dx \tag{19}$$

Substituting equation (9) into (19), we obtain

$$\beta_1 \int_{\Omega_F} \nabla\phi_j^i \nabla v_j dx + \beta_2 \int_{\Omega_F} \nabla\phi_j^i v_j dx + \int_{\Omega_F} \left(\beta_3\phi_j^{i+1} + \beta_4\phi_j^i + \beta_5\phi_j^{i-1}\right) v_j dx = \beta_6 \int_{\Omega_F} \tilde{f}^i v_j dx \tag{20}$$

where $\tilde{f}^i \in L^2$ is an extension of $f(x, y^i)$ in the fictitious domain, Ω_F.

To obtain approximate solution of the DP in equation (18), we employ Daubechies scaling function. We seek a solution that is written as a linear combination of scaling function and coefficient. Thus, we define the FDFDWM solution for resolution m with a scaling parameter k at a fixed $y-$coordinate, y^i, as

$$\phi_w(x, y^i) = 2^{\frac{m}{2}} \sum_k \xi_{k,m}^i \varphi(2^m x - k) \tag{21}$$

where ϕ_w is the FDFDWM solution and $\xi_{k,m}^i$ are the scaling coefficient values to be determined.

Now, we set up a linear system for equation (20) to solve for $\xi_{k,m}^i$. Substituting equation (21) into (20) and simpling gives,

$$\begin{aligned}
&2^{\frac{m}{2}}\beta_1 \sum_k \xi_{k,m}^i \int \varphi'(2^m x - k)\varphi'(2^m x - j)dx \\
&+2^{\frac{m}{2}}\beta_2 \sum_k \xi_{k,m}^i \int \varphi'(2^m x - k)\varphi(2^m x - j)dx \\
&+2^{\frac{m}{2}}\beta_3 \sum_k \xi_{k,m}^{i+1} \int \varphi(2^m x - k)\varphi(2^m x - j)dx
\end{aligned}$$

$$+2^{\frac{m}{2}}\beta_4\sum_k \xi^i_{k,m}\int \varphi^(2^m x-k)\varphi(2^m x-j)dx$$
$$+2^{\frac{m}{2}}\beta_5\sum_k \xi^{i-1}_{k,m}\int \varphi(2^m x-k)\varphi(2^m x-j)dx$$
$$=2^{\frac{m}{2}}\beta_6\tilde{f}^i\int \varphi(2^m x-k)\varphi(2^m x-j)dx \tag{22}$$

We set up a linear system from equation (22) by introducing connection coefficients.

That is,

$$\Omega^2_{k-j}=\int_{-\infty}^{\infty}\varphi'(X-k)\varphi'(X-j)dx, \quad \text{for derivative } d=2,$$
$$\Omega_{k-j}=\int_{-\infty}^{\infty}\varphi'(X-k)\varphi(X-j)dx, \quad \text{for derivative } d=1$$

and the Kronecker-delta function,

$$\delta_{k,j}=\int_{-\infty}^{\infty}\varphi(X-k)\varphi(X-j)dx$$

where we set $X=2^m x$

We handle the exterior nodes of $\xi^{i+1}_{k,m}$ and $\xi^{i-1}_{k,m}$ by applying a shift to the scaling functions in the following manner:

$$\xi^{i+1}_{k,m}\varphi(2^m x-k)\varphi(2^m x-j)=\xi^i_{k,m}\varphi(2^m x-k+1)\varphi(2^m x-j) \tag{23}$$

and

$$\xi^{i-1}_{k,m}\varphi(2^m x-k)\varphi(2^m x-j)=\xi^i_{k,m}\varphi(2^m x-k-1)\varphi(2^m x-j) \tag{24}$$

Substituting equations (23) and (24) into (22), we obtain

$$\sum_k\left[\xi^i_{k,m}\Omega^2_{k-j}+\beta_2\xi^i_{k,m}\Omega_{k-j}+\beta_3\xi^i_{k,m}\delta_{k+1,j}+\beta_4\xi^i_{k,m}\delta_{k,j}+\beta_5\xi^i_{k,m}\delta_{k-1,j}\right]=\beta_6\tilde{f}^i\delta_{k,j} \tag{25}$$

We write equation (25) as a linear system in a vector form, given by

$$A_1\vec{\xi}+A_2\vec{\xi}+A_3\vec{\xi}+A_4\vec{\xi}+A_5\vec{\xi}=F \tag{26}$$

where

$$A_1=\beta_1\sum_k\Omega^2_{k-j},\quad A_2=\beta_2\sum_k\Omega_{k-j},\quad A_3=\beta_3\sum_k\delta_{k+1,j},$$
$$A_4=\beta_4\sum_k\delta_{k,j},\quad A_5=\beta_5\sum_k\delta_{k-1,j} \text{ and } F=\beta_6\tilde{f}^i\delta_{k,j}$$

