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Skewed Double Exponential Distribution and Its Stochastic Representation

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Abstract. Definitions of the skewed double exponential (SDE) distribution in terms of a mixture of double exponential distributions as well as in terms of a scaled product of a c.d.f. and a p.d.f. of double exponential random variable are proposed. Its basic properties are studied. Multi-parameter versions of the skewed double exponential distribution are also given. Characterization of the SDE family of distributions and stochastic representation of the SDE distribution are derived.

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Key words: Symmetric distributions, Skew distributions, Stochastic representation, Linear combination of random variables, Characterizations, Skew Normal distribution.

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1. Introduction

The double exponential distribution was first published as Laplace's first law of error in the year 1774 and stated that the frequency of an error could be expressed as an exponential function of the numerical magnitude of the error, disregarding sign.

This distribution comes up as a model in many statistical problems. It is also considered in robustness studies, which suggests that it provides a model with different characteristics

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than other symmetric distributions. In particular, the tails are thicker than those of the normal distribution, but not as thick as the Cauchy distribution. This distribution has not gained much exposure, possibly partly due to a lack of available statistical techniques and partly due to the sharpness of the peak. Many applications are concerned with tail probabilities and the double exponential distribution would be a good choice when exponential tails are required.

The skewed double exponential distribution (SDE), proposed here, is a good addition to the family of skewed distributions in the sense that, like the skew normal distribution, it is a skewed distribution that possesses many of the properties that symmetric distributions do, while at the same time, being skewed, provides a better fit to real life data that is seldom symmetric in nature.

There are many SDE type random variables in literature including those introduced by McGill(1962), Holla and Bhattacharya(1968), Lingappaiah(1988), Poiraud-Cassanova and Thomas-Agnan(2000), Hinkley and Revankar(1977), and Kozubowski and Podgórski(2000) to name a few. One of the approaches that we take in this paper is to look at the mixture of a double exponential random variable and an exponential random variable. The other approach is similar to that of Azzalini(1985) in generating skew normal random variables. There are advantages and disadvantages to both models as will be evident in later sections of this paper.

Section 2 deals with preliminary results necessary for results in this paper. Section 3 talks about the SDE_1 distribution and its various representations. Section 4 discusses the SDE_2 distribution and its characterization in terms of the double exponential distribution.

2. Preliminary Results and Definitions

Definiton 1. Consider a random variable U distributed as a double exponential distribution ($U \sim DE(\eta, \theta)$) if it has probability density function(p.d.f.) given by

$$f_U(u) = \frac{1}{2\theta} e^{|u-\eta|/\theta}, \; u \in \mathbb{R}.$$

where $\eta \in \mathbb{R}$ and $\theta > 0$. The corresponding cumulative distribution function(c.d.f.) is given by

$$F_U(u) = \begin{cases} \frac{1}{2}e^{(u-\eta)/\theta} & if \ \ u < \eta \\ 1 - \frac{1}{2}e^{-(u-\eta)/\theta} & if \ \ u \geq \eta \end{cases}$$

Note that if $U \sim DE(0,1)$, then $V = \theta(U) + \eta$ has a double exponential (η, θ) distribution, i.e $V \sim DE(\eta, \theta)$. As a special case, let U be a standard double exponential (DE) random variable, i.e $U \sim DE(0,1)$. Then, $|U|$ is distributed as $|U| \sim Exp(1)$.

There are two approaches taken in generating the SDE random variables. The first (SDE_1) involves the scaled mixture of exponential and double exponential random variables. The second (SDE_2) deals with the product of the p.d.f. and scaled c.d.f. of the double exponential distribution(DE(0,1)). The following sections treat each of the two methods in detail.

3. The SDE_1 Distribution : Definition and Stochastic Representation

In this section, we present the distribution function of the different models of the skewed double exponential(SDE_1) distribution.

Theorem 3.1. *Consider two i.i.d random variables U,V distributed as $DE(0,1)$. Then, the random variable Y defined as $a|U| + bV$, for $a > 0$, $b \in \mathbb{R}^+$, $a \neq b$ is defined to be distributed as $SDE_1(a,b)$ and has the c.d.f.*

$$F_Y(y) = \begin{cases} \frac{b}{2(a+b)}e^{y/b}, & if \ \ y < 0 \\ 1 + \frac{a^2}{b^2-a^2}e^{-y/a} - \frac{b}{2(b-a)}e^{-y/b}, & if \ \ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{e^{y/b}}{2(a+b)}, & if \ \ y < 0 \\ \frac{e^{-y/b}}{2(b-a)} - \frac{a}{(b^2-a^2)}e^{-y/a}, & if \ \ y \geq 0 \end{cases} .$$

If $a = b$, then the c.d.f. is

$$F_Y(y) = \begin{cases} \frac{1}{4}e^{y/a}, & if \ \ y < 0 \\ 1 - \left(\frac{3a+2y}{4a}e^{-y/a}\right), & if \ \ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{e^{y/a}}{4a}, & if\ y < 0 \\ (\frac{a+2y}{4a^2})e^{-y/a}, & if\ y \geq 0 \end{cases} .$$

Proof. We give a proof that is similar to the one given by Henze(1986). Consider

$$\begin{aligned} \mathbb{P}[Y \leq y] &= \int_0^\infty \mathbb{P}[V \leq \frac{y-au}{b}] \cdot f_{|U|}(u)\,du \\ &= \int_0^\infty \left[\int_{-\infty}^{\frac{y-au}{b}} \frac{1}{2}\ e^{-|v|}\,dv\right]\ e^{-u}\,du. \end{aligned} \tag{3.1}$$

If $y < 0$, then, since $a > 0$ and the support of $|U|$ is $(0,\infty)$, $\dfrac{y-au}{b} < 0\ \ ,\forall u$. Hence, (1) becomes

$$\begin{aligned} \mathbb{P}[Y \leq y] &= \int_0^\infty \left[\int_{-\infty}^{\frac{y-au}{b}} \frac{1}{2}\ e^{v}\,dv\right]\quad e^{-u}\,du \\ &= \frac{b}{2(a+b)}\ e^{y/b} \quad , y < 0. \end{aligned}$$

