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Forcing Subsets for $\gamma^*_{t_{pw}}$-sets in Graphs

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Abstract. In this paper, the lower and upper bounds of the forcing total dr-power domination number of any graph are determined. Total dr-power domination number of some special graphs such as complete graphs, star, fan and wheel graphs are shown. Moreover, the forcing total dr-power domination number of these graphs, together with paths and cycles, are determined.

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1. Introduction

Let $G = (V, E)$ be a graph representing the electrical power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. In order to monitor the power system, some measurement devices must be placed at selected locations so that all the state variables of the system can be measured. A Phase Measurement Unit (PMU) is a measurement device placed on a vertex and has the ability to measure the state of the vertex and the edges connected to the vertex. The vertices and edges that are measured by PMU's are said to be observed. In this study, it is necessary that each vertex with PMU is adjacent to another vertex with PMU also. But because of the high cost value of a PMU, it is desirable to minimize their number while maintaining the ability to monitor the entire power system.

All graphs considered in this study are simple, undirected and without loops or multiple edges.

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v in G is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For $X \subseteq V(G)$, the *open neighborhood* of X is the set $N(X) = \cup_{v \in X} N_G(v)$ and its *closed neighborhood* is the set $N[X] = N(X) \cup X$.

A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The *domination number* $\gamma(G)$ (resp. *total domination number*

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$\gamma_t(G)$) of G is the minimum cardinality of a dominating set (resp. total dominating set). If S is a dominating set (resp. a total dominating set) with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$), then we call S a γ-set (resp. a γ_t-set) of G.

Let $G = (V, E)$ be a simple graph. Let $P \subseteq V(G)$. An edge $e = uv$ of G is directly observed by P if $u \in P$ or $v \in P$. A vertex u of G is directly observed if u is incident to a directly observed edge. An edge $e' = xy$ is remotely observed by P if $x, y \notin P$ and x, y are directly observed vertices or at least one of x and y is incident to k edges where $k - 1$ of these edges are directly observed by P. Clearly, k is a positive integer, $k > 1$, and k is not constant for any pair of vertices x and y. A non-directly observed vertex u of G which is incident to a remotely observed edge is called remotely observed vertex. Let $O_V^P(G)$ be the set of all directly and remotely observed vertices and $O_E^P(G)$ be the set of all directly and remotely observed edges. Then $P \subseteq V(G)$ is a *dr*-power dominating set (*dr*-pds) of G if $O_V^P(G) = V(G)$ and $O_E^P(G) = E(G)$. The minimum cardinality of a *dr*-power dominating set is called the *dr-power domination number of* G and is denoted by $\gamma_{pw}^*(G)$. A subset P of $V(G)$ with cardinality $\gamma_{pw}^*(G)$ is called a γ_{pw}^**-set of* G. A *dr*-power dominating set D is said to be a total *dr*-power dominating set(t*dr*-pds) if the induced subgraph $\langle D \rangle$ has no isolated vertex. The minimum cardinality of a total *dr*-power dominating set (t*dr*-pds) is called the *total dr-power domination number of* G and is denoted by $\gamma_{t_{pw}}^*(G)$. A subset T of $V(G)$ with cardinality $\gamma_{t_{pw}}^*(G)$ is called a $\gamma_{t_{pw}}^*$*-set of* G. Moreover, there exists a connected graph G such that $2 \leq \gamma_{t_{pw}}^*(G) \leq \gamma_t(G)$.

Let S be a $\gamma_{t_{pw}}^*$-set of a graph G. A subset D of S is said to be a *forcing subset* for S if S is the unique $\gamma_{t_{pw}}^*$-set containing D. The *forcing total dr-power domination number of* S is given by $f\gamma_{t_{pw}}^*(S) = \min\{|D| : D$ is a forcing subset for $S\}$. The *forcing total dr-power domination number of* G is given by

$$f\gamma_{t_{pw}}^*(G) = \min\{f\gamma_{t_{pw}}^*(S) : S \text{ is a } \gamma_{t_{pw}}^*\text{-set of } G\}.$$

The *join* of two graphs G and H, denoted by $G + H$ is the graph with vertex set

$$V(G + H) = V(G) \cup V(H)$$

and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The total domination is studied by Amos [1]. Chartrand et al. [5] investigated the relation between forcing and domination concepts and defined "forcing domination number". Canoy, et al studied the following concepts: total *dr*-power domination [6], forcing domination number of graphs under some binary operations [7], forcing total domination number and forcing connected domination number under the lexicographic product of graphs [8], forcing independent domination number of a graph [4], and A-differential of graphs [3]. Also, Armada [2] studied the forcing total *dr*-power domination of graphs under some binary operations.