The indexes k and j are delimited to the whole domain. We recall that, the original domain is discretized with N_x functions. The fictitious domain approach requires that

$N-1$ scaling functions be added to each end of the domain, then the new domain, spanned by the fictitious domain will expand from $-(N-1)$ to $N_x-1+(N-1)$. This will result in a linear system with size, $N_x+2(N-1)$. Thus, $-(N-1)\leq k\leq N_x-1+(N-1)$ and $-(N-1)\leq j\leq N_x-1+(N-1)$, and so equation (26) will be an $(N_x+2(N-1))\times(N_x+2(N-1))$ linear system.

We use Daubechies scaling functions of order six (i.e $DN6$) to construct matrices for the linear system. The matrices A_1 and A_2 are $(2N-3)$-diagonal $(N_x+2(N-1))\times(N_x+2(N-1))$ matrices. We reserve the first and last rows for the enforcement of the Dirichlet boundary condition. We treat thoroughly the implementation of the boundary conditions in section 2.4

We present matrices, A_1 and A_2 as follows:

$$A_1=\beta_1\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ \Omega^2_{-1} & \Omega^2_0 & \Omega^2_1 & \Omega^2_2 & \Omega^2_3 & \Omega^2_4 & 0 & \cdots & & \\ \Omega^2_{-2} & \Omega^2_{-1} & \Omega^2_0 & \Omega^2_1 & \Omega^2_2 & \Omega^2_3 & \Omega^2_4 & 0 & \cdots & \\ \Omega^2_{-3} & \Omega^2_{-2} & \Omega^2_{-1} & \Omega^2_0 & \Omega^2_1 & \Omega^2_2 & \Omega^2_3 & \Omega^2_4 & 0 & \cdots \\ \Omega^2_{-4} & \Omega^2_{-3} & \Omega^2_{-2} & \Omega^2_{-1} & \Omega^2_0 & \Omega^2_1 & \Omega^2_2 & \Omega^2_3 & \Omega^2_4 & 0 \\ 0 & \Omega^2_{-4} & \Omega^2_{-3} & \Omega^2_{-2} & \Omega^2_{-1} & \Omega^2_0 & \Omega^2_1 & \Omega^2_2 & \Omega^2_3 & \Omega^2_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \tag{27}$$

$$A_2=\beta_2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & \\ \Omega_{-1} & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & 0 & \cdots & & \\ \Omega_{-2} & \Omega_{-1} & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & 0 & \cdots & \\ \Omega_{-3} & \Omega_{-2} & \Omega_{-1} & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & 0 & \cdots \\ \Omega_{-4} & \Omega_{-3} & \Omega_{-2} & \Omega_{-1} & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & 0 \\ 0 & \Omega_{-4} & \Omega_{-3} & \Omega_{-2} & \Omega_{-1} & \Omega_0 & \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix} \tag{28}$$

It is important to note that the values of the connection coefficient are precomputed. In this case a parallelization may be applied in the implementation.

The rest of the matrices, A_3, A_4 and A_5 are obtained from the Kronecker-delta function as super-diagonal, diagonal and subdiagonal matrices respectively. That is,

$$A_3=\begin{bmatrix} 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_3 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{29}$$

$$A_4 = \begin{bmatrix} \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \beta_4 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \beta_4 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{30}$$

and

$$A_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \beta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \beta_5 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \beta_5 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \beta_5 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{31}$$

Combining equation (27), (28), (29), (30), and (31), we obtain a linear system

$$A\vec{\xi} = F \tag{32}$$

where $A = \sum_{i=1}^{5} A_i$

In order to construct the weak formulation for the function $f(x, y^i)$ on the right hand side of equation (32), we give the definitions and state the theorems relevant for the weak formulation.

Definition 1. *The orthogonal projection of the function $f(x, y)$ at a fixed $y-$ coordinate, y^i onto a subspace V_m is defined by*

$$P^m \left\{f(x, y^i)\right\} = \sum_k \left(\int f(x, y^i)\varphi_{m,k}(x)dx\right) \varphi_{m,k}(x) \tag{33}$$

Definition 2. *The function $f(x, y)$ sampled at a fixed $y-$ coordinate, y^i is also defined by*

$$S^m \left\{f(x, y^i)\right\} = \sum_k 2^{\frac{-m}{2}} f(k/2^m)\varphi_{m,k}(x) \tag{34}$$

Now, state two theorems relevant to the approximation of the function on the right hand side.