On the other hand, if $y \geq 0$, then either $\dfrac{y-au}{b} < 0$ for $u > y/a$ or $\dfrac{y-au}{b} > 0$ for $u < y/a$. So, (1) is equivalent to

$$\begin{aligned} \mathbb{P}[Y \leq y] &= \underbrace{\int_0^{\frac{y}{a}} [\frac{1}{2} + \int_0^{\frac{y-au}{b}} \frac{1}{2}\ e^{-v}\,dv]\ e^{-u}\,du}_{(i)} \\ &\quad + \underbrace{\int_{\frac{y}{a}}^\infty \left[\int_{-\infty}^{\frac{y-au}{b}} \frac{1}{2}\ e^{v}\,dv\right]\quad e^{-u}\,du}_{(ii)}. \end{aligned} \tag{3.2}$$

Consider :

$$
\begin{aligned}
(i) &= \int_0^{\frac{y}{a}} \frac{1}{2}\ e^{-u}\,du + \int_0^{\frac{y}{a}} \int_0^{\frac{y-au}{b}} \frac{1}{2}\ e^{-v}\,dv\ e^{-u}\,du \\
&= \frac{1}{2}(1-e^{-y/a}) + \int_0^{\frac{y}{a}} \frac{1}{2}(1-e^{-\frac{y-au}{b}})e^{-u}\,du \\
&= 1-e^{-y/a} - \frac{b}{2(b-a)}e^{-y/b} + \frac{b}{2(b-a)}e^{-y/a}. \qquad (3.3)
\end{aligned}
$$

Also,

$$
\begin{aligned}
(ii) &= \int_{\frac{y}{a}}^{\infty} \left[\int_{-\infty}^{\frac{y-au}{b}} \frac{1}{2}\ e^{v}\,dv\right]\ e^{-u}\,du \\
&= \frac{b}{2(a+b)}\ e^{-y/a}. \qquad (3.4)
\end{aligned}
$$

Adding the right hand sides of (3) and (4), as required by (2), we get via some elementary algebra that the c.d.f. of the $SDE_1(a,b)$ is as follows:

$$
F_Y(y) = \begin{cases} \frac{b}{2(a+b)}e^{y/b}, & if\ y<0 \\ 1+\frac{a^2}{(b^2-a^2)}e^{-y/a} - \frac{b}{2(b-a)}e^{-y/b}, & if\ y\geq 0 \end{cases} . \qquad (3.5)
$$

Differentiating (3.5) with respect to y, we get the p.d.f. of an $SDE_1(a,b)$ random variable to be

$$
f_Y(y) = \begin{cases} \frac{e^{y/b}}{2(a+b)}, & if\ y<0 \\ \frac{e^{-y/b}}{2(b-a)} - \frac{a}{(b^2-a^2)}e^{-y/a}, & if\ y\geq 0 \end{cases} .
$$

A similar approach can be taken in the case when $a=b$ to find the c.d.f. and p.d.f. of the $SDE_1(a,b)$ random variable.

Note. In the above theorem the value of "b" is restricted to the positive half line. Since $V\sim DE(0,1)$, $bV \stackrel{d}{=} -bV$. Thus, we can restrict our attention to the case when $b\geq 0$ as the results will be the same when $b<0$.

The following result given by Theorem(3.2) is obtained by using the fact that

$$F_{Y_{a,b}}(y) = \mathbb{P}(a|U| + bV \leq y) = \mathbb{P}(-a|U| + bV \geq -y) = 1 - F_{Y_{-a,b}}(-y)$$

and the result of Theorem(3.1).

Theorem 3.2. *Consider two i.i.d random variables U,V distributed as $DE(0,1)$. Then, the random variable Y defined as $a|U| + bV$, for $a < 0$, $b \in \mathbb{R}^+$, $a \neq -b$ has the c.d.f.*

$$F_Y(y) = \begin{cases} \frac{b}{2(a+b)} e^{y/b} - \frac{a^2}{b^2-a^2} e^{-y/a}, & if\ y < 0 \\ 1 - \frac{b}{2(b-a)} e^{-y/b}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{e^{y/b}}{2(a+b)} + \frac{a}{(b^2-a^2)} e^{-y/a}, & if\ y < 0 \\ \frac{e^{-y/b}}{2(b-a)}, & if\ y \geq 0 \end{cases} .$$

If $a = -b$, then the c.d.f. is

$$F_Y(y) = \begin{cases} (\frac{3a+2y}{4a}) e^{-y/a}, & if\ y < 0 \\ 1 - \frac{1}{4} e^{y/a}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} -(\frac{a+2y}{4a^2}) e^{-y/a}, & if\ y < 0 \\ -\frac{1}{4a} e^{y/a}, & if\ y \geq 0 \end{cases} .$$

Corollary 3.1. *Consider the random variable Y as defined in Theorem 3.1. Then the random variable X defined as $X = \theta(Y) + (a+b)\eta$, for $a > 0$, $b \in \mathbb{R}^+$ with $a \neq b$ has c.d.f.*

$$F_X(x) = \begin{cases} \frac{b}{2(a+b)} e^{\frac{x-(a+b)\eta}{b\theta}}, & if\ x < (a+b)\eta \\ 1 + \frac{a^2}{(b^2-a^2)} e^{-\frac{x-(a+b)\eta}{a\theta}} - \frac{b}{2(b-a)} e^{-\frac{x-(a+b)\eta}{b\theta}}, & if\ x \geq (a+b)\eta \end{cases}$$

and it's p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{2\theta(a+b)} e^{\frac{x-(a+b)\eta}{b\theta}}, & if\ x < (a+b)\eta \\ \frac{1}{2\theta(b-a)} e^{-\frac{x-(a+b)\eta}{b\theta}} - \frac{a}{\theta(b^2-a^2)} e^{-\frac{x-(a+b)\eta}{a\theta}}, & if\ x \geq (a+b)\eta \end{cases} .$$

If $a = b$, then the c.d.f. is given by

$$F_X(x) = \begin{cases} \frac{1}{4}e^{\frac{x-2a\eta}{a\theta}}, & if\ x < 2a\eta \\ 1 - (\frac{a(3\theta-4\eta)+2x}{4a\theta})e^{-\frac{x-2a\eta}{a\theta}}, & if\ x \geq 2a\eta \end{cases}$$

and it's p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{4a\theta}e^{\frac{x-2a\eta}{a\theta}}, & if\ x < 2a\eta \\ (\frac{a(\theta-4\eta)+2x}{4a^2\theta^2})e^{-\frac{x-2a\eta}{a\theta}}, & if\ x \geq 2a\eta \end{cases} .$$

Y is said to have a skewed double exponential distribution with parameters η and θ, i.e $Y \sim SDE_1(a, b, \eta, \theta)$.

