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Properties of nilpotent evolution algebras with no maximal nilindex

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Abstract. As a system of abstract algebra, evolution algebras are commutative and non-associative algebras. There is no deep structure theorem for general non-associative algebras. However, there are deep structure theorem and classification theorem for evolution algebras because it has been introduced concepts of dynamical systems to evolution algebras. Recently, in [25], it has been studied some properties of nilpotent evolution algebra with maximal index ($\dim \mathbf{E^2} = \dim \mathbf{E} - 1$). This paper is devoted to studying nilpotent finite-dimensional evolution algebras $\mathbf{E}$ with $\dim \mathbf{E^2} = \dim \mathbf{E} - 2$. We describe Lie algebras related to the evolution of algebras. Moreover, this result allowed us to characterize all local and 2-local derivations of the considered evolution algebras. All automorphisms and local automorphisms of the nilpotent evolution algebras are found.

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1. Introduction

The departure point of a new type of evolution algebra has been introduced by [34]. This algebra is motivated by some evolution laws of genetics. The study of evolution algebras serves as a foundation of a new research area in algebra and the theory of dynamic systems. Many related open problems have to be addressed to develop research in this area (for further details, we refer to [33]).

We note that evolution algebras are not defined by identities, and therefore they do not form a type of non-associative algebras, such as Lie, Jordan, or alternative algebras. Thus, to investigate such algebras, a different approach has to be used (see [7, 9, 12]).

In [12], the relationships among nil, right nilpotent evolution algebras, which are defined by an upper triangular matrix of structural constants, have been found. A further

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problem which has been addressed in [10, 15, 18, 19, 27] is the classification of low-dimensional evolution algebras. Nevertheless, a full classification of nilpotent evolution algebras is a tricky task. [20] have investigated certain properties of nilpotent evolution algebras with maximal nilindex. In the current study, we analyze some propensities of nilpotent evolution algebras whose index of nilpotency is $2^{n-2}+1$.

The derivation of non-associative algebra forms the Lie algebra, which is considered as one of the important tools for studying its structure. Extensive work has been conducted on the subject of derivations of genetic algebras ([13], [17], [20], [28],[16],[1]). since the multiplication is trivial then set of all until is invertible In fact, [7, 14] have investigated several properties of derivations of n-dimensional complex evolution algebras,depending on the rank of the appropriate matrices. Recently, many paper have been devoted to study the derivation of evolution algebras see for instance [2, 26, 30, 31]. Other properties of evolution algebra have been investigated in [5, 6, 8, 9, 11, 23, 29]. In [25], it has been study the properties of the nilpotent finite-dimensional evolution algebras with maximal nil index such as derivation, local derivation, automorphism, and local automorphism.

In the present study, we explicitly describe the space of derivations of evolution algebras with nilindex $2^{(n-2)}+1$, which allows us to study further properties of the evolution algebras.Moreover, we describe all local and 2-local derivations of the considered algebra.We stress that the notions of local automorphism and local derivation were introduced and investigated independently by Kadison [22] and Larson and Sourour [24]. Subsequently, P. Šemrl [32] introduced the concepts of 2-local automorphisms and 2-local derivations. The preceding studies have led to a series of works devoted to description of mappings which are close to automorphisms and derivations of C^*-algebras and operator algebras. For details and the survey, we refer to the work of [3, 4].

The paper is organized as follows. Section 2 provides preliminary information about evolution algebras. Derivations of non-associative algebras form the Lie algebra; thus, so, in section 3 we describe the Lie algebra associated with evolution algebras whose nilindex is $2^{(n-2)}+1$. Furthermore,based on results in section 3, section 4 describes local and 2-local derivations of the considered evolution algebras. In section 5, we find all automorphisms and local automorphisms of the nilpotent evolution algebras with nilindex $2^{(n-2)}+1$.

2. Evolution algebras

Recall the definition of evolution algebras. Let $\mathbf{E}$ be a vector space over a field $\mathbb{K}$. In the follows, we always assume that $\mathbb{K}$ has characteristic zero. The vector space $\mathbf{E}$ is called *evolution algebra* w.r.t. *natural basis* $\{\mathbf{e}_1, \mathbf{e}_2, ...\}$ if a multiplication rule $\cdot$ on $\mathbf{E}$ satisfies

$$\mathbf{e}_i \cdot \mathbf{e}_j = \mathbf{0}, \ i \neq j,$$

$$\mathbf{e}_i \cdot \mathbf{e}_i = \sum_k a_{ik}\mathbf{e}_k, \ i \geq 1.$$

From the preceding definition, it follows that evolution algebras are commutative (therefore, flexible).

We denote by $A = (a_{ij})_{i,j=1}^n$ the matrix of the structural constants of the finite-dimensional evolution algebra $\mathbf{E}$. Obviously, $rankA = \dim(\mathbf{E}\cdot\mathbf{E})$. Thus, for finite-dimensional evolution algebra, the rank of the matrix does not depend on choice of natural basis. In the following, for convenience, we write $\mathbf{uv}$ instead $\mathbf{u}\cdot\mathbf{v}$ for any $\mathbf{u},\mathbf{v}\in\mathbf{E}$ and we write $\mathbf{E^2}$ instead of $\mathbf{E}\cdot\mathbf{E}$.

A linear map $\psi:\mathbf{E}_1\to\mathbf{E}_2$ is called a *homomorphism* of evolution algebras if $\psi(\mathbf{uv}) = \psi(\mathbf{u})\psi(\mathbf{v})$ for any $\mathbf{u},\mathbf{v}\in\mathbf{E}_1$. Moreover, if ψ is bijective, then it is called an *isomorphism.* In this case, the last relation is denoted by $\mathbf{E}_1\cong\mathbf{E}_2$.

For an evolution algebra $\mathbf{E}$, we introduce the following sequence, $k\geq 1$

$$\mathbf{E}^k = \sum_{i=1}^{k-1}\mathbf{E}^i\mathbf{E}^{k-i}. \tag{1}$$

As $\mathbf{E}$ is commutative algebra, we obtain

$$\mathbf{E}^k = \sum_{i=1}^{\lfloor k/2\rfloor}\mathbf{E}^i\mathbf{E}^{k-i},$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Definition 1. *An evolution algebra $\mathbf{E}$ is called nilpotent if some $m\in\mathbb{N}$ such that $\mathbf{E}^m=\mathbf{0}$. The smallest m such that $\mathbf{E}^m=\mathbf{0}$ is called the index of nilpotency.*

Theorem 1. *[12] An n-dimensional evolution algebra $\mathbf{E}$ is nilpotent iff it admits a natural basis such that the matrix of the structural constants corresponding to $\mathbf{E}$ on this basis is represented in the form*

$$\tilde{A} = \begin{pmatrix} 0 & \tilde{a}_{12} & \tilde{a}_{13} & \vdots & \tilde{a}_{1n} \\ 0 & 0 & \tilde{a}_{23} & \vdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & \tilde{a}_{n-1,n} \\ 0 & 0 & 0 & \vdots & 0 \end{pmatrix}$$

.

Due to Theorem 1, any nilpotent evolution algebra $\mathbf{E}$ with $\dim(\mathbf{E}^2) = n-2$ has the following form:

$$\mathbf{e}_i^2 = \begin{cases} \sum\limits_{j=i+1}^{n} a_{ij}\mathbf{e}_j, & i\leq n-2; \\ \mathbf{0}, & i\in\{n-1,n\}. \end{cases} \tag{2}$$

where $a_{ij}\in\mathbb{K}$ and $a_{i,i+1}\neq 0$ for any $i<n-1$.

Theorem 2. *[11] Let $\mathbf{E}$ be a nilpotent evolution algebra. Then, $\mathbf{E}$ has maximal index of nilpotency $2^{(n-2)}+1$ if and only if the multiplication table of $\mathbf{E}$ is given by (2).*

In the following, we will work with nilpotent evolution algebras with $2^{(n-2)}+1$ index of nilpotency. Due to the last theorem, we only consider evolution algebras with the multiplication table given by (2).

Lemma 1. *Let* $\mathbf{E}_1$, $\mathbf{E}_2$ *be two isomorphic evolution algebras. Then,* $Der(\mathbf{E}_1)\cong Der(\mathbf{E}_2)$.

Lemma 2. *Let* $\mathbf{E}$ *and* $\mathbf{E}'$ *be evolution algebras with basis* $\{\mathbf{e}_i\}_{i=1}^n$ *and* $\{\mathbf{f}_i\}_{i=1}^n$ *respectively, defined by*

$$\mathbf{e}_i^2=\begin{cases} a_{i,i+1}\mathbf{e}_{i+1}+a_{in-1}\mathbf{e}_{n-1}+a_{in}\mathbf{e}_n, & i<n-1;\\ \mathbf{0}, & i\in\{n-1,n\}.\end{cases}\qquad \mathbf{f}_i^2=\begin{cases}\mathbf{f}_{i+1}, & i<n-1;\\ \mathbf{0}, & i\in\{n-1,n\}.\end{cases}$$

If $a_{i,i+1}\neq 0$ *for every* $i<n-1$, *then* $\mathbf{E}\cong\mathbf{E}'$.