Theorem 1. *If $m_1(\tau) = 0$ for $\tau = 0, 1, \ldots, M$, then L^2 error is:*

$$\varepsilon_1 = \|P^m \left\{f(x, y^i)\right\} - f(x, y^i)\|_2 \leq \lambda_1 2^{-m(M+1)}$$

where λ_1 is a constant dependent on $f(x, y^i)$ and the scaling functions but independent of m and M.

The proofs of theorems 1 and 2 can be found in [3]. From theorem 1, we observe that for a given Daubechies scaling function, $M+1$ represents the number of vanishing moments (i.e. $N = M+1$). Following from definition 2, we obtain theorem 2.

Theorem 2. *If $m_2(\tau) = 0$ for $\tau = 0, 1, \ldots, M$, then L^2 error is:*

$$\varepsilon_2 = \|P^m \left\{f(x, y^i)\right\} - S^m \left\{f(x, y^i)\right\} \|_2 \leq \lambda_2 2^{-m(M+1)}$$

where λ_2 is a constant independent of m and M but dependent on $f(x, y^i)$ and the wavelet system.

We note that the L^2 errors, ε_1 and ε_2 in theorems 1 and 2 respectively are both bounded above by $2^{-m(M+1)}$. This means nodal values of the original function at fixed $y-$ coordinate, $f(x, y^i)$ can be used rather than the sampling values from $S^m \left\{f(x, y^i)\right\}$ which are yet to be determined. Therefore the $f(x, y^i)$ nodal values can be utilized at the right hand side of equation (32).

2.4. Incorporating Dirichlet Boundary Conditions

One of the main challenges with Wavelet - Galerkin is the treatment of boundary conditions. In this paper, we use the fictitious boundary condition approach to enforce the Dirichlet boundary condition [11]. We handle the left boundary by replacing the first equation of (32) by the following equation.

$$\tilde{\phi}(x, y^i) = \sum_k \xi^i_{k,m} \varphi(-k) = g(y^i)$$

We take inner product of the RHS with $\varphi(-j)$ to obtain,

$$\sum_k \xi^i_{k,m} \int_{-\infty}^{\infty} \varphi(-k)\varphi(-j) dx = g(y^i)$$

This results in a Kronecker-delta function on the left,

$$\sum_k \xi^i_{k,m} \delta_{k,j} = g(y^i) \tag{35}$$

Evaluating the Dirichlet boundary expressed in equation (35), we obtain an identity whose Nth column has the value 1, and $g(y^i)$ representing the first element of the right hand side. In similar vein the right boundary of the system is handled.

3. Numerical Results

In this section we present the results obtained from numerical experiments carried out using the FDFDWM on some linear elliptic PDEs with Dirichlet boundary conditions. The results from the FDFDWM are compared with the results from two traditional methods; FDM and FEM, to determine the level of accuracy of our method. The outcome of the experiments are presented in a form of two dimensional graphs and tables. All numerical experiments are performed using MATLAB.

3.1. The FDFDWM Test Cases

Now, we consider numerical experiments performed using FDFDWM approach on PDEs with Dirichlet boundary conditions on rectangular domains. In each of the two cases presented here, the domain of the boundary value problem is placed in a slightly larger rectangular domain referred to as the fictitious domain. The Daubechies scaling functions of orders $D6$, $D8$, $D10$, $D12$, $D14$, $D16$, $D18$ and $D20$ with varying levels of resolution (i.e. $m = 0, m = 1, m = 2$ and $m = 3$) are used and analyzed in the experiments.

Test Case 1

A Dirichlet problem defined on a rectangular domain, $\Omega = [-5, 5] \times [-5, 5]$ embedded in a fictitious domain $\Omega_F = [-8, 8] \times [-8, 8]$ with a Dirichlet boundary condition $g = \sin(x + y)$ is considered for the first numerical test. The problem is given as

$$\begin{cases} -\Delta\phi + \phi = 3\sin(x + y) & \text{in } \Omega \\ \phi(x, y) = g & \text{on } \partial\Omega \end{cases} \tag{36}$$

The Dirichlet problem (36) is first solved using FDFDWM with $D6$ at varying levels of resolution, starting from $m = 0$ to $m = 3$, corresponding to a basis of 8, 16, 32 and 64 translated scaling functions with support intersecting with $[-5, 5]$ on both the horizontal and vertical axes. The values of the approximate solutions are used to generate two dimensional coordinates surface graphs shown in figures 1. This is first to illustrate the ability of the FDFDWM to approximate the solution to the Dirichlet boundary value problem (36). We demonstrate further the level of accuracy of the FDFDWM by comparing the approximate solutions of FDM, FEM and FDFDWM with the exact solution of problem (36).