Proof. The proof of this assertion is an exercise in transformations of random variables. We are transform the variable Y as in Theorem 3.1 and Theorem 3.2 to $\dfrac{Y-(a+b)\eta}{\theta}$. The transformed random variable will have the appropriate c.d.f. and p.d.f. as mentioned in the statement of the corollary.

In the case when $a < 0$ in the above Corollary, we have the following corollary to Theorem 3.2, which we state without proof.

Corollary 3.2. *Consider the random variable Y as defined in Theorem 3.1. Then the random variable X defined as $X = \theta(Y) + (a+b)\eta$, for $a < 0$, $b \in \mathbb{R}^+$ with $a \neq -b$ has c.d.f.*

$$F_X(x) = \begin{cases} \frac{b}{2(a+b)}e^{\frac{x-(a+b)\eta}{b\theta}} - \frac{a^2}{(b^2-a^2)}e^{-\frac{x-(a+b)\eta}{a\theta}}, & if\ x < (a+b)\eta \\ 1 - \frac{b}{2(b-a)}e^{-\frac{x-(a+b)\eta}{b\theta}}, & if\ x \geq (a+b)\eta \end{cases}$$

and it's p.d.f. is given by

$$f_X(x) = \begin{cases} \frac{1}{2\theta(a+b)}e^{\frac{x-(a+b)\eta}{b\theta}} + \frac{a}{\theta(b^2-a^2)}e^{-\frac{x-(a+b)\eta}{a\theta}}, & if\ x < (a+b)\eta \\ \frac{1}{2\theta(b-a)}e^{-\frac{x-(a+b)\eta}{b\theta}}, & if\ x \geq (a+b)\eta \end{cases} .$$

If $a = -b$ then the c.d.f. is given by

$$F_X(x) = \begin{cases} (\frac{a(3\theta-4\theta+2x}{4a\theta})e^{-\frac{x-2a\eta}{a\theta}}, & if\ x < 2a\eta \\ 1 - \frac{1}{4}e^{\frac{x-2a\eta}{a\theta}}, & if\ x \geq 2a\eta \end{cases}$$

and it's p.d.f. is given by

$$f_X(x) = \begin{cases} -\left(\frac{a(\theta-4\eta)+2x}{4a^2\theta^2}\right)e^{-\frac{x-2a\eta}{a\theta}}, & if\ x < 2a\eta \\ -\frac{1}{4a\theta}e^{\frac{x-2a\eta}{a\theta}}, & if\ x \geq 2a\eta \end{cases} .$$

In Fig.1 through Fig.4, graphs of the $SDE_1(a,b)$ density functions are presented for $a = -90, -20, -5, 5, 20, 90$ and $b = .1, 1, 10, 50$. Fig.1 deals with the graphs of $SDE_1(a,b)$ density functions when $a = -90$ while Fig.2 deals with the case when $a = -20, -5$. Fig.3 portrays the case when $a = 5, 20$ and Fig.4 deals with the case when $a = 90$.

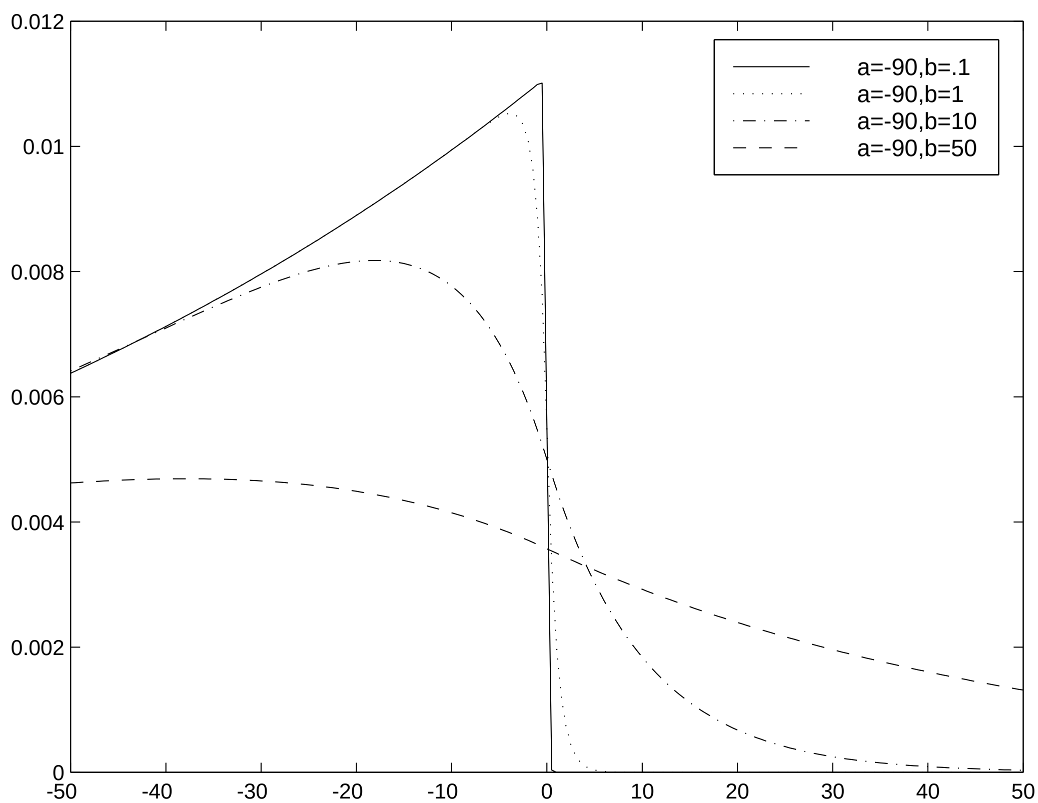

Figure 1: Graphs of the $SDE_1(a,b)$ density function for a=-90 and b = .1,1,10,50.