Proof. Let $a_{i,i+1}\neq 0$ for every $i<n-1$. If $n=3$ after changing the basis $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ to $\mathbf{f}_1=\mathbf{e}_1$, $\mathbf{f}_2=\mathbf{e}_1^2$, and $\mathbf{f}_3=\mathbf{e}_3$, we immediately get $\mathbf{E}'$.

So. let us suppose $n\geq 4$. Then, the linear mapping $\varphi:\mathbf{E}\to\mathbf{E}'$ defined by

$$\varphi:\begin{cases}\mathbf{f}_1=\mathbf{e}_1\\ \mathbf{f}_2=\mathbf{e}_1^2\\ \mathbf{f}_{i+1}=\prod_{k=1}^{i-1}a_{k,k+1}^{2^{i-k}}\mathbf{e}_i^2, \quad 2\leq i<n-1\\ \mathbf{f}_n=\mathbf{e}_n\end{cases}\tag{3}$$

is an isomorphism from $\mathbf{E}$ to $\mathbf{E}'$.

3. Derivations

In this section, we consider derivations of nilpotent evolution algebras with $2^{n-2}+1$ index of nilpotency.

Recall that derivation of an evolution algebra $\mathbf{E}$ is a linear mapping $d:\mathbf{E}\to\mathbf{E}$ such that $d(\mathbf{uv})=d(\mathbf{u})\mathbf{v}+\mathbf{u}d(\mathbf{v})$ for all $\mathbf{u},\mathbf{v}\in\mathbf{E}$.

We note that for any algebra, the space $Der(\mathbf{E})$ of all derivations is a Lie algebra w.r.t. the commutator multiplication:

$$[d_1,d_2]=d_1d_2-d_2d_1,\quad \forall d_1,d_2\in Der(\mathbf{E}).$$

For a given structural matrix $A=(a_{ij})_{i,j\geq 1}^n$ of nilpotent evolution algebra $\mathbf{E}$ with $dim(\mathbf{E}^2)=n-2$, we denote

$$I_A=\{(i,j): i+1<j<n-1,\ a_{ij}\neq 0\}.\tag{4}$$

Theorem 3. *Let* $\mathbf{E}$ *be an evolution algebra with structural matrix* $A=(a_{ij})_{i,j\geq 1}^n$ *in a natural basis* $\{\mathbf{e}_i\}_{i=1}^n$. *If* $\mathbf{E}$ *is a nilpotent with* $rankA=n-2$, *then the following statements hold* *(i) if $I_A \neq \emptyset$, then*

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} 0 & 0 & \dots & d_{1n-1} & d_{1n} \\ 0 & 0 & \dots & d_{2n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1n-1} & d_{n-1n} \\ 0 & 0 & \dots & d_{nn-1} & d_{nn} \end{pmatrix} \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= -a_{n-2,n} d_{n,n-1}; \\ d_{n-1,n} &= -a_{n-2,n} d_{nn}; \\ d_{im} &= -\sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,m}, \ m \in \{n-1, n\} \end{aligned}$$

(ii) if $I_A = \emptyset$, then

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & \beta & \gamma \\ 0 & 2\alpha & \dots & d_{2,n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-2,n-1} & d_{n-2,n} \\ 0 & 0 & \dots & d_{n-1,n-1} & d_{n-1,n} \\ 0 & 0 & \dots & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K} \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= 2^{n-2}\alpha - a_{n-2,n} s \\ d_{n-1,n} &= (2^{n-2} - t) a_{n-2,n} \\ d_{i,n-1} &= (2^{i-1} - 2^{n-2}) \alpha a_{i-1,n-1} + (a_{i-1,n-1} a_{n-2,n} - a_{i-1,n}) s \\ d_{i,n} &= a_{i-1,n-1} d_{n-1,n} + a_{n-2,n}(2^{i-1}\alpha - t), \quad 2 \leq i < n-1. \end{aligned}$$

Proof. The (i) and (ii) are easy to check for $n = 3, 4$. Thus, we consider only the case $n > 4$. Let d be a derivation. We represent d in a matrix form based on $\{\mathbf{e}_i\}_{i=1}^n$ as follows: $d(\mathbf{e}_i) = \sum_{j=1}^n d_{ij} \mathbf{e}_j$. Then, we have $d_{ji} \mathbf{e}_i^2 + d_{ij} \mathbf{e}_j^2 = \mathbf{0}$ for all $1 \leq i < j \leq n$. As $\mathbf{e}_i^2$ and $\mathbf{e}_j^2$ are linearly independent, then $d_{ij} = d_{ji} = 0$ for any $1 \leq i < j < n-1$. If we take $m \in \{n-1, n\}$, then considering that $\mathbf{e}_{n-1}^2 = \mathbf{e}_n^2 = \mathbf{0}$ from $d_{mi} \mathbf{e}_i^2 + d_{im} \mathbf{e}_m^2 = \mathbf{0}$ one has $d_{mi} = 0$ for any $i < m$.

Thus, we have shown the following:

$$d_{ij} = 0, \quad \text{if } i \neq j, \ i \leq n, \ j < n-1. \tag{5}$$

On the other hand, we have $d(\mathbf{e}_i^2) = 2d_{ii}\mathbf{e}_i^2$ for any $i \leq n$. Then, for $i = n-2$ using (2), we obtain $d(\mathbf{e}_{n-1} + a_{n-2,n}\mathbf{e}_n) = 2d_{n-2,n-2}\mathbf{e}_{n-2}^2$. Then, we have the following system:

$$\begin{aligned} d_{n-1,n-1} + a_{n-2,n}d_{n,n-1} &= 2d_{n-2,n-2} \\ d_{n-1,n} + a_{n-2,n}d_{n,n} &= 2a_{n-2,n}d_{n-2,n-2}. \end{aligned} \tag{6}$$

Furthermore, we assume that $i < n-2$. Then, one finds

$$\begin{aligned} d(\mathbf{e}_i^2) &= d\left(\sum_{j=i+1}^{n} a_{ij}\mathbf{e}_j\right) = \sum_{j=i+1}^{n} a_{ij}d(\mathbf{e}_j) \\ &= \sum_{j=i+1}^{n-2} a_{ij}d_{jj}\mathbf{e}_j + \sum_{j=i+1}^{n} a_{ij}d_{j,n-1}\mathbf{e}_{n-1} + \sum_{j=i+1}^{n} a_{ij}d_{jn}\mathbf{e}_n. \end{aligned} \tag{7}$$

On the other hand, from

$$d(\mathbf{e}_i^2) = 2d_{ii}\mathbf{e}_i^2 = 2d_{ii}\sum_{j=i+1}^{n} a_{ij}\mathbf{e}_j$$

with (7), one finds

$$2d_{ii} = d_{i+1,i+1}, \quad 1 \leq i < n-2 \tag{8}$$

$$a_{ij}d_{jj} = 2a_{ij}d_{ii}, \quad i+2 \leq j \leq n-2 \tag{9}$$

$$\sum_{j=i+1}^{n} a_{ij}d_{j,n-1} = 2d_{ii}a_{i,n-1}, \quad 1 \leq i < n-2 \tag{10}$$

$$\sum_{j=i+1}^{n} a_{ij}d_{jn} = 2d_{ii}a_{in}, \quad 1 \leq i < n-2. \tag{11}$$

From (8),(9), we can easily derive

$$d_{jj} = 2^{j-1}d_{11}, \qquad 2 \leq j \leq n-2 \tag{12}$$

$$a_{ij}d_{11} = 0, \quad i+2 \leq j \leq n-1. \tag{13}$$

Now, we consider (10), (11).

$$\begin{aligned} d_{i+1,n-1} &= 2a_{i,n-1}d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i}d_{k+i,n-1} \\ d_{i+1,n} &= 2a_{i,n}d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i}d_{k+i,n}. \end{aligned} \tag{14}$$

Thus, from (5),(6),(12),(13) and (14), we conclude that d is a derivation of evolution algebra given by (2) if and only if

$$d_{ij} = d_{n-1,i} = d_{ni} = 0, \quad 1 \leq i \neq j \leq n-2 \tag{15}$$

$$d_{jj} = 2^{j-1} d_{11}, \quad 2 \leq j \leq n-2 \tag{16}$$

$$a_{ij} d_{11} = 0, \quad i+2 \leq j \leq n-2 \tag{17}$$

$$d_{i+1,n-1} = 2a_{i,n-1} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n-1} \tag{18}$$

$$d_{i+1,n} = 2a_{i,n} d_{ii} - \sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,n}. \tag{19}$$

Case $I_A \neq \emptyset$. In this case, we have $a_{i_0 j_0} \neq 0$ for a pair (i_0, j_0) that satisfies $i_0 + 2 \leq j_0 < n-1$. Then, from (17), one finds $d_{11} = 0$. Plugging this fact into (16),(18), and (19), we obtain

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} 0 & 0 & \dots & d_{1n-1} & d_{1n} \\ 0 & 0 & \dots & d_{2n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-1n-1} & d_{n-1n} \\ 0 & 0 & \dots & d_{nn-1} & d_{nn} \end{pmatrix} \right\}$$

where

$$\begin{aligned} d_{n-1,n-1} &= -a_{n-2,n} d_{n,n-1}; \\ d_{n-1,n} &= -a_{n-2,n} d_{nn}; \\ d_{im} &= -\sum_{k=1}^{n-i} a_{i-1,k+i} d_{k+i,m}, \; m \in \{n-1, n\}. \end{aligned}$$

Case $I_A = \emptyset$. In this case, (17) is true for any $d_{11} \in \mathbb{K}$. Thus, from (15),(16), (18), and (19), we conclude that

$$Der(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & \beta & \gamma \\ 0 & 2\alpha & \dots & d_{2,n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & d_{n-2,n-1} & d_{n-2,n} \\ 0 & 0 & \dots & d_{n-1,n-1} & d_{n-1,n} \\ 0 & 0 & \dots & s & t \end{pmatrix} : \; \alpha, \beta, s, t \in \mathbb{K} \right\}$$

where

$$d_{n-1,n-1} \;=\; 2^{n-2}\alpha - a_{n-2,n} s$$

$$\begin{aligned} d_{n-1,n} &= (2^{n-2}-t)a_{n-2,n} \\ d_{i,n-1} &= (2^{i-1}-2^{n-2})\alpha a_{i-1,n-1} + (a_{i-1,n-1}a_{n-2,n} - a_{i-1,n})s \\ d_{i,n} &= a_{i-1,n-1}d_{n-1,n} + a_{n-2,n}(2^{i-1}\alpha - t), \quad 2 \le i < n-1. \end{aligned}$$

The proof is complete.