Illustration 1.1. *Consider the cycle graph* $C_5 = [u_1, u_2, u_3, u_4, u_5, u_1]$. *Let* $P \subseteq V(C_5)$. *Pick* $u_2, u_3 \in P$. *Then* u_1u_2, u_2u_3 *and* u_3u_4 *are directly observed edges in* C_5. *Clearly,* u_1, u_2, u_3 *and* u_4 *are incident to a directly observed edge, and so,* u_1, u_2, u_3 *and* u_4 *are directly observed vertices. The edges* u_1u_5 *and* u_4u_5 *are remotely observed edges since* $u_1, u_4, u_5 \notin P$ *and there are* $k = 2$ *incident edges to the vertices* u_1 *and* u_4 *such that* $k - 1 = 2 - 1 = 1$ *edge is directly observed by* P *which are* u_1u_2 *and* u_3u_4. *Since* u_5 *is incident to a remotely observed edge* u_1u_5 *or* u_4u_5, *then* u_5 *is a remotely observed vertex. Clearly,* $O_V^P(C_5) = V(C_5)$ *and* $O_E^P(C_5) = E(C_5)$, *that is,* P *is a dr-power dominating set of* C_5. *Since the induced subgraph* $\langle P \rangle$ *has no isolated vertex,* P *is a total dr-power dominating set of* C_5. *Note that for any connected graph* G, $\gamma^*_{t_{pw}}(G) \geq 2$ *and since* $|P| = 2$, P *is a* $\gamma^*_{t_{pw}}$*-set of* C_5 *and* $\gamma^*_{t_{pw}}(C_5) = 2$. *Clearly, any pair of adjacent vertices in* C_5 *is a* $\gamma^*_{t_{pw}}$*-set of* C_5, *that is,* $S_1 = \{u_1, u_2\}$, $S_2 = P = \{u_2, u_3\}$, $S_3 = \{u_3, u_4\}$, $S_4 = \{u_4, u_5\}$, $S_5 = \{u_5, u_1\}$, *are the only* $\gamma^*_{t_{pw}}$*-sets of* C_5. *Clearly, for all* $i = 1, 2, \ldots, 5$, *no subset* $\{u_i\}$ *is contained in a unique* $\gamma^*_{t_{pw}}$*-set* S_j *for all* $j = 1, 2, \ldots, 5$ *and so,* $f\gamma^*_{t_{pw}}(S_j) \neq 1$. *Therefore, for all* $j = 1, 2, \ldots, 5$, $f\gamma^*_{t_{pw}}(S_j) = |S_j| = 2 = f\gamma^*_{t_{pw}}(C_5)$.

2. Known Results

This section contains known results involving *dr*-power domination, total domination and total *dr*-power domination numbers of a graph G that are useful in proving the main results of this study.

Remark 2.1. [6] For any graph G without isolated vertices,

$$\gamma^*_{pw}(G) \leq \gamma^*_{t_{pw}}(G) \leq \gamma_t(G).$$

Theorem 2.2. [6] Let n be a positive integer with $n \geq 5$. Then

$$\gamma^*_{t_{pw}}(P_n) = \begin{cases} \frac{2n}{5}, & n \equiv 0 (\text{mod } 5) \\ \frac{2n-2}{5}, & n \equiv 1 (\text{mod } 5) \\ \frac{2n+1}{5}, & n \equiv 2 (\text{mod } 5) \\ \frac{2n+4}{5}, & n \equiv 3 (\text{mod } 5) \\ \frac{2n+2}{5}, & n \equiv 4 (\text{mod } 5) \end{cases}$$