Remark 3.1. Appealing to the fact that the measure of skewness of a distribution is invariant to changes of scale, we can reparametrize the mixing parameters a and b to $\frac{a}{a^2+b^2}$ and $\frac{b}{a^2+b^2}$. If we substitute the variable "c" for $\frac{a}{a^2+b^2}$, we obtain that $\frac{b}{a^2+b^2} = \sqrt{1-c^2}$.

Notice that the variable c is now bounded between -1 and 1. This model is easier to use in estimating the skewness as it has the advantage of having one less parameter. We present the c.d.f. and p.d.f. of this model in the following theorem.

Theorem 3.3. *Consider two i.i.d random variables U,V distributed as* $DE(0,1)$*. Then, the random variable Y defined as* $c|U| + \sqrt{1-c^2}V$*, for* $-1 < c < 1$.

If $0 < c < 1$ *,* $c \neq \sqrt{1/2}$*, then* Y *has c.d.f.*

$$F_Y(y) = \begin{cases} \frac{\sqrt{1-c^2}}{2(c+\sqrt{1-c^2})} e^{y/\sqrt{1-c^2}}, & if\ y < 0 \\ 1 + \frac{c^2}{1-2c^2} e^{-y/c} - \frac{\sqrt{1-c^2}}{2(\sqrt{1-c^2}-c)} e^{-y/\sqrt{1-c^2}}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{e^{y/\sqrt{1-c^2}}}{2(c+\sqrt{1-c^2})}, & if\ y < 0 \\ \frac{e^{-y/\sqrt{1-c^2}}}{2(\sqrt{1-c^2}-c)} - \frac{c}{(1-2c^2)} e^{-y/c}, & if\ y \geq 0 \end{cases} .$$

If $-1 < c < 0$ *,* $c \neq -\sqrt{1/2}$*, then* Y *has c.d.f.*

$$F_Y(y) = \begin{cases} \frac{\sqrt{1-c^2}}{2(c+\sqrt{1-c^2})} e^{y/\sqrt{1-c^2}} - \frac{c^2}{1-2c^2} e^{-y/c}, & if\ y < 0 \\ 1 - \frac{\sqrt{1-c^2}}{2(\sqrt{1-c^2}-c)} e^{-y/\sqrt{1-c^2}}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{e^{y/\sqrt{1-c^2}}}{2(c+\sqrt{1-c^2})} + \frac{c}{(1-2c^2)} e^{-y/c}, & if\ y < 0 \\ \frac{e^{-y/\sqrt{1-c^2}}}{2(\sqrt{1-c^2}-c)}, & if\ y \geq 0 \end{cases} .$$

If $c = \sqrt{1/2}$*, then* Y *has c.d.f.*

$$F_Y(y) = \begin{cases} \frac{1}{4} e^{\sqrt{2}y}, & if\ y < 0 \\ 1 - (\frac{3+2\sqrt{2}y}{2\sqrt{2}}) e^{-\sqrt{2}y}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{2}} e^{\sqrt{2}y}, & if\ y < 0 \\ (\frac{1+\sqrt{2}y}{\sqrt{2}}) e^{-\sqrt{2}y}, & if\ y \geq 0 \end{cases} .$$

If $c = -\sqrt{1/2}$, then Y has c.d.f.

$$F_Y(y) = \begin{cases} (\frac{3-2\sqrt{2}y}{4})e^{\sqrt{2}y}, & if\ y < 0 \\ 1 - \frac{1}{4}e^{-\sqrt{2}y}, & if\ y \geq 0 \end{cases}$$

and it's p.d.f. is given by

$$f_Y(y) = \begin{cases} (\frac{1-2\sqrt{2}y}{2\sqrt{2}})e^{\sqrt{2}y}, & if\ y < 0 \\ \frac{1}{2\sqrt{2}}e^{-\sqrt{2}y}, & if\ y \geq 0 \end{cases} .$$

Remark 3.2. Notice that the proofs of the above theorems mirror the proofs of Theorem 3.1 and Theorem 3.2 respectively. Further, the generalizations that were made following Theorem 3.1 and Theorem 3.2 can also be made in this incarnation of the SDE_1 distribution.

4. The SDE_2 Distribution : Definition and Stochastic Representation

Azzalini(1985) introduced the skew-normal distribution, i.e ($f_Y(y) = 2\phi_X(y)\Phi_X(\lambda y)$) where ϕ and Φ represent the standard normal p.d.f. and c.d.f. respectively. Although credit is given to Azzalini, Roberts(1966) had already used this distribution in studying twin data. Gupta, Nguyen and Sanqui(2004) have given a characterization of this distribution. Azzalini however, introduced a way to "skew" symmetric distributions by considering similar products of p.d.f.'s and c.d.f.'s of symmetric distributions. Also see Gupta et al.(2002) and Gupta and Chang(2003). We can utilize Azzalini's method to produce the SDE_2 distributions.

Before presenting the c.d.f. and p.d.f. of the SDE_2 distribution, we state the following (Azzalini(1985), Gupta et. al.(2002))

Lemma 4.1. *Let f be a density function symmetric about 0, and G an absolutely continuous distribution function such that G' is symmetric about 0. Then,*

$$2f(y)G(\lambda y)\ \ (-\infty < y < \infty) \tag{4.1}$$

is a density function for any real λ.

The above result is used in the proof of the following result.