Test Case 2

In the second test, we consider another Dirichlet problem defined on a rectangular domain, $\Omega = [-3, 3] \times [-3, 3]$ embedded in a fictitious domain $\Omega_F = [-5, 5] \times [-5, 5]$ with a Dirichlet boundary condition $g = x^2 + y^2$. We write the problem as

$$\begin{cases} -\Delta\phi + \phi = x^2 + y^2 - 4 & \text{in } \Omega \\ \phi(x, y) = g & \text{on } \partial\Omega \end{cases} \tag{37}$$

Similar procedure as used for equation (36) is also applied to equation (37) to generate two dimensional coordinates surface graphs shown in figures 3.

Test Case 3

A Poisson equation is considered for the third numerical test. It is defined on a rectangular domain, $\Omega = [0, 1] \times [0, 1]$ embedded in a fictitious domain $\Omega_F = [-0.5, 1.5] \times [-0.5, 1.5]$ with a Dirichlet boundary condition $g = \sin\pi x + \sin\pi y$. The problem is given as

$$\begin{cases} -\Delta\phi = \sin\pi x + \sin\pi y & \text{in } \Omega \\ \phi(x, y) = 0 & \text{on } \partial\Omega \end{cases} \tag{38}$$

The FDFDWM is used to compute the approximate solution for problem (38) in similar manner, using wavelet order $D6$ and resolution at levels, $m = 1, m = 2, m = 3$ and

$m = 4$ respectively. The resulting approximate solution is presented in two dimensional co-ordinates surface graph shown in figure 5.

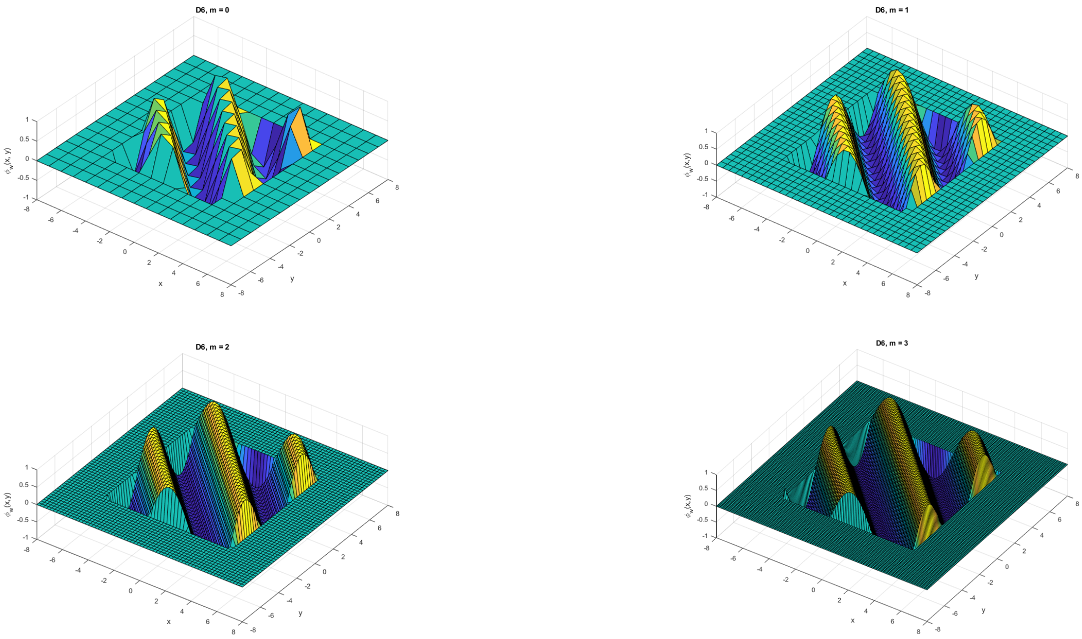

Figure 1: The FWDFDM approximation on a rectangular domain for $D6$ and $m = 0, 1, 2$ and 3