Remark 1. *i. In [25], It has been considered the nilpotent evolution algebras with maximal nil index and they found* $1 \le \dim Der(\mathbf{E}) \le 2$.

ii. From the proved theorem, we infer that $1 \le \dim Der(\mathbf{E}) \le 5$. *This type of result can be proved using the work of Jacobson [21]. However, the advantage of Theorem 3 is that it fully describes the structure of the derivations on a natural basis.*

Corollary 1. *Lie algebras*

$$\mathbf{E} = \left\{ \begin{pmatrix} \alpha & 0 & \vdots & \beta & \gamma \\ 0 & 2\alpha & \vdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & 2^{n-2}\alpha & 0 \\ 0 & 0 & \vdots & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K} \right\}$$

and

$$\mathbf{E}' = \left\{ \begin{pmatrix} \alpha & 0 & \vdots & \beta & \gamma \\ 0 & 2\alpha & \vdots & d_{2,n-1} & d_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & d_{n-1,n-1} & d_{n-1,n} \\ 0 & 0 & \vdots & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K} \right\}$$

are isomorphic for any $d_{i,n-1}, d_{i,n} \in \mathbb{K}, i = \overline{2, n-1}$.

Remark 2. *We stress that isomorphisms of Lie algebras do not imply isomorphism of the corresponding evolution algebras (see lemma 1).*

4. Local and 2-local derivations for evolution algebras

The results of section 3 allow us to describe local and 2-local derivations of nilpotent evolution algebra. In this section, we want to fully describe local and 2-local derivations of nilpotent evolution algebras with $2^{n-2}+1$ index of nilpotency.

Recall that a linear mapping Δ on $\mathbf{E}$ is called *local derivation* if for every $\mathbf{u} \in \mathbf{E}$, a derivation $d_{\mathbf{u}}$ exists such that $\Delta(\mathbf{u}) = d_{\mathbf{u}}(\mathbf{u})$. A mapping (not necessary linear) $D : \mathbf{E} \to \mathbf{E}$ is called 2-*local derivation* of algebra $\mathbf{E}$ if for every $\mathbf{u}, \mathbf{v} \in \mathbf{E}$ there exists a derivation $d_{\mathbf{u},\mathbf{v}}$ of $\mathbf{E}$ such that $D(\mathbf{u}) = d_{\mathbf{u},\mathbf{v}}(\mathbf{u})$ and $D(\mathbf{v}) = d_{\mathbf{u},\mathbf{v}}(\mathbf{v})$. Therefore, it is natural to find all local derivations of $\mathbf{E}$.

Theorem 4. *Let* $\mathbf{E}$ *be an* n*-dimensional nilpotent evolution algebra with* $2^{(n-2)}+1$ *index of nilpotency. Then, the following statements hold:*

(*i*) *If* $n = 3$*, then the space of all local derivations has the following form:*

$$\left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \delta & 0 \\ 0 & s & t \end{pmatrix} : \alpha, \beta, \gamma, \delta, s, t \in \mathbb{K} \right\}. \tag{20}$$

(*ii*) *If* $n > 3$*, then every local derivation of* $\mathbf{E}$ *is a derivation.*

Proof. (*i*) Let $n = 3$. Due to Lemma 2, we may assume that an evolution algebra $\mathbf{E}$ is given by $\mathbf{e}_1^2 = \mathbf{e}_2$ and $\mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{0}$. Take an arbitrary linear map Δ on $\mathbf{E}$, i.e.,

$$\Delta(\mathbf{u}) = (\Delta_{11}u_1+\Delta_{21}u_2+\Delta_{31}u_3)\mathbf{e}_1+(\Delta_{12}u_1+\Delta_{22}u_2+\Delta_{32}u_3)\mathbf{e}_2+(\Delta_{13}u_1+\Delta_{23}u_2+\Delta_{33}u_3)\mathbf{e}_3,$$

$$\forall \mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3.$$

If Δ is a local derivation, then for any $\mathbf{u}$, there exist $\alpha_{\mathbf{u}}, \beta_{\mathbf{u}}, s_{\mathbf{u}}$, and $t_{\mathbf{u}}$ such that

$$\begin{aligned} &\Delta_{11}u_1 + \Delta_{21}u_2 + \Delta_{31}u_3 = \alpha_{\mathbf{u}}u_1 \\ &\Delta_{12}u_1 + \Delta_{22}u_2 + \Delta_{32}u_3 = \beta_{\mathbf{u}}u_1 + 2\alpha_{\mathbf{u}}u_2 + s_{\mathbf{u}}u_3 \\ &\Delta_{13}u_1 + \Delta_{23}u_2 + \Delta_{33}u_3 = \gamma_{\mathbf{u}}u_1 + t_{\mathbf{u}}u_3; \end{aligned}$$

From the first equation, we get $\Delta_{21} = \Delta_{31} = 0$. If we take $\mathbf{u}$ such that $u_1 = u_3 = 0$, then from the second equation, we immediately find $\Delta_{22} \in \{0, 2\Delta_{11}\}$. we find that Δ is a derivation of $\mathbf{E}$ if $\Delta_{22} = 2\Delta_{11}$.

Suppose $\Delta_{22} = 0$ and $\Delta_{11} \neq 0$. Then, for every $\mathbf{u}$, we can find that derivation $d_{\mathbf{u}}$ satisfies $\Delta(\mathbf{u}) = d_{\mathbf{u}}(\mathbf{u})$ as follows:

$$d_{\mathbf{u}} = \begin{cases} \begin{pmatrix} \Delta_{11} & \Delta_{12} - \frac{2\Delta_{11}u_2}{u_1} & \Delta_{13} \\ 0 & 2\Delta_{11} & 0 \\ 0 & \Delta_{32} & \Delta_{33} \end{pmatrix}, & \text{if } u_1, u_3 \neq 0, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \text{if } u_1 = u_3 = 0. \end{cases}$$

This result means that a linear mapping defined by

$$\Delta = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K} \tag{21}$$

is a local derivation of $\mathbf{E}$. Finally, as every derivation of algebra $\mathbf{E}$ is local derivation and due to (21), one obtains (20).

(ii) Let Δ be a non-zero local derivation given by matrix $(\Delta_{ij})_{i,j\geq 1}^{n}$. Assume that $I_A \neq \emptyset$. Then, due to $\Delta(\mathbf{e}_i) = d_{\mathbf{e}_i}(\mathbf{e}_i)$, for any $i \leq n$, we immediately obtain

$$\Delta_{1,n-1} = d_{1,n-1}^{(\mathbf{e_1})}, \quad \Delta_{1n} = d_{1,n}^{(\mathbf{e_1})}; \tag{22}$$

$$\Delta_{im} = d_{im}^{(\mathbf{e_i})}, \quad m \in \{n-1, n\},\ 2 \leq i < n-1; \tag{23}$$

$$\Delta_{n-1,n-1} = d_{n-1,n-1}^{(\mathbf{e_{n-1}})}, \quad \Delta_{n-1,n} = d_{n-1,n}^{(\mathbf{e_{n-1}})}; \tag{24}$$

$$\Delta_{n,n-1} = d_{n,n-1}^{(\mathbf{e_n})}, \quad \Delta_{n,n} = d_{nn}^{(\mathbf{e_n})}; \tag{25}$$

$$\Delta_{ij} = 0, \quad otherwise. \tag{26}$$

Taking $\mathbf{u} = \sum_{k=2}^{n-2} \mathbf{e}_k$. we obtain $\Delta_{im} = -\sum_{k=1}^{n-i} a_{i-1,k+i}\Delta_{k+i,m}$. We consider that, $\mathbf{v} = \mathbf{e_{n-1}} + a_{n-2,n}\mathbf{e_n}$, and then a derivation $d_{\mathbf{v}}$ exists such that $\Delta(\mathbf{v}) = \mathbf{d_v}(\mathbf{v})$. Thus,

$$\begin{aligned}&(-a_{n-2,n}d_{n,n-1}^{(\mathbf{e_{n-1}})} + a_{n-2,n}d_{n,n-1}^{(\mathbf{e_n})})\mathbf{e_{n-1}} + (-a_{n-2,n}d_{n,n}^{(\mathbf{e_{n-1}})} + a_{n-2,n}d_{nn}^{(\mathbf{e_n})})\mathbf{e_n}\\ &= (-a_{n-2,n}d_{n,n-1}^{(\mathbf{u})} + a_{n-2,n}d_{n,n-1}^{(\mathbf{u})})\mathbf{e_{n-1}} + (-a_{n-2,n}d_{n,n}^{(\mathbf{u})} + a_{n-2,n}d_{nn}^{(\mathbf{u})})\mathbf{e_n}.\end{aligned}$$

This result implies $d_{n,n-1}^{(\mathbf{e_{n-1}})} = d_{n,n-1}^{(\mathbf{e_n})}$, $\ d_{n,n}^{(\mathbf{e_{n-1}})} = d_{nn}^{(\mathbf{e_n})}$. Therefore, one has $\Delta_{n-1,n-1} = -a_{n-1,n}\Delta_{n,n-1}$. Suppose that $I_A = \emptyset$.We establish that $\Delta \in Der(\mathbf{E})$ for any local derivation Δ. Due to Lemmas 2 and 1, to show every local derivation can be a derivation we need to check only for evolution algebra $\mathbf{E}'$ (see Lemma 2).