Theorem 4.1. Gupta et al. (2002) *Consider a random variable $X \sim DE(0,1)$. Then, the variable Y with p.d.f. defined as $f_Y(y) = 2f_X(y)F_X(\lambda y)$, for $\lambda \in \mathbb{R}$ has the p.d.f. given by*

$$f_Y(y) = \frac{1}{2}e^{-|y|}\big(1 + sign(\lambda y)(1 - e^{-|\lambda y|})\big)$$

If Y has the form above, we will say $Y \sim SDE_2(\lambda)$.

Graphs of the $SDE_2(\lambda)$ density function for values of $\lambda = -100, -50, -10, -1, 1, 10, 50, 100$ are given in Fig. 5.

Definiton 2. Generator of $SDE_2(\lambda)$.
A distribution function F is said to be a generator of the $SDE_2(\lambda)$ distribution if the function $2f(\cdot)F(\lambda\cdot)$, where $f(\cdot)$ is the p.d.f. of the distribution function F, is the density function of the $SDE_2(\lambda)$ distribution.

The following theorem is a statement of the uniqueness of the generator of the $SDE_2(\lambda)$ family of distributions.

Theorem 4.2. *The double exponential distribution is the unique generator of the $SDE_2(\lambda)$ distribution.*

Proof. Let F be a generator of the $SDE_2(\lambda)$ family with p.d.f. f, that is $2f(x)F(\lambda x) = g(x)$, for all λ, where $g(x)$ is the p.d.f. of the $SDE_2(\lambda)$ distribution.
Consider first the case when $x < 0$.
Then,

$$\begin{aligned} 2\,f(x)\,F(\lambda x) &= 2\,\big(\frac{1}{2}e^{x}\big)\,\big(\frac{1}{2}e^{\lambda x}\big) \\ \Rightarrow \frac{4f(x)}{e^x} &= \frac{e^{\lambda x}}{F(\lambda x)}. \end{aligned} \tag{4.2}$$

Since $\dfrac{4f(x)}{e^x} = \dfrac{e^{\lambda x}}{F(\lambda x)}$, none of the terms in the above expression depend on λ. Hence, we can set $\dfrac{4f(x)}{e^x} = \dfrac{e^{\lambda x}}{F(\lambda x)} = c(x)$, where $c(x)$ is a function of x only. Further, without loss of generality, we can set $\lambda = 1$. Now, (4.2) leads to the following equations.

$$f(x) = \frac{c(x)\, e^x}{4} \tag{4.3}$$

$$F(x) = \frac{e^x}{c(x)}. \tag{4.4}$$

Taking derivatives of (4.4) with respect to x, we get

$$f(x) = \frac{c(x)\, e^x - e^x\, c'(x)}{(c(x))^2}. \tag{4.5}$$

Equating (4.3) and (4.5), we get

$$\frac{c(x)\, e^x}{4} = \frac{c(x)\, e^x - e^x\, c'(x)}{(c(x))^2}. \tag{4.6}$$

Solving the above differential equation, we get that

$$c(x) = \frac{e^x}{\sqrt{\frac{e^{2x}}{4} + c}} \tag{4.7}$$

where c is a constant. Using the value of $c(x)$ obtained in (4.7) in (4.4), we get that

$$F(x) = \sqrt{\frac{e^{2x}}{4} + c}.$$

Taking limits as $x \to -\infty$, we get

$$\lim_{x \to -\infty} F(x) = \sqrt{c}. \tag{4.8}$$

Then, from (4.8) and (4.7), $c = 0$ and $c(x) = 2$. Hence, from (4.3), $f(x) = \frac{1}{2}e^x$.

Now, if $x > 0$, then

$$\begin{aligned} 2\, f(x)\, F(\lambda x) &= 2\left(\frac{1}{2}e^{-x}\right)\left(1 - \frac{1}{2}e^{-\lambda x}\right) \\ \Rightarrow \frac{2f(x)}{e^{-x}} &= \frac{2 - e^{-\lambda x}}{2} F(\lambda x). \end{aligned} \tag{4.9}$$

Again, for the same reasons as above, we can set $\frac{2f(x)}{e^{-x}} = \frac{2 - e^{-\lambda x}}{F(\lambda x)} = c(x)$, where $c(x)$ is a function of x only. Further, without loss of generality, we can set $\lambda = 1$.

(4.9) leads to the following equations.

$$f(x) = \frac{c(x)\,e^{-x}}{2} \tag{4.10}$$

$$F(x) = \frac{2 - e^{-x}}{2c(x)}. \tag{4.11}$$

Taking derivatives of (4.11) with respect to x, we get

$$f(x) = \frac{c(x)\,e^{-x} - (2 - e^{-x})\,c'(x)}{2(c(x))^2}. \tag{4.12}$$

Equating (4.10) and (4.12), we get

$$\frac{c(x)\,e^{-x}}{2} = \frac{c(x)\,e^{-x} - (2 - e^{-x})\,c'(x)}{2(c(x))^2}. \tag{4.13}$$

Solving the above differential equation, we get that

$$c(x) = \frac{2 - e^{-x}}{\sqrt{-e^{-x}(4 - e^{-x}) + c}} \tag{4.14}$$

where c is a constant. Using the value of $c(x)$ obtained in (4.14) in (4.11), we get that

$$F(x) = \frac{\sqrt{-e^{-x}(4 - e^{-x}) + c}}{2}. \tag{4.15}$$

Taking limits as $x \to \infty$, we get

$$\lim_{x\to\infty} F(x) = \frac{\sqrt{c}}{2}.$$

Then, from (4.16) and (4.14), $c = 4$ and $c(x) = 1$. Hence, from (4.10), $f(x) = \frac{1}{2}e^{-x}$.

Thus, $f(x) = \frac{1}{2}e^{-|x|}$, and F is a $DE(0,1)$ distribution. Hence, the $DE(0,1)$ is the unique generator of the $SDE_2(\lambda)$ distribution.

The direct simulation of random variables from the SDE_2 distribution tends to be problematic due to the difficulties with inversion of the c.d.f. In order to simulate a SDE_2 random variable with ease, we present the following stochastic representation.