It is evident from figures 1 and 3 that, as the level of the resolution increases the FDFDWM solutions appear to better approximate the exact solution in both test cases. This clearly indicates that the FDFDWM provides reasonable approximation to the PDEs under consideration. Although the results from the FDFDWM approximations look good, we wish to know the level of accuracy of our method in relation to traditional methods like the FDM and FEM. We ascertain this fact by comparing the solution of FDFDWM, FDM and FEM with the exact solution. In order to appreciate the comparison, a two dimensional line graph is plotted at fixed y - coordinate $(y = 0)$ to provide a cross-sectional view. The graphs presented in figures 2, 4 and 6 reveal that, the FDFDWM performs better in terms of accuracy than the FEM followed by the FDM. Observing closely figures 2, 4 and 6, we realized that as the resolution, number of basis functions or discretization points increases the accuracy of all the methods improves. However, by inspecting the graphs at resolution $m = 3$, the FDFDWM approximation and the exact solution are indistinguishable. This can be attributed to the use of the Daubechies scaling functions which offer more accurate and stable approximation as opposed to piece basis functions or quadrature methods often used for the FEM and the differencing operators used in the discretization process of FDM. The outcome of these results are in consonance with the findings of a number of related studies including; [6],[8] and [9].

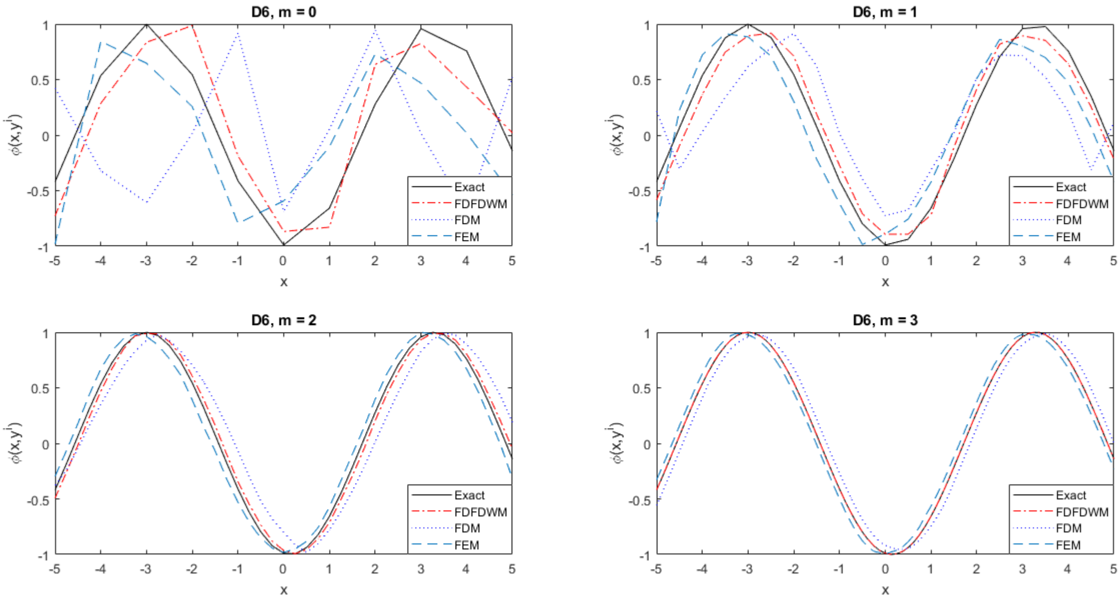

Figure 2: Comparison of FWDFDM, FDM and FEM solutions with exact solution $\phi = \sin(x+y)$ on a rectangular domain for $D6$ and $m = 1, 2$ and 3