As $\Delta(\mathbf{e}_i) = d_{\mathbf{e}_i}(\mathbf{e}_i)$, for any $i \leq n$, we can easily find

$$\begin{array}{ll}\Delta_{ii} = d_{ii}^{(\mathbf{e}_i)}, & i \leq n-1\\ \Delta_{1,n-1} = d_{1,n-1}^{(\mathbf{e}_1)} & \\ \Delta_{1n} = d_{1n}^{(\mathbf{e}_1)} & \\ \Delta_{n,n-1} = d_{n,n-1}^{(\mathbf{e}_n)} & \\ \Delta_{n,n} = d_{n,n}^{(\mathbf{e}_n)} & \\ \Delta_{ij} = 0, & \text{otherwise.}\end{array} \tag{27}$$

Taking $\mathbf{u} = \sum_{k=1}^{n-2} \mathbf{e}_k$, we obtain

$$\Delta_{ii} = 2^{i-1}\Delta_{11}, \quad i < n-1. \tag{28}$$

Consider $\mathbf{v} = \mathbf{e}_2 + \mathbf{e}_{n-1}$. Then, a derivation $d_{\mathbf{v}}$ exists such that $\Delta(\mathbf{v}) = d_{\mathbf{v}}(\mathbf{v})$. Due to the assumption (ii) of Theorem 3, we have

$$\Delta_{22}\mathbf{e}_2 + \Delta_{n-1,n-1}\mathbf{e}_n = 2d_{11}^{(\mathbf{v})}\mathbf{e}_2 + 2^{n-2}d_{11}^{(\mathbf{v})}\mathbf{e}_n.$$

This result implies

$$\begin{aligned}&2d_{11}^{(\mathbf{v})} = \Delta_{22}\\ &2^{n-2}d_{11}^{(\mathbf{v})} = \Delta_{n-1,n-1}.\end{aligned}$$

Adding the preceding equations into (28), we obtain $\Delta_{ii} = 2^{i-1}\Delta_{11}$. Then, using (27), one finds

$$\begin{array}{ll} \Delta_{ii} = d_{ii}^{(\mathbf{e}_1)}, & i \leq n-1 \\ \Delta_{1,n-1} = d_{1,n-1}^{(\mathbf{e}_1)} & \\ \Delta_{1n} = d_{1n}^{(\mathbf{e}_1)} & \\ \Delta_{n,n-1} = d_{n,n-1}^{(\mathbf{e}_n)} & \\ \Delta_{n,n} = d_{n,n}^{(\mathbf{e}_n)} & \\ \Delta_{ij} = 0, & \text{otherwise.} \end{array}$$

Thus, due to Theorem 3, we conclude that Δ is a derivation. The proof is complete.

Remark 3. *In [25], it was proven that if $n > 2$ then all local derivation is derivation, in the above theorem we find the if $n > 3$ then all local derivation is derivation.*

Theorem 5. *Every 2-local derivation of nilpotent evolution algebras with $2^{n-2}+1$ index of nilpotency is a derivation.*

Proof. Let D be a non-zero 2-local derivation of $\mathbf{E}$. Denote $\Gamma_1 = \{\mathbf{u} \in \mathbf{E} : u_1 \neq 0\}$, $\Gamma_2 = \{\mathbf{u} \in \mathbf{E} : u_n \neq 0\}$.

Case $I_A = \emptyset$. By definition, functionals $\alpha_{\mathbf{u},\mathbf{v}}, \beta_{\mathbf{u},\mathbf{v}}, \gamma_{\mathbf{u},\mathbf{v}}, s_{\mathbf{u},\mathbf{v}}$ and $t_{\mathbf{u},\mathbf{v}}$ exist such that

$$\begin{aligned} D(\mathbf{u}) &= \sum_{k=1}^{n-2} 2^{k-1}\alpha_{\mathbf{u},\mathbf{v}}u_k\mathbf{e}_k + \\ &\left(\beta_{\mathbf{u},\mathbf{v}}u_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - M_{n-1}s_{\mathbf{u},\mathbf{v}}\right)u_{n-1} + s_{\mathbf{u},\mathbf{v}}u_n + \sum_{i=2}^{n-2}\left(K_i\alpha_{\mathbf{u},\mathbf{v}} + M_i s_{\mathbf{u},\mathbf{v}}\right)u_i\right)\mathbf{e_{n-1}} \\ &+ \left(\gamma_{\mathbf{u},\mathbf{v}}u_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - N_{n-1}t_{\mathbf{u},\mathbf{v}}\right)u_{n-1} + t_{\mathbf{u},\mathbf{v}}u_n + \sum_{i=2}^{n-2}\left(L_i\alpha_{\mathbf{u},\mathbf{v}} + N_i t_{\mathbf{u},\mathbf{v}}\right)u_i\right)\mathbf{e_n} \\ D(\mathbf{v}) &= \sum_{k=1}^{n-2} 2^{k-1}\alpha_{\mathbf{u},\mathbf{v}}v_k\mathbf{e}_k + \\ &\left(\beta_{\mathbf{u},\mathbf{v}}v_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - M_{n-1}s_{\mathbf{u},\mathbf{v}}\right)v_{n-1} + s_{\mathbf{u},\mathbf{v}}v_n + \sum_{i=2}^{n-2}\left(K_i\alpha_{\mathbf{u},\mathbf{v}} + M_i s_{\mathbf{u},\mathbf{v}}\right)v_i\right)\mathbf{e_{n-1}} \\ &+ \left(\gamma_{\mathbf{u},\mathbf{v}}v_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - N_{n-1}t_{\mathbf{u},\mathbf{v}}\right)v_{n-1} + t_{\mathbf{u},\mathbf{v}}v_n + \sum_{i=2}^{n-2}\left(L_i\alpha_{\mathbf{u},\mathbf{v}} + N_i t_{\mathbf{u},\mathbf{v}}\right)v_i\right)\mathbf{e_n} \end{aligned} \tag{29}$$

where $\mathbf{u} = \sum_{k=1}^{n} u_k\mathbf{e}_k$ and $\mathbf{v} = \sum_{k=1}^{n} v_k\mathbf{e}_k$. Take an arbitrary non-zero $\mathbf{u} \in \mathbf{E}$. Then, for any $\mathbf{v}, \mathbf{v'} \in \mathbf{E}$ from the preceding equations, we find

$$
\begin{aligned}
&\sum_{k=1}^{n-2} 2^{k-1}\alpha_{\mathbf{u},\mathbf{v}} u_k \mathbf{e}_k + \\
&\left(\beta_{\mathbf{u},\mathbf{v}} u_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - M_{n-1}s_{\mathbf{u},\mathbf{v}}\right) u_{n-1} + s_{\mathbf{u},\mathbf{v}} u_n + \sum_{i=2}^{n-2} \left(K_i\alpha_{\mathbf{u},\mathbf{v}} + M_i s_{\mathbf{u},\mathbf{v}}\right) u_i\right) \mathbf{e_{n-1}} \\
&+ \left(\gamma_{\mathbf{u},\mathbf{v}} u_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - N_{n-1}t_{\mathbf{u},\mathbf{v}}\right) u_{n-1} + t_{\mathbf{u},\mathbf{v}} u_n + \sum_{i=2}^{n-2} \left(L_i\alpha_{\mathbf{u},\mathbf{v}} + N_i t_{\mathbf{u},\mathbf{v}}\right) u_i\right) \mathbf{e_n} \\
&= \sum_{k=1}^{n-2} 2^{k-1}\alpha_{\mathbf{u},\mathbf{v'}} u_k \mathbf{e}_k + \\
&\left(\beta_{\mathbf{u},\mathbf{v'}} u_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v'}} - M_{n-1}s_{\mathbf{u},\mathbf{v'}}\right) u_{n-1} + s_{\mathbf{u},\mathbf{v'}} u_n + \sum_{i=2}^{n-2} \left(K_i\alpha_{\mathbf{u},\mathbf{v'}} + M_i s_{\mathbf{u},\mathbf{v'}}\right) u_i\right) \mathbf{e_{n-1}} \\
&+ \left(\gamma_{\mathbf{u},\mathbf{v'}} u_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v'}} - N_{n-1}t_{\mathbf{u},\mathbf{v'}}\right) u_{n-1} + t_{\mathbf{u},\mathbf{v'}} u_n + \sum_{i=2}^{n-2} \left(L_i\alpha_{\mathbf{u},\mathbf{v'}} + N_i t_{\mathbf{u},\mathbf{v'}}\right) u_i\right) \mathbf{e_n},
\end{aligned} \tag{30}
$$