Theorem 4.3. *Let X be a random variable with c.d.f. F_X and p.d.f. f. Suppose $V = |X|$ with c.d.f. G_V and p.d.f. g. Define the variable $S|V$ by*

$$S|V = \begin{cases} -1, & \text{with probability } 1 - F_X(\lambda v) \\ 1, & \text{with probability } F_X(\lambda v) \end{cases}. \tag{4.16}$$

Then, the random variable $Y = (S|V)V$ is distributed as SDE_2.

Proof. Consider

$$\begin{aligned}\mathbb{P}(Y \leq y) &= \int_0^\infty F_{S|V}(y/v)\, g(v)\, dv \\ &= \int_{y/v<-1} 0 \cdot g(v)\, dv + \int_{-1\leq y/v<1} (1 - F_X(\lambda v)) \cdot g(v)\, dv \\ &\quad + \int_{y/v\geq 1} 1 \cdot g(v)\, dv. \end{aligned} \tag{4.17}$$

If $y > 0$, then $y/v > 0$. Then the condition $-1 \leq y/v < 1$ is equivalent to $0 \leq y/v < 1$, which in turn is equivalent to $v > y$. The condition $y/v \geq 1$ is then equivalent to $v \leq y$. So, (8) becomes

$$\mathbb{P}(Y \leq y) = \int_0^y 1 \cdot g(v)\, dv + \int_y^\infty (1 - F_X(\lambda v)) \cdot g(v)\, dv.$$

Taking derivatives on both sides of the above equation, we get

$$\begin{aligned}\frac{d}{dy}\mathbb{P}(Y \leq y) &= g(y) - [g(y) - F_X(\lambda y) \cdot g(y)] \\ &= g(y) F_X(\lambda y).\end{aligned}$$

Recall that $V = |X|$. Hence, $g(y) = 2f(y)$ and we get that $f_Y(y) = 2f(y)F(\lambda y)$. So, in the case when $y > 0$, $Y \sim SDE_2(\lambda)$.

If $y < 0$, then $y/v < 0$. Then the condition $y/v \geq 1$ is never satisfied and the condition $-1 \leq y/v < 1$ is equivalent to $v \geq -y$. So, (8) becomes

$$\mathbb{P}(Y \leq y) = \int_{-y}^\infty (1 - F_X(\lambda v)) \cdot g(v)\, dv.$$

Taking derivatives on both sides again, we get

$$\frac{d}{dy}\mathbb{P}(Y \leq y) = -\big(1 - F_X(-\lambda y)\big) \cdot g(-y).$$

Again, $-g(-y) = g(y) = 2f(y)$ since $f(.)$ is a symmetric p.d.f. Also, $1 - F(-\lambda y) = F(\lambda y)$ and so, $f_Y(y) = 2f(y)F(\lambda y)$ and $Y \sim SDE_2(\lambda)$.

Note. The above theorem is helpful in obtaining a stochastic representation of any symmetric distribution function F, that we wish to use as a generator of a skewed family of distributions.