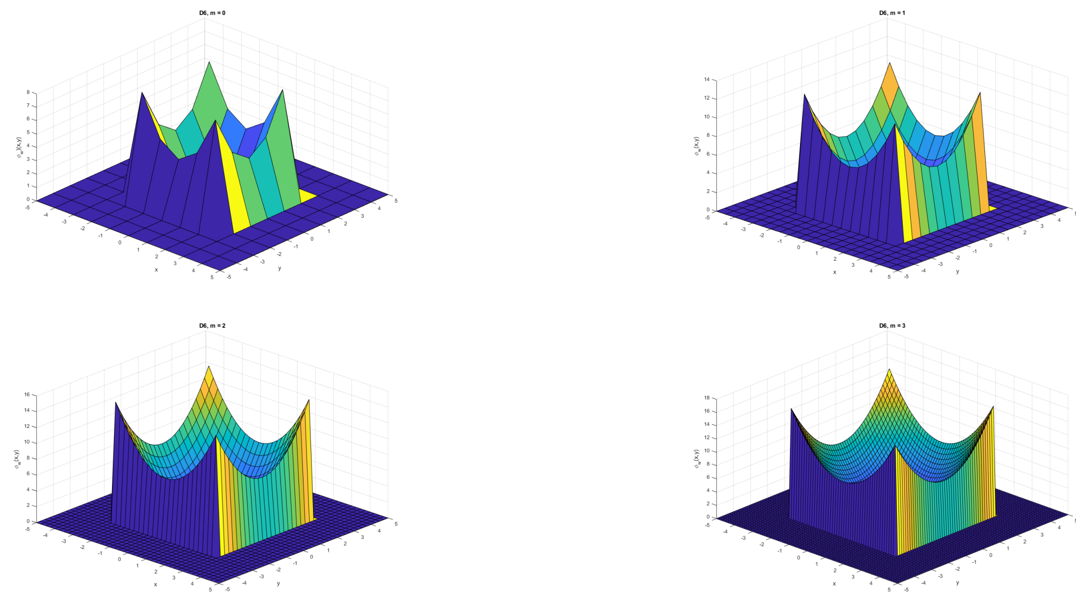

Figure 3: The FWDFDM approximation on a rectangular domain for $D6$ and $m = 0, 1, 2$ and 3

3.2. Error Analysis for FDFDWM

Here, the relative error using L^2 vector norm is computed for each of the test cases with varying resolutions (i.e $m = 0, 1, 2, 3, 4, 5, 6, 7$) and scaling function order $D6$. The

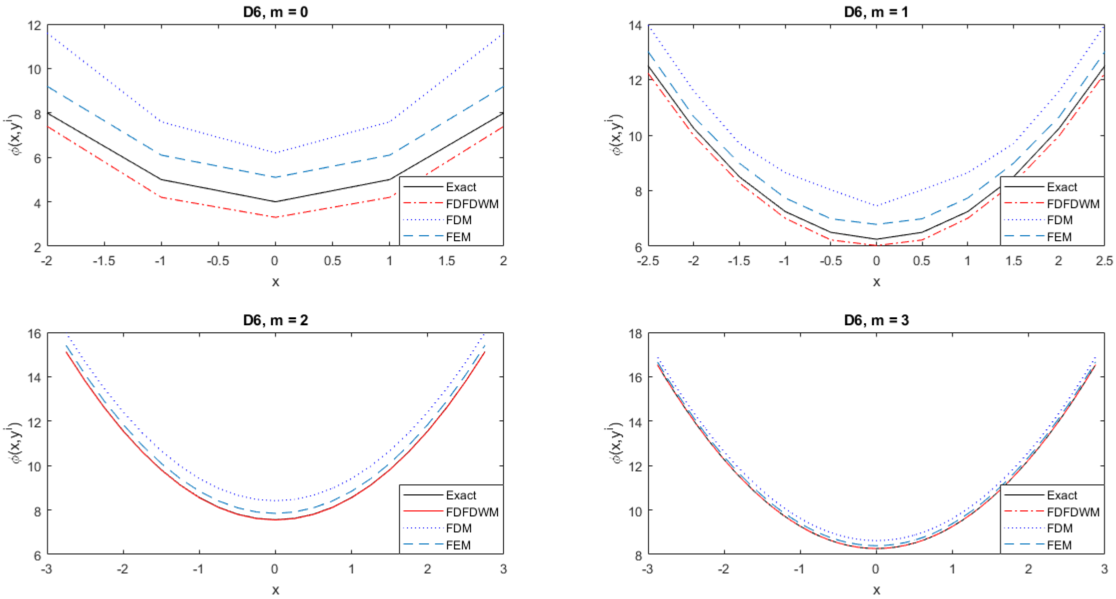

Figure 4: Comparison of FWDFDM, FDM and FEM solutions with exact solution $\phi = x^2 + y^2$ on a rectangular domain for $D6$ and $m = 0, 1, 2$ and 3