which is equivalent to

$$
\begin{aligned}
&\alpha_{\mathbf{u},\mathbf{v}} u_k = \alpha_{\mathbf{u},\mathbf{v'}} u_k, \quad k = \overline{1, n-2} \\
&\left(\beta_{\mathbf{u},\mathbf{v}} u_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - M_{n-1}s_{\mathbf{u},\mathbf{v}}\right) u_{n-1} + s_{\mathbf{u},\mathbf{v}} u_n + \sum_{i=2}^{n-2} \left(K_i\alpha_{\mathbf{u},\mathbf{v}} + M_i s_{\mathbf{u},\mathbf{v}}\right) u_i\right) = \\
&\left(\beta_{\mathbf{u},\mathbf{v'}} u_1 + \left(K_{n-1}\alpha_{\mathbf{u},\mathbf{v'}} - M_{n-1}s_{\mathbf{u},\mathbf{v'}}\right) u_{n-1} + s_{\mathbf{u},\mathbf{v'}} u_n + \sum_{i=2}^{n-2} \left(K_i\alpha_{\mathbf{u},\mathbf{v'}} + M_i s_{\mathbf{u},\mathbf{v'}}\right) u_i\right) \\
&\left(\gamma_{\mathbf{u},\mathbf{v}} u_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v}} - N_{n-1}t_{\mathbf{u},\mathbf{v}}\right) u_{n-1} + t_{\mathbf{u},\mathbf{v}} u_n + \sum_{i=2}^{n-2} \left(L_i\alpha_{\mathbf{u},\mathbf{v}} + N_i t_{\mathbf{u},\mathbf{v}}\right) u_i\right) = \\
&\left(\gamma_{\mathbf{u},\mathbf{v'}} u_1 + \left(L_{n-1}\alpha_{\mathbf{u},\mathbf{v'}} - N_{n-1}t_{\mathbf{u},\mathbf{v'}}\right) u_{n-1} + t_{\mathbf{u},\mathbf{v'}} u_n + \sum_{i=2}^{n-2} \left(L_i\alpha_{\mathbf{u},\mathbf{v'}} + N_i t_{\mathbf{u},\mathbf{v'}}\right) u_i\right).
\end{aligned}
$$

As, $\mathbf{u} \neq \mathbf{0}$, we obtain $\alpha_{\mathbf{u},\mathbf{v}} = \alpha_{\mathbf{u},\mathbf{v'}}$ for any $\mathbf{v}, \mathbf{v'} \in \mathbf{E}$. This result means that

$$\alpha_{\mathbf{u},\mathbf{v}} =: \alpha_{\mathbf{u}}. \tag{31}$$

Moreover, if $\mathbf{u} \in \Gamma$, then one finds

$$\beta_{\mathbf{u},\mathbf{v}} =: \beta_{\mathbf{u}},\ \gamma_{\mathbf{u},\mathbf{v}} =: \gamma_{\mathbf{u}}. \tag{32}$$

Now, if $\mathbf{u} \in \mathbf{\Gamma_2}$, then one finds

$$s_{\mathbf{u},\mathbf{v}} =: s_{\mathbf{u}}, \quad t_{\mathbf{u},\mathbf{v}} =: t_{\mathbf{u}}. \tag{33}$$

Taking (31),(32),(33) into (36), we conclude that mapping D can be defined as

$$D(\mathbf{u}) = \begin{cases} \sum_{k=1}^{n-2} 2^{k-1}\alpha_{\mathbf{u}} u_k \mathbf{e}_k + \\ \left(\beta_{\mathbf{u}} u_1 + (K_{n-1}\alpha_{\mathbf{u}} - M_{n-1}s_{\mathbf{u}})\, u_{n-1} + s_{\mathbf{u}} u_n + \sum_{i=2}^{n-2} (K_i\alpha_{\mathbf{u}} + M_i s_{\mathbf{u}})\, u_i\right) \mathbf{e_{n-1}} \\ + \left(\gamma_{\mathbf{u}} u_1 + (L_{n-1}\alpha_{\mathbf{u}} - N_{n-1}t_{\mathbf{u}})\, u_{n-1} + t_{\mathbf{u}} u_n + \sum_{i=2}^{n-2} (L_i\alpha_{\mathbf{u}} + N_i t_{\mathbf{u}})\, u_i\right) \mathbf{e_n} \\ \text{if } \mathbf{u} \in \Gamma_1 \cup \Gamma_2 \\ \sum_{k=2}^{n-2} 2^{k-1}\alpha_{\mathbf{u}} u_k \mathbf{e}_k + \left((K_{n-1}\alpha_{\mathbf{u}} - M_{n-1}s_{\mathbf{u}})\, u_{n-1} + \sum_{i=2}^{n-2} (K_i\alpha_{\mathbf{u}} + M_i s_{\mathbf{u}})\, u_i\right) \mathbf{e_{n-1}} \\ + \left((L_{n-1}\alpha_{\mathbf{u}} - N_{n-1}t_{\mathbf{u}})\, u_{n-1} + \sum_{i=2}^{n-2} (L_i\alpha_{\mathbf{u}} + N_i t_{\mathbf{u}})\, u_i\right) \mathbf{e_n} \text{ if } \mathbf{u} \notin \mathbf{\Gamma_1} \cup \mathbf{\Gamma_2}. \end{cases} \tag{34}$$

Then, for any $\mathbf{u}', \mathbf{v}' \in \mathbf{E}$, we can find derivation d given by

$$d_{ij} = \begin{cases} 2^{i-1}\alpha, & \text{if } 1 \le i = j < n-1 \\ \beta, & \text{if } i = 1, j = n-1 \\ \gamma, & \text{if } i = 1, j = n \\ s, & \text{if} i = n, j = n-1 \\ t, & \text{if} i = n, j = n \\ K_{n-1}\alpha - M_{n-1}s, & \text{if } i = n-1, j = n-1 \\ L_{n-1}\alpha - N_{n-1}t, & \text{if } i = n-1, j = n \\ K_i\alpha + N_i s, & \text{if } 2 \le i \le n-2,\ j = n-1 \\ L_i\alpha + N_i t, & \text{if } 2 \le i \le n-2,\ j = n \\ 0, & \text{otherwise} \end{cases}$$

such that $D(\mathbf{u}') = d(\mathbf{u}')$, $D(\mathbf{v}') = d(\mathbf{v}')$. Then, from (34) one obtains

$$\alpha_{\mathbf{u}'} = \alpha_{\mathbf{v}'} = \alpha \quad \text{for any } \mathbf{u}', \mathbf{v}' \in \mathbf{E}. \tag{35}$$

This result means that functional $\alpha_{\mathbf{u}}$ is a constant.

To complete the proof, we show $\beta_{\mathbf{u}}, \gamma_{\mathbf{u}}, s_{\mathbf{u}}$, and $t_{\mathbf{u}}$ are constants for any $\mathbf{u} \in \mathbf{E}$. We consider non-zero points $\mathbf{u}, \mathbf{v} \in \Gamma_1 \cup \Gamma_2$. Using the first equality of (34) and noting (35) by definition of 2-local derivation, we obtains $\beta_{\mathbf{u}} = \beta_{\mathbf{v}}, \gamma_{\mathbf{u}} = \gamma_{\mathbf{v}}, s_{\mathbf{u}} = s_{\mathbf{v}}$, and $t_{\mathbf{u}} = t_{\mathbf{v}}$. This result means that $\beta_{\mathbf{u}}, \gamma_{\mathbf{u}}, s_{\mathbf{u}}$ and $t_{\mathbf{u}}$ do not depend on $\mathbf{u}$, i.e., $\beta_{\mathbf{u}} = \beta, \gamma_{\mathbf{u}} = \gamma, s_{\mathbf{u}} = s$ and $t_{\mathbf{u}} = t$ for any $\mathbf{u} \in \mathbf{\Gamma_1} \cup \mathbf{\Gamma_2}$. Placing this result and (35) into (34) yields D, which has the following form:

$$
\begin{aligned}
D(\mathbf{u}) = &\sum_{\mathbf{k=1}}^{\mathbf{n-2}} \mathbf{2^{k-1}} \alpha \mathbf{u_k e_k} + \\
&\left(\beta u_1 + (K_{n-1}\alpha - M_{n-1}s)\, u_{n-1} + s u_n + \sum_{i=2}^{n-2} (K_i \alpha + M_i s)\, u_i\right) \mathbf{e_{n-1}} \\
&+ \left(\gamma u_1 + (L_{n-1}\alpha - N_{n-1}t)\, u_{n-1} + t u_n + \sum_{i=2}^{n-2} (L_i \alpha + N_i t)\, u_i\right) \mathbf{e_n}.
\end{aligned} \tag{36}
$$

Due to Theorem 3 (ii), D is a derivation.