Theorem 2.3. [6] Let n be a positive integer with $n \geq 5$. Then

$$\gamma^*_{t_{pw}}(C_n) = \begin{cases} \frac{2n}{5}, & n \equiv 0 (\text{mod } 5) \\ \frac{2n+3}{5}, & n \equiv 1 (\text{mod } 5) \\ \frac{2n+6}{5}, & n \equiv 2 (\text{mod } 5) \\ \frac{2n+4}{5}, & n \equiv 3 (\text{mod } 5) \\ \frac{2n+2}{5}, & n \equiv 4 (\text{mod } 5) \end{cases}$$

Proposition 2.4. [1] *The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is given by*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0(mod\ 4), \\ \frac{n+2}{2}, & n \equiv 2(mod\ 4), \\ \frac{n+1}{2}, & otherwise. \end{cases}$$

Theorem 2.5. [6] Let G and H be any graphs. Then $P \subseteq V(G+H)$ is a total dr-power dominating set of $G+H$ if and only if it satisfies one of the following conditions:
(i) $P \subseteq V(G)$ and is a total dominating set, provided that G is a graph with no isolated vertex;
(ii) $P \subseteq V(H)$ and is a total dominating set, provided that H is a graph with no isolated vertex; or
(iii) $P = P_1 \cup P_2$, where $\varnothing \neq P_1 \subseteq V(G)$ and $\varnothing \neq P_2 \subseteq V(G)$.

Corollary 2.6. [6] Let G and H be any graphs. Then

$$\gamma^*_{t_{pw}}(G+H) = 2.$$

3. Main Results

This section contains the lower and upper bounds of $f\gamma^*_{t_{pw}}(G)$ and the forcing total dr-power domination number of some special graphs such as path, cycle, complete graph, fan, star and wheel graphs.

Theorem 3.1. *Let G be a graph. Then*

(i) $f\gamma^*_{t_{pw}}(G) = 0$ *if and only if G has a unique $\gamma^*_{t_{pw}}$-set.*

(ii) $f\gamma^*_{t_{pw}}(G) = 1$ *if and only if G has at least two $\gamma^*_{t_{pw}}$-sets and there exists a vertex v which is contained in exactly one $\gamma^*_{t_{pw}}$-set of G.*

Proof. (i) Suppose that $f\gamma^*_{t_{pw}}(G) = 0$. It follows that $\varnothing$ is the forcing subset for a $\gamma^*_{t_{pw}}$-set, say P, in G. Suppose that there is another $\gamma^*_{t_{pw}}$-set of G, say R. Note that $\varnothing$ is also a subset for R, a contradiction since $\varnothing$ is a forcing subset for P. Thus, G has a unique $\gamma^*_{t_{pw}}$-set. Conversely, if G has a unique $\gamma^*_{t_{pw}}$-set, say Q. Clearly, $\varnothing$ is a forcing subset for Q. Consequently, $|\varnothing| = 0 = f\gamma^*_{t_{pw}}(Q) = f\gamma^*_{t_{pw}}(G)$.
(ii) Suppose that $f\gamma^*_{t_{pw}}(G) = 1$. Hence, G has at least two $\gamma^*_{t_{pw}}$-sets by part (i) and there exists a $\gamma^*_{t_{pw}}$-set, say P, and $v \in P$ such that $\{v\}$ is a forcing subset for P and $f\gamma^*_{t_{pw}}(P) = |\{v\}| = 1$, that is, there exists a vertex which is contained in exactly one $\gamma^*_{t_{pw}}$-set of G. Conversely, if G has at least two $\gamma^*_{t_{pw}}$-sets, then $f\gamma^*_{t_{pw}}(G) > 0$ by part (i). By assumption, there exists a vertex, say x, which is contained in exactly one $\gamma^*_{t_{pw}}$-set of G, say T, that is, $\{x\}$ is a forcing subset for T. Therefore, $f\gamma^*_{t_{pw}}(T) = |\{x\}| = 1 = f\gamma^*_{t_{pw}}(G)$. □

The next result is a direct consequence of Theorem 3.1 and definition of forcing total dr-power domination.