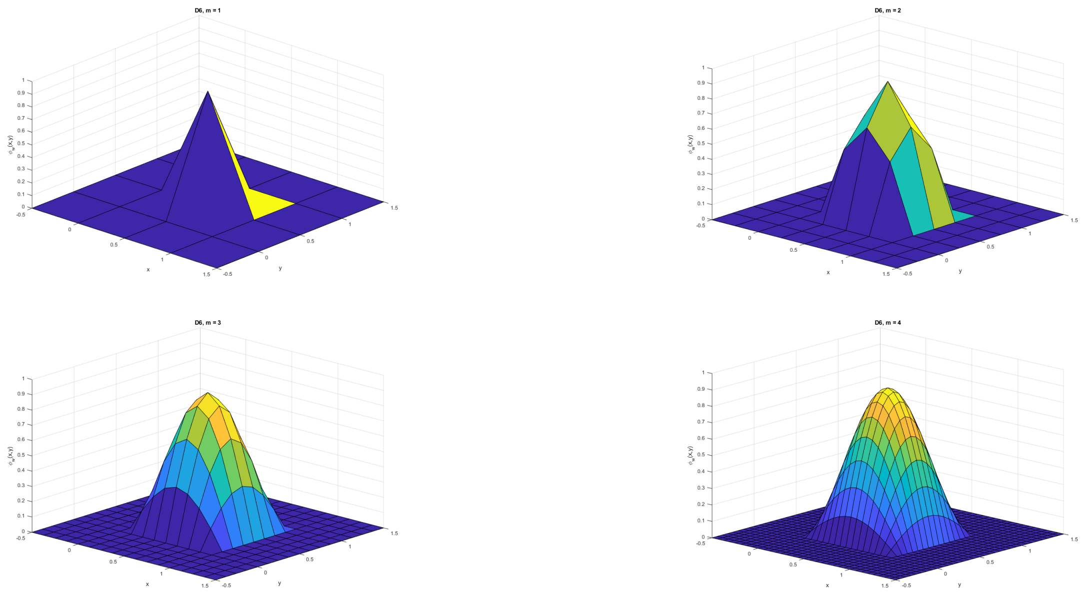

Figure 5: The FWDFDM approximation on a rectangular domain for $D6$ and $m = 1, 2, 3$ and 4

relative error used is provided as

$$E = \frac{\|\phi - \tilde{\phi}\|_{L^2(\Omega)}}{\|\phi\|_{L^2(\Omega)}}$$

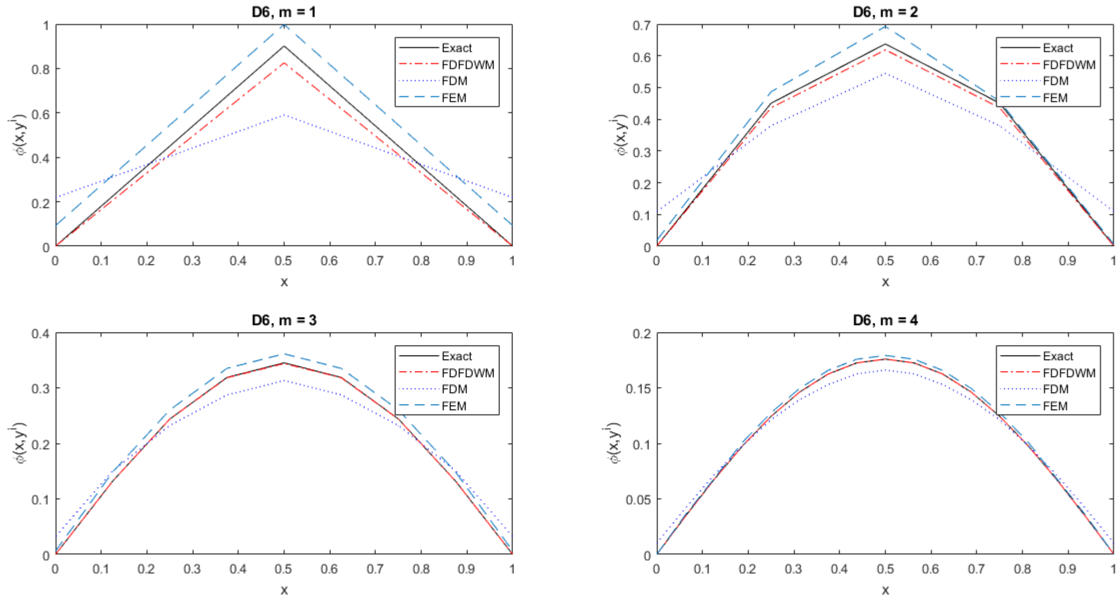

Figure 6: Comparison of FWDFDM, FDM and FEM solutions with exact solution $(\sin \pi x + \sin \pi y)/-2\pi^2$ on a rectangular domain for $D6$ and $m = 1, 1, 3$ and 4