Case $I_A \neq \emptyset$. By definition, there exist functionals $\beta_{\mathbf{u},\mathbf{v}}, \gamma_{\mathbf{u},\mathbf{v}}, s_{\mathbf{u},\mathbf{v}}$ and $t_{\mathbf{u},\mathbf{v}}$ such that

$$
\begin{aligned}
D(\mathbf{u}) &= \left(\beta_{\mathbf{u},\mathbf{v}} u_1 + s_{\mathbf{u},\mathbf{v}} u_n + s_{\mathbf{u},\mathbf{v}} \sum_{i=2}^{n-2} d_i u_i\right) \mathbf{e_{n-1}} + \left(\gamma_{\mathbf{u},\mathbf{v}} u_1 + t_{\mathbf{u},\mathbf{v}} u_n + t_{\mathbf{u},\mathbf{v}} \sum_{i=2}^{n-2} d_i u_i\right) \mathbf{e_n} \\
D(\mathbf{v}) &= \left(\beta_{\mathbf{u},\mathbf{v}} v_1 + s_{\mathbf{u},\mathbf{v}} v_n + s_{\mathbf{u},\mathbf{v}} \sum_{i=2}^{n-2} d_i v_i\right) \mathbf{e_{n-1}} + \left(\gamma_{\mathbf{u},\mathbf{v}} v_1 + t_{\mathbf{u},\mathbf{v}} v_n + t_{\mathbf{u},\mathbf{v}} \sum_{i=2}^{n-2} d_i v_i\right) \mathbf{e_n}
\end{aligned} \tag{37}
$$

where $\mathbf{u} = \sum_{k=1}^{n} u_k \mathbf{e}_k$ and $\mathbf{v} = \sum_{k=1}^{n} v_k \mathbf{e}_k$.

Take arbitrary $\mathbf{u} \in \Gamma_1 \cup \Gamma_2$. Then from the first equation of (37) we obtain $\beta_{\mathbf{u},\mathbf{v}} = \beta_{\mathbf{u},\mathbf{v'}}, s_{\mathbf{u},\mathbf{v}} = s_{\mathbf{u},\mathbf{v'}}$, $t_{\mathbf{u},\mathbf{v}} = t_{\mathbf{u},\mathbf{v'}}$ for any $\mathbf{v}, \mathbf{v'} \in \mathbf{E}$. This result means that $\beta_{\mathbf{u},\mathbf{v}}, t_{\mathbf{u},\mathbf{v}}$, and $t_{\mathbf{u},\mathbf{v}}$ do not depend on $\mathbf{v}$, i.e., $\beta_{\mathbf{u},\mathbf{v}} = \beta_{\mathbf{u}}, s_{\mathbf{u},\mathbf{v}} = s_{\mathbf{u}}, t_{\mathbf{u},\mathbf{v}} = t_{\mathbf{u}}$, $\forall \mathbf{u} \in \Gamma_1 \cup \Gamma_2$. On the other hand, from the second equation of (37) we obtains $\beta_{\mathbf{u},\mathbf{v}} = \beta_{\mathbf{v}}, s_{\mathbf{u},\mathbf{v}} = s_{\mathbf{v}}$ and $t_{\mathbf{u},\mathbf{v}} = t_{\mathbf{v}}$ for any $\mathbf{v} \in \Gamma_1 \cup \Gamma_2$. These facts yield that $\beta_{\mathbf{u}} =: \beta, s_{\mathbf{u}} =: s$ and $t_{\mathbf{u}} =: t$ for any $\mathbf{u}, \mathbf{v} \in \mathbf{E}$. Consequently, we have

$$
D(\mathbf{u}) = \left(\beta u_1 + s u_n + s \sum_{i=2}^{n-2} d_i u_i\right) \mathbf{e_{n-1}} + \left(\gamma u_1 + t u_n + t \sum_{i=2}^{n-2} d_i u_i\right) \mathbf{e_n}.
$$

Due to Theorem 3 (i), we obtain $D \in Der(\mathbf{E})$.

5. Automorphisms and local automorphisms

Recall that by an *automorphism* of an evolution algebra $\mathbf{E}$, we mean an isomorphism of $\mathbf{E}$ into itself. The set of all automorphisms is denoted by $Aut(\mathbf{E})$. It is known that $Aut(\mathbf{E})$ is a group. In this section, to describe $Aut(\mathbf{E})$ of nilpotent evolution algebras with maximal index of nilpotency.

If $I_A \neq \emptyset$, then by η we denote the largest common divisor of all numbers $2^{j-1} - 2^i$ where $(i, j) \in I_A$, i.e.,

$$
\eta = LCD_{(i,j) \in I_A}(2^{j-1} - 2^i). \tag{38}
$$

Theorem 6. *Let* $\mathbf{E}$ *be an* n*-dimensional nilpotent evolution algebra with* $2^{n-2} + 1$ *index of nilpotency and* $A = (a_{ij})_{i,j=1}^n$ *be its structural matrix in a natural basis* $\{\mathbf{e}_i\}_{i=1}^n$*. Then, the following statements hold:*

(*i*) *if* $I_A \neq \emptyset$ *then*

$$Aut(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & 0 & \beta & \gamma \\ 0 & \alpha^2 & \dots & 0 & \varphi_{2,n-1} & \varphi_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & \alpha^{2^{n-2}} & \varphi_{n-2,n-1} & \varphi_{n-2,n} \\ 0 & 0 & \dots & 0 & \varphi_{n-1,n-1} & \varphi_{n-1,n} \\ 0 & 0 & \dots & 0 & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K}, \ \alpha^{\eta} = 1 \right\}$$

where η *is defined as* (38), *and* $\varphi_{in-1}, \varphi_{in}$ *is given by the following recurrence formula.*

$$\begin{aligned} \varphi_{n-1,n-1} &= \alpha^{2^{n-2}} - a_{n-2,n}s, \\ \varphi_{n-1,n} &= a_{n-2,n}\big(\alpha^{2^{n-2}} - t\big), \\ \varphi_{i,n-1} &= a_{i-1,n-1}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n-1}, \quad 1 < i < n-1, \\ \varphi_{i,n} &= a_{i-1,n}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n}, \quad 1 < i < n-1. \end{aligned}$$

(*ii*) *if* $I_A = \emptyset$ *then*

$$Aut(\mathbf{E}) = \left\{ \begin{pmatrix} \alpha & 0 & \dots & 0 & \beta & \gamma \\ 0 & \alpha^2 & \dots & 0 & \varphi_{2,n-1} & \varphi_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & \alpha^{2^{n-2}} & \varphi_{n-2,n-1} & \varphi_{n-2,n} \\ 0 & 0 & \dots & 0 & \varphi_{n-1,n-1} & \varphi_{n-1,n} \\ 0 & 0 & \dots & 0 & s & t \end{pmatrix} : \alpha, \beta, \gamma, s, t \in \mathbb{K}, \ \alpha \neq 0 \right\}$$

where $\varphi_{in-1}, \varphi_{in}$ *is given by the following recurrence formula:*

$$\begin{aligned} \varphi_{n-1,n-1} &= \alpha^{2^{n-2}} - a_{n-2,n}s, \\ \varphi_{n-1,n} &= a_{n-2,n}\big(\alpha^{2^{n-2}} - t\big), \\ \varphi_{i,n-1} &= a_{i-1,n-1}\big(\alpha^{2^{i-1}} - \varphi_{n-1,n-1}\big) - a_{i-1,n}s, \quad 1 < i < n-1, \\ \varphi_{i,n} &= a_{i-1,n}\big(\alpha^{2^{i-1}} - t\big) - a_{i-1,n-1}\varphi_{n-1,n}, \quad 1 < i < n-1. \end{aligned}$$

Proof. Let φ be a linear mapping on $\mathbf{E}$. Now, we represent φ on the basis elements as follows:

$$\varphi(\mathbf{e}_i) = \sum_{j=1}^{n} \varphi_{ij}\mathbf{e}_j, \quad 1 \leq i \leq n.$$

We want to describe matrix $(\varphi_{ij})_{i,j=1}^n$ when φ is an automorphism of $\mathbf{E}$. Suppose that φ is an automorphism. Then, we have

$$\begin{aligned} \varphi(\mathbf{e}_i)\varphi(\mathbf{e}_j) &= \mathbf{0}, \quad i \neq j \\ \varphi(\mathbf{e}_i^2) &= [\varphi(\mathbf{e}_i)]^2, \quad 1 \leq i \leq n \end{aligned}$$

which is equivalent to the followings:

$$\sum_{k=1}^{n-1} \varphi_{ik}\varphi_{jk}\mathbf{e}_k^2 = \mathbf{0}, \quad i \neq j \tag{39}$$

$$\sum_{j=i+1}^{n} a_{ij} \sum_{k=1}^{n} \varphi_{jk}\mathbf{e}_k = \sum_{k=1}^{n-1} \varphi_{ik}^2 \mathbf{e}_k^2, \quad i \leq n-2 \tag{40}$$

$$a_{n-1,n}(\sum_{k=1}^{n} \varphi_{nk}\mathbf{e}_k) = \sum_{k=1}^{n-1} \varphi_{n-1,k}^2 \mathbf{e}_k^2, \tag{41}$$

$$\sum_{k=1}^{n-1} \varphi_{nk}^2 \mathbf{e}_k^2 = 0. \tag{42}$$

The linear independence of $\{\mathbf{e}_1^2, \mathbf{e}_2^2, \cdots, \mathbf{e}_{n-2}^2\}$ together with (39),(42) implies

$$\varphi_{ik}\varphi_{jk} = 0, \quad i \neq j, \ k \leq n-2 \tag{43}$$

$$\varphi_{n-1,k} = \varphi_{nk} = 0, \qquad k \leq n-2. \tag{44}$$

We find that $\varphi_{n-2,n-2} \neq 0$. Plugging (44) into (41), we find

$$\begin{aligned} &\varphi_{n-2,n-2} = \varphi_{n-3,n-3}^2 \\ &\varphi_{n-3,k} = 0, \qquad\qquad k \leq n-3 \end{aligned} \tag{45}$$