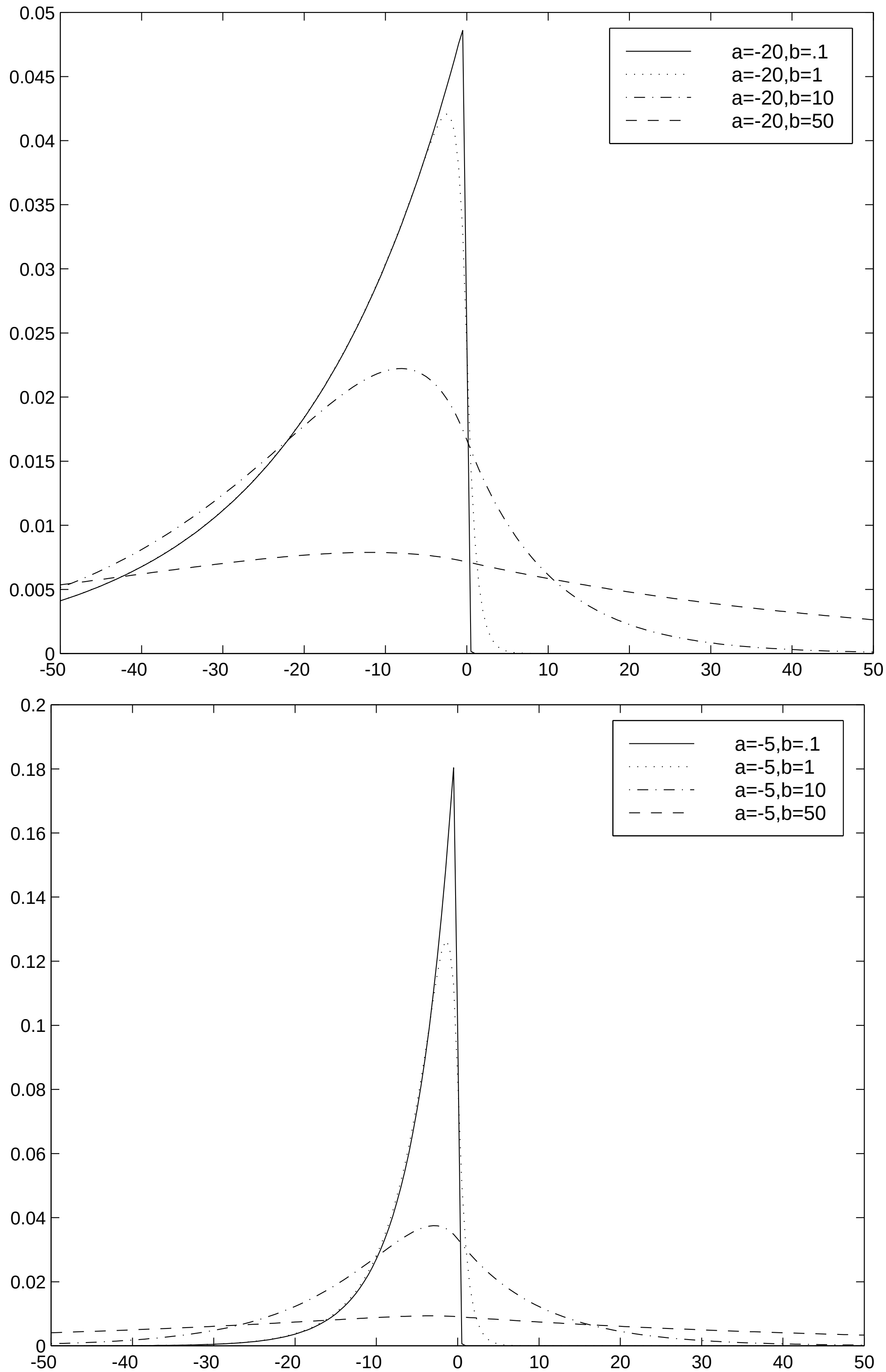

Figure 2: Graphs of the $SDE_1(a,b)$ density function for a=-20, -5 and b = .1,1,10,50.

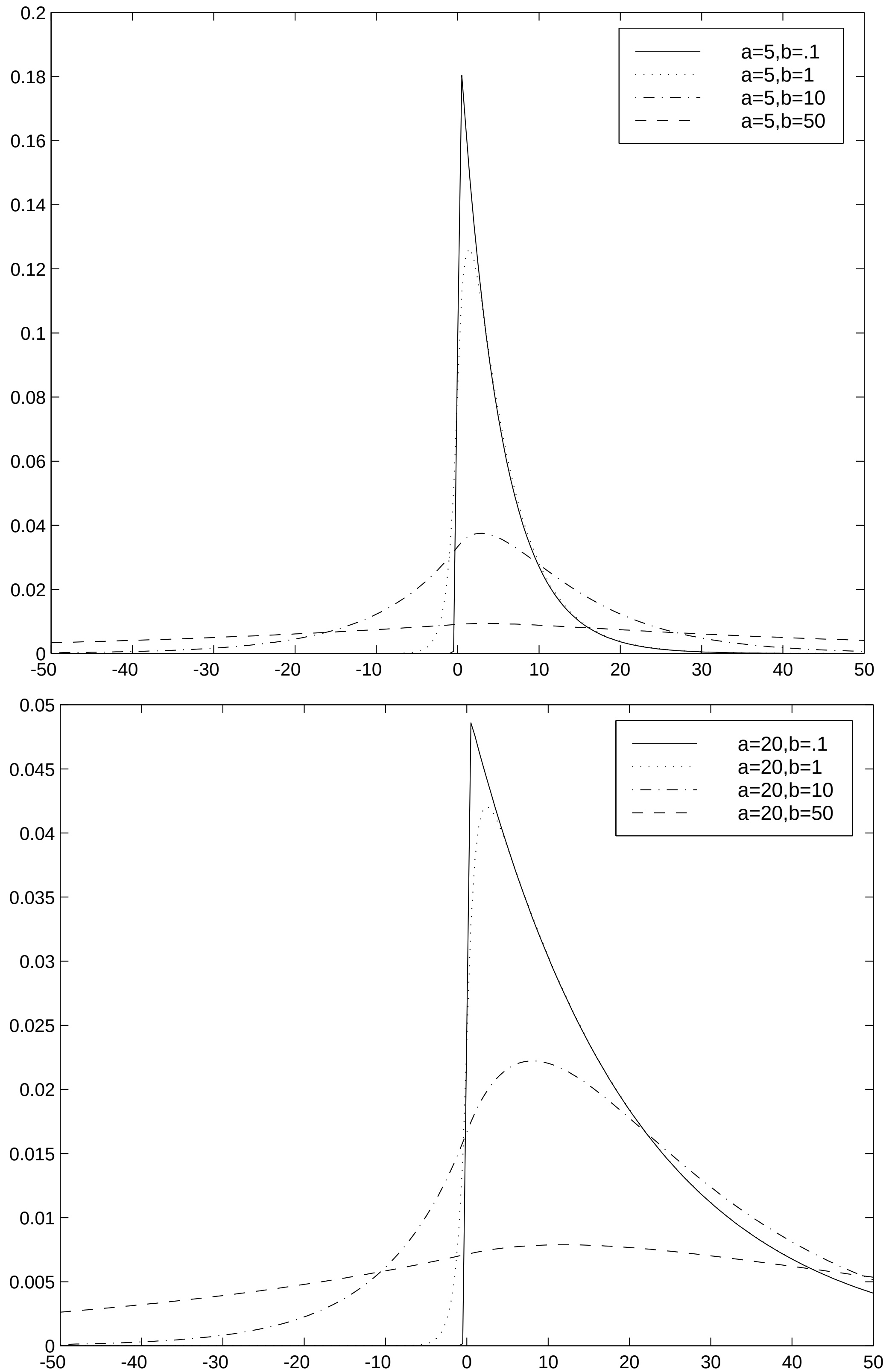

Figure 3: Graphs of the $SDE_1(a,b)$ density function for a=5,20 and b = .1,1,10,50.