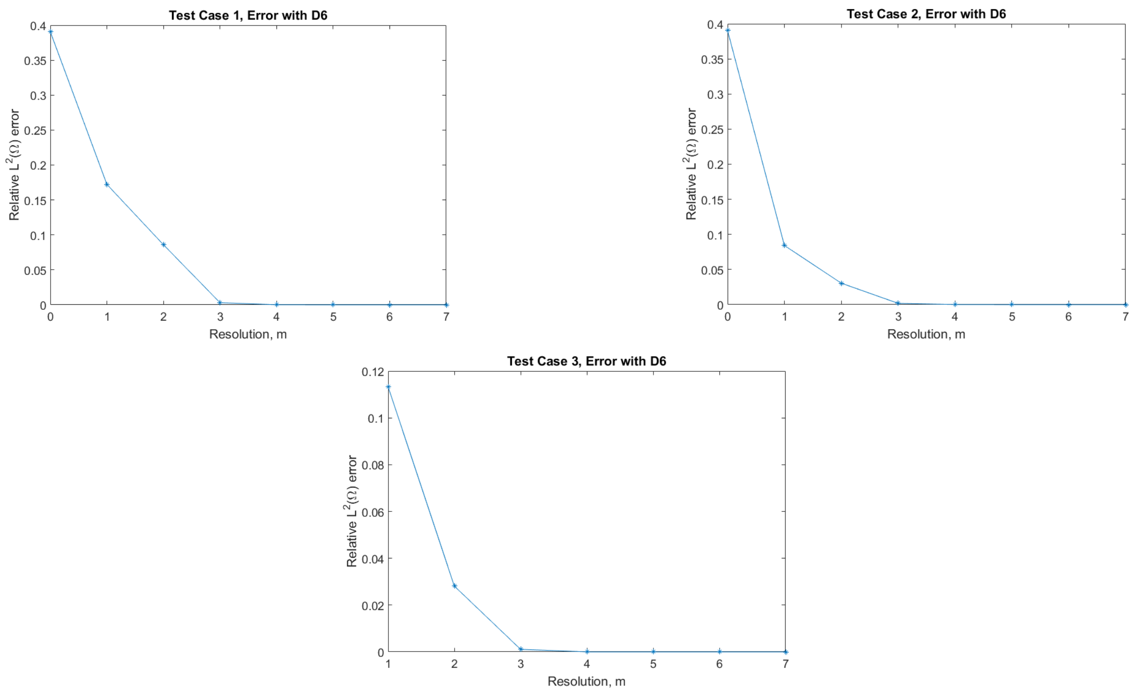

Figure 7: Relative L^2 error for Test Case 1, 2 and 3 using $D6$ and varying resolutions, $m = 0$ to $m = 7$

It is apparent from figure 7 that, as the resolution increases the error decays rapidly, indicating good convergence of the approximate solution to the exact solution. Although using Daubechies scaling functions with genus $D6$ at varying resolutions appears to generate reasonable approximation to the PDEs under consideration using the FDFDWM, it is essential to examine instances where the genus of the scaling functions increases.

Table 1: Relative L^2 error for $\phi = (\sin \pi x + \sin \pi y)/ - 2\pi^2$ approximation and condition number of FDFDWM's system matrix

	$m = 1$		$m = 2$	
DN	Error (L^2)	Cond. Num	Error (L^2)	Cond. Num.
6	1.72273872e-01	266545.642	8.59266326e-02	1216547.557
8	1.34574728e-01	299652.198	5.82274887e-02	2549819.826
10	1.00084497e-01	313878.814	2.37372575e-02	4877499.258
12	9.03869299e-02	452164.135	1.00522059e-02	6225570.392
14	8.17337237e-02	724845.847	6.08648401e-03	9913426.592
16	7.99390005e-02	1095616.081	4.19176077e-03	12874663.138
18	7.62888260e-02	1479906.308	1.24158634e-03	19988975.378
20	7.39675096e-02	2129604.614	1.02026987e-04	31017665.097

In table 1, we display the condition number of the coefficient matrix of FDFDWM provided in equation (32) together with the relative L^2 norm error at $m = 1$ and $m = 2$, and varying scaling function genus from $D6$ to $D20$ for test case 1. The table reveals that the approximation of the FDFDWM gets better as the genus of the scaling functions increases from $D6$ to $D20$. However, we realized that for $D12$ and above, the effect of the increment in the genus on the accuracy of the approximation diminishes. We noticed that the condition number grows as the resolution increases and same when the genus of the scaling function increases. Observably, we believe that the condition number is one of the contributing sources of error in the FDFDWM.

4. Conclusion

We have demonstrated in this paper that the FDFDWM is comparable to the traditional FEM and FDM approaches, and has inherently better accuracy to the approximation of the Dirichlet problem for linear elliptic partial differential equation in two dimension. It is much easier increasing the Daubechies scaling function order and resolution to obtain more accurate solutions. Also the reduction of the problem from 2D to 1D minimizes the complexities associated with solving PDEs in higher order.

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