Inserting $\mathbf{e}_l^2 = \sum_{j=l+1}^{n} a_{lj}\mathbf{e}_j, \ l \leq n-2$ into (40), we obtain

$$\sum_{j=i+1}^{n} a_{ij}\varphi_{jl} = \sum_{j=1}^{l-1} a_{jl}\varphi_{ij}^2, \quad i \leq n-2, \ l \geq 2 \tag{46}$$

$$\sum_{j=i+1}^{n} a_{ij}\varphi_{j1} = 0, \qquad i \leq n-2 \tag{47}$$

We claim:

$$\begin{aligned} &\varphi_{il} = 0, \qquad\qquad l+1 \leq i \\ &\varphi_{j+1,j+1} = \varphi_{jj}^2, \quad j \leq n-2. \end{aligned} \tag{48}$$

Let us prove the last relations by induction. Due to (44),(45), the first step is satisfied. We take an arbitrary $i_0 > 1$, and assume that for any $i > i_0$, assertion (48) holds.

We must prove that $\varphi_{i_0 l} = 0$ for any $l \leq i_0 - 1$ and $\varphi_{i_0 i_0} = \varphi_{i_0-1,i_0-1}^2$. Rewriting (46) for $i = i_0 > 1$, we find

$$\sum_{j=i_0+1}^{n} a_{i_0 j}\varphi_{jl} = \sum_{j=1}^{l-1} a_{jl}\varphi_{i_0 j}^2, \quad l \geq 2 \tag{49}$$

If $j > i_0$, then due to the assumption, we have $\varphi_{jl} = 0$ for any $l \leq i_0$. As, for any $l \leq i_0$ the left side of (49) is equals to zero. Thus,

$$\sum_{j=1}^{l-1} \varphi_{i_0 j}^2 a_{jl} = 0, \quad 2 \leq l \leq i_0. \tag{50}$$

If $l = 2$ then from (50) we obtain $\varphi_{i_0 1} = 0$. Suppose that $\varphi_{i_0,l} = 0$ for every $l < l_0 \leq i_0$. Then this fact together with (50) for $l = l_0$ implies $\varphi_{i_0,l_0} = 0$. Thus, we have shown that $\varphi_{i_0,l} = 0$ for every $l \leq i_0$. From the arbitrary-ness of $i_0 > 1$, we conclude that

$$\varphi_{il} = 0, \quad l + 1 < i. \tag{51}$$

On the other hand, rewriting (46) for $l = i + 1$ and keeping in mind (51), we obtain

$$\varphi_{i+1,i+1} = \varphi_{ii}^2, \quad i \leq n - 2.$$

The last equality yields $\varphi_{i+1,i+1} = \varphi_{ii}^2$ for every $i \leq n-2$. This together with (45) implies

$$\varphi_{ii} = \varphi_{11}^{2^{i-1}} \neq 0, \quad i \leq n - 2. \tag{52}$$

Thus, from (51) and (52), it follows (48).

Plugging (51) into (43), we obtain

$$\varphi_{ij} = 0, \quad i < j < n - 1. \tag{53}$$

Let us consider (46) for $l > i + 1$. Then, for every $i \leq n - 2$, we obtain

$$a_{il}\varphi_{ll} = a_{il}\varphi_{ii}^2, \quad i + 1 < l < n - 1 \tag{54}$$

$$\sum_{j=i+1}^{n} a_{ij}\varphi_{j,n-1} = a_{i,n-1}\varphi_{ii}^2, \quad l = n - 1. \tag{55}$$

$$\sum_{j=i+1}^{n} a_{ij}\varphi_{jn} = a_{in}\varphi_{ii}^2, \quad l = n. \tag{56}$$

From (56) with (52) we obtain a recurrence formula for $\varphi_{in-1}, \varphi_{in}$ as follows:

$$\begin{aligned}
&\varphi_{n-1,n-1} = \alpha^{2^{n-2}} - a_{n-2,n}s, \\
&\varphi_{n-1,n} = a_{n-2,n}\left(\alpha^{2^{n-2}} - t\right), \\
&\varphi_{i,n-1} = a_{i-1,n-1}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n-1}, \quad 1 < i < n - 1, \\
&\varphi_{i,n} = a_{i-1,n}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n}, \qquad 1 < i < n - 1.
\end{aligned} \tag{57}$$

Hence, we infer that φ is an automorphism of evolution algebra (2) if and only if the followings holds:

$$\begin{aligned}
&\varphi_{ij} = 0, && i \neq j,\ j < n-1\\
&\varphi_{ii} = \varphi_{11}^{2^{i-1}}, && i \leq n-2\\
&a_{il}\varphi_{ll} = a_{il}\varphi_{ii}^2, && i+1 < l < n-1\\
&\varphi_{n-1,n-1} = \alpha^{2^{n-2}} - a_{n-2,n}s, &&\\
&\varphi_{n-1,n} = a_{n-2,n}\big(\alpha^{2^{n-2}} - t\big), &&\\
&\varphi_{i,n-1} = a_{i-1,n-1}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n-1}, && 1 < i < n-1,\\
&\varphi_{i,n} = a_{i-1,n}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n}, && 1 < i < n-1.
\end{aligned} \tag{58}$$

Now let us consider two cases w.r.t.I_A.

Case $I_A \neq \emptyset$. For the sake of convenience, we denote $\varphi_{11} = \alpha \neq 0$. Then, from (58) one obtains

$$\begin{aligned}
&\varphi_{ij} = \varphi_{ji} = 0, && i \neq j,\ j < n-1\\
&\varphi_{ii} = \alpha^{2^{i-1}}, && i \leq n-2\\
&\alpha^{2^{l-1}-2^i} = 1, && (i,l) \in I_A\\
&\varphi_{1,n-1} = \gamma,\ \varphi_{1n} = \beta,\ \varphi_{n,n-1} = s,\ \varphi_{n,n} = t &&\\
&\varphi_{n-1,n-1} = \alpha^{2^{n-2}} - a_{n-2,n}s, &&\\
&\varphi_{n-1,n} = a_{n-2,n}\big(\alpha^{2^{n-2}} - t\big), &&\\
&\varphi_{i,n-1} = a_{i-1,n-1}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n-1}, && 1 < i < n-1,\\
&\varphi_{i,n} = a_{i-1,n}\alpha^{2^{i-1}} - \sum_{j=i+1}^{n-i} a_{i-1,j}\varphi_{j,n}, && 1 < i < n-1.
\end{aligned}$$

where $\alpha, \beta, \gamma, s, t \in \mathbb{K}$, and $\alpha^{\eta} = 1$, which implies the assertion.

Case $I_A = \emptyset$. For the automorphism φ, we have

$$\begin{aligned}
&\varphi_{ij} = \varphi_{ji} = 0, && i \neq j,\ j < n-1\\
&\varphi_{ii} = \alpha^{2^{i-1}}, && 1 \leq i \leq n-2\\
&\varphi_{1,n-1} = \gamma,\ \varphi_{1n} = \beta,\ \varphi_{n,n-1} = s,\ \varphi_{n,n} = t &&\\
&\varphi_{n-1,n-1} = \alpha^{2^{n-2}} - a_{n-2,n}s, &&\\
&\varphi_{n-1,n} = a_{n-2,n}\big(\alpha^{2^{n-2}} - t\big), &&\\
&\varphi_{i,n-1} = a_{i-1,n-1}\big(\alpha^{2^{i-1}} - \varphi_{n-1,n-1}\big) - a_{i-1,n}s, && 1 < i < n-1,\\
&\varphi_{i,n} = a_{i-1,n}\big(\alpha^{2^{i-1}} - t\big) - a_{i-1,n-1}\varphi_{n-1,n}, && 1 < i < n-1.
\end{aligned}$$

where $\alpha, \beta, \gamma, s, t \in \mathbb{K}$, which implies the assertion.

The proof is complete.

5.1. Local automorphisms of evolution algebras

In the previous section, we have been able to find the set of all automorphisms of evolution algebra (2). Now, we show that every local automorphism is an automorphism if evolution algebra is defined by (2) with $n > 3$. Recall that a linear mapping ψ from $\mathbf{E}$ to $\mathbf{E}$ is called *local automorphism* if for every $\mathbf{u} \in \mathbf{E}$ there exists an automorphism $\varphi_{\mathbf{u}} \in Aut(\mathbf{E})$ such that $\psi(\mathbf{u}) = \varphi_{\mathbf{u}}(\mathbf{u})$.