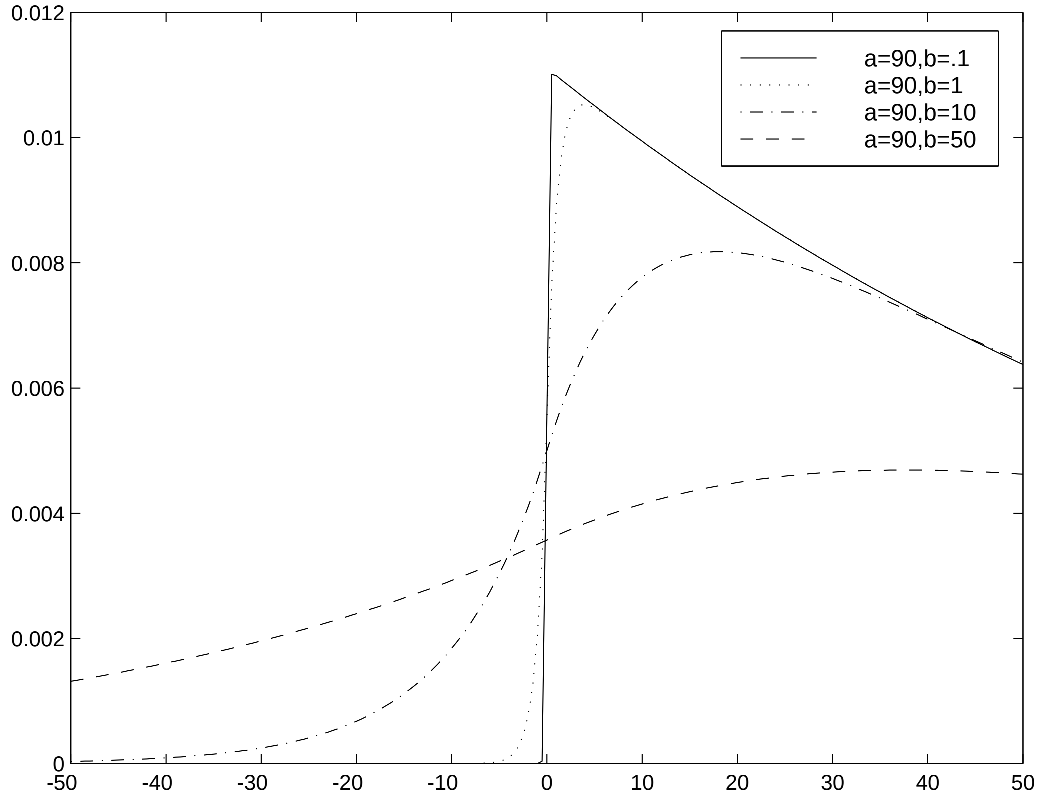

Figure 4: Graphs of the $SDE_1(a,b)$ density function for a=90 and b = .1,1,10,50.

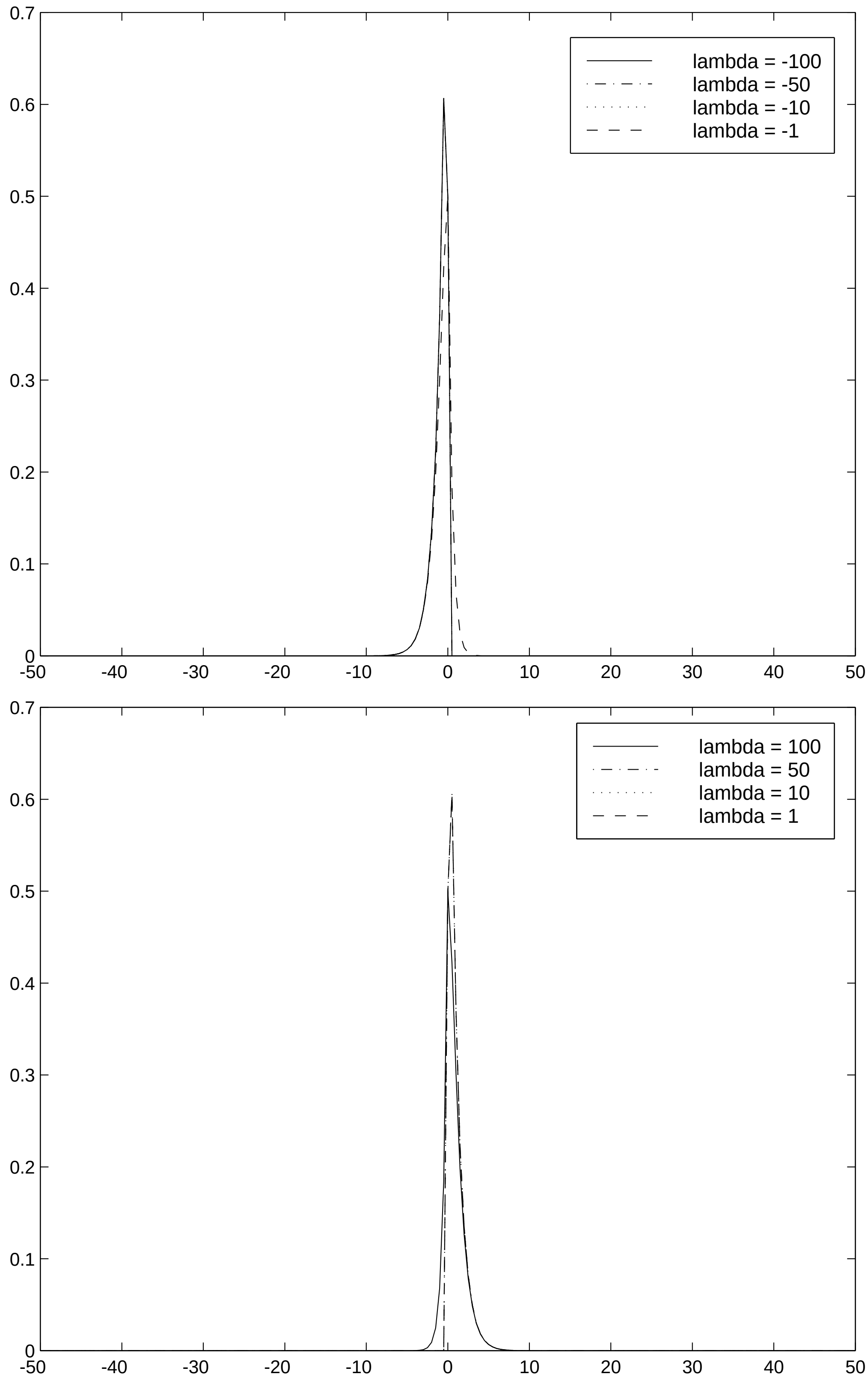

Figure 5: Graphs of the $SDE_2(\lambda)$ density function for $\lambda = -100, -50, -10, -1, 1, 10, 50, 100$.

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