Theorem 7. *Let* $\mathbf{E}$ *be an* n*-dimensional nilpotent evolution algebra with* $2^{n-2}+1$ *index of nilpotency. Then, the following statements hold:*

(*i*) *If* $n = 3$*, then the set of all local automorphisms has the following form:*

$$\left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & l^2 & 0 \\ 0 & s & T \end{pmatrix} : \alpha, \beta, \gamma, \ l, s, t \in \mathbb{K}, \ \alpha l \neq 0 \right\}. \tag{59}$$

(*ii*) *If* $n > 3$*, then every local automorphism of* $\mathbf{E}$ *is an automorphism.*

Proof. (*i*) Let $n = 3$. Due to Lemma 2, we may assume that an evolution algebra $\mathbf{E}$ is given by $\mathbf{e}_1^2 = \mathbf{e}_2$ and $\mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{0}$. Take an arbitrary linear map ψ on $\mathbf{E}$, i.e.,

$$\begin{aligned} \psi(\mathbf{u}) \ &= \ (\psi_{11}u_1 + \psi_{21}u_2 + \psi_{31}u_3)\mathbf{e}_1 + (\psi_{12}u_1 + \psi_{22}u_2 + \psi_{32}u_3)\mathbf{e}_2 + (\psi_{13}u_1 + \psi_{23}u_2 + \psi_{33}u_3)\mathbf{e}_3 \\ & \quad \forall \mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3. \end{aligned}$$

If ψ is a local automorphism, then for any $\mathbf{u}$, there exist $\alpha_{\mathbf{u}}, \beta_{\mathbf{u}}, \gamma_{\mathbf{u}}, s_{\mathbf{u}}$ and $t_{\mathbf{u}}$, such that

$$\begin{aligned} \psi_{11}u_1 + \psi_{21}u_2 + \psi_{31}u_3 &= \alpha_{\mathbf{u}}u_1 \\ \psi_{12}u_1 + \psi_{22}u_2 + \psi_{32}u_3 &= \beta_{\mathbf{u}}u_1 + \alpha_{\mathbf{u}}^2 u_2 + s_{\mathbf{u}}u_3 \\ \psi_{13}u_1 + \psi_{23}u_2 + \psi_{33}u_3 &= \gamma_{\mathbf{u}}u_1 + t_{\mathbf{u}}u_3. \end{aligned}$$

From the first equation, we obtain $\psi_{21} = \psi_{31} = 0$, and from the third equation, we have $\psi_{23} = 0$. If we take $\mathbf{u}$ such that $u_1 = u_3 = 0$, then from the second equation, we immediately find $\psi_{22} = \alpha_{\mathbf{u}}^2$. It yields that if ψ is a local automorphism, it has the following form:

$$\left\{ \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & l^2 & 0 \\ 0 & s & T \end{pmatrix} : \alpha, \beta, \gamma, \ l, s, t \in \mathbb{K}, \ \alpha l \neq 0 \right\} \tag{60}$$

We show that (60) is indeed a local automorphism of (2). In fact, for any $\mathbf{u} \in \mathbf{E}$, we may take an automorphism $\varphi_{\mathbf{u}}$ of (2) as follows:

$$\varphi_{\mathbf{u}} = \begin{cases} \begin{pmatrix} \alpha & \beta + \frac{(l^2-\alpha^2)u_2}{u_1} & \gamma \\ 0 & \alpha^2 & 0 \\ 0 & s & t \end{pmatrix}, & \text{if } u_1u_3 \neq 0 \\ \begin{pmatrix} l & 0 & 0 \\ 0 & l^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \text{if } u_1 = u_3 = 0 \end{cases}$$

From this, one can check that $\psi(\mathbf{u}) = \varphi_{\mathbf{u}}(\mathbf{u})$.

(ii) Let $n > 3$. Let ψ be a local automorphism for (2). By definition of local automorphism, for every $\mathbf{u} \in \mathbf{E}$ we have $\psi(\mathbf{u}) = \varphi_{\mathbf{u}}(\mathbf{u})$, where $\varphi_{\mathbf{u}}$ is an automorphism. Then, theorem 6 implies $\psi_{ij} = 0$ for every $i \neq j$, $j < n-1$. On the other hand, taking $\mathbf{u} = \mathbf{e}_i$, $i \leq n$, we conclude that the local automorphism ψ has the following form:

$$\psi = \begin{pmatrix} \alpha_{\mathbf{e}_1} & 0 & 0 & \vdots & 0 & \beta_{\mathbf{e}_1} & \gamma\mathbf{e_1} \\ 0 & \alpha^2_{\mathbf{e}_2} & 0 & \vdots & 0 & \varphi^{(\mathbf{e}_2)}_{2,n-1} & \varphi^{(\mathbf{e}_2)}_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \vdots & \alpha^{2^{n-2}}_{\mathbf{e}_{n-2}} & \varphi^{(\mathbf{e}_{n-2})}_{n-2,n-1} & \varphi^{(\mathbf{e}_{n-2})}_{n-2,n} \\ 0 & 0 & 0 & \vdots & 0 & \varphi^{(\mathbf{e}_{n-1})}_{n-1,n-1} & \varphi^{(\mathbf{e}_{n-1})}_{n-1,n} \\ 0 & 0 & 0 & \vdots & 0 & s_{\mathbf{e}_n} & t_{\mathbf{e}_n} \end{pmatrix}$$

Now, we take arbitrary $\mathbf{v} = \sum_{i=1}^{n} v_i\mathbf{e}_i$. Then, from $\psi(\mathbf{v}) = \varphi_{\mathbf{v}}(\mathbf{v})$, we obtain

$$\alpha^{2^{i-1}}_{\mathbf{e}_i} v_i = \alpha^{2^{i-1}}_{\mathbf{v}} v_i, \quad i < n-1 \tag{61}$$

$$\beta_{\mathbf{e}_1} v_1 + s_{\mathbf{e}_n} v_n + \sum_{k=2}^{n-1} \varphi^{(\mathbf{e}_k)}_{k,n-1} v_k = \beta_{\mathbf{v}} v_1 + s_{\mathbf{v}} v_n + \sum_{k=2}^{n-1} \varphi^{(\mathbf{v})}_{k,n-1} v_k \tag{62}$$

$$\gamma_{\mathbf{e}_1} v_1 + t_{\mathbf{e}_n} v_n + \sum_{k=2}^{n-1} \varphi^{(\mathbf{e}_k)}_{k,n} v_k = \beta_{\mathbf{v}} v_1 + t_{\mathbf{v}} v_n + \sum_{k=2}^{n-1} \varphi^{(\mathbf{v})}_{k,n} v_k. \tag{63}$$

From (61) we find

$$\alpha^{2^{i-1}}_{\mathbf{e}_i} = \alpha^{2^{i-1}}_{\mathbf{e}_1}, \qquad i < n-1 \tag{64}$$

Consequently, $\varphi^{(\mathbf{e}_k)}_{k,n-1} = \varphi^{(\mathbf{e}_1)}_{k,n-1}$ and $\varphi^{(\mathbf{e}_k)}_{kn} = \varphi^{(\mathbf{e}_1)}_{kn}$ for any $k < n$. Based on this fact, the following is obtained from (62) and (63),

$$\gamma_{\mathbf{e}_1} v_1 + s_{\mathbf{e}_n} v_n = \gamma_{\mathbf{v}} v_1 + s_{\mathbf{v}} v_n, \quad \beta_{\mathbf{e}_1} v_1 + t_{\mathbf{e}_n} v_n = \beta_{\mathbf{v}} v_1 + s_{\mathbf{v}} v_n. \tag{65}$$

Finally, taking $\mathbf{v}' = \mathbf{e}_2 + \mathbf{e}_n$, we obtain $\gamma_{\mathbf{e}_1} = \gamma_{\mathbf{v}}$, $\beta_{\mathbf{e}_1} = \beta_{\mathbf{v}}$, $s_{\mathbf{e}_n} = s_{\mathbf{v}}$, and $t_{\mathbf{e}_n} = t_{\mathbf{v}}$, for any $\mathbf{v} \in \mathbf{E}$.

Thus, we conclude that local automorphism $\psi = (\varphi_{ij})$ has the following form:

$$\varphi_{ij} = \begin{cases} \alpha_{\mathbf{e}_1}^{2^{i-1}}, & i = j > n-1 \\ \gamma_{\mathbf{e}_1}, & i = 1,\ j = n-1 \\ \beta_{\mathbf{e}_1}, & i = 1,\ j = n \\ \varphi_{i,n-1}^{(\mathbf{e}_1)}, & i > 1,\ j = n-1 \\ \varphi_{in}^{(\mathbf{e}_1)}, & i > 1,\ j = n \\ s_{\mathbf{e}_1}, & i = n,\ j = n-1 \\ t_{\mathbf{e}_1}, & i = n,\ j = n \\ 0, & \text{otherwise.} \end{cases}$$

Thus, theorem 6 implies that the local automorphism ψ is an automorphism. The proof is complete.

Remark 4. *In [25], it was proven that if $n > 2$ then all local automorphism is automorphism, in the above theorem we find that if $n > 3$ then all local automorphism is automorphism.*

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