Corollary 3.2. *Let G be a connected graph. Then*

$$0 \leq f\gamma^*_{t_{pw}}(G) \leq \gamma^*_{t_{pw}}(G).$$

Theorem 3.3. *Let G be a nontrivial graph. Then $f\gamma^*_{t_{pw}}(G) = \gamma^*_{t_{pw}}(G)$ if and only if for every $\gamma^*_{t_{pw}}$-set P of G and for each $v \in P$, there exists $u \in V(G)\backslash P$ such that $[P\backslash\{v\}] \cup \{u\}$ is a $\gamma^*_{t_{pw}}$-set of G.*

Proof. Suppose that $f\gamma^*_{t_{pw}}(G) = \gamma^*_{t_{pw}}(G)$. Let P be a $\gamma^*_{t_{pw}}$-set of G such that $f\gamma^*_{t_{pw}}(G) = |P| = \gamma^*_{t_{pw}}(G)$, that is, P is the only forcing subset for P. Let $v \in P$. Since $P\backslash\{v\}$ is not a forcing subset for P, there exists a $u \in V(G)\backslash P$ such that $[P\backslash\{v\}] \cup \{u\}$ is a $\gamma^*_{t_{pw}}$-set of G.

Conversely, suppose that every $\gamma^*_{t_{pw}}$-set P' of G satisfies the given condition. Let P be a $\gamma^*_{t_{pw}}$-set of G such that $f\gamma^*_{t_{pw}}(G) = f\gamma^*_{t_{pw}}(P)$. Moreover, suppose that P has a forcing subset R with $|R| < |P|$, that is, $P = R \cup S$, where $S = \{u \in P : u \notin R\}$. Pick $u \in S$. By assumption, there exists $v \in V(G)\backslash P$ such that $[P\backslash\{u\}] \cup \{v\} = Q$ is a $\gamma^*_{t_{pw}}$-set of G. Thus, $Q = R \cup T$, where $T = [S\backslash\{u\}] \cup \{v\}$, that is, Q is a $\gamma^*_{t_{pw}}$-set containing R, a contradiction. Thus, $|R| = |P|$ and P is the only forcing subset for P. Therefore, $f\gamma^*_{t_{pw}}(G) = |P| = \gamma^*_{t_{pw}}(G)$. □

Theorem 3.4. *Let n be a positive integer with $n \geq 5$. Then*

$$f\gamma^*_{t_{pw}}(P_n) = \begin{cases} 0, & n = 7 \text{ or } n \equiv 1(mod\ 5) \\ 2, & n \equiv 3(mod\ 5) \\ 1, & otherwise. \end{cases}$$

Proof. Let the path $P_n = [u_1, u_2, \ldots, u_n]$. Note that $deg(u_1) = 1 = deg(u_n)$ and $deg(u_i) = 2$ for all $i = 2, 3, \ldots, n-1$. By definition of total dr-power dominating set, a choosen vertex in a $\gamma^*_{t_{pw}}$-set, say D, of P_n must always have an adjacent vertex in D, say u_i and u_{i+1} and by Theorem 2.2, the choosen vertex u_{i+1} must be of at most distance 4 to the next choosen vertex in D since if the next vertex to be choosen is of distance 5, say $u_1, u_2 \in D$ and choose u_7 to be the next vertex, then u_3 and u_6 are directly observed vertices while u_4 and u_5 are remotely observed vertices but the edge u_4u_5 is neither directly nor remotely observed edge which is a contradiction. Also, the starting vertex of a $\gamma^*_{t_{pw}}$-set D, of P_n must be u_1, u_2 or u_3 since if it starts with u_4, then u_3 is a directly observed vertex while u_2 becomes a remotely observed vertex but the vertex u_1 is neither directly nor remotely observed vertex which is a contradiction. Now, consider the following cases:

Case 1: Suppose that $n = 7$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_7) = \frac{2(7)+1}{5} = 3$. Clearly, $S = \{u_3, u_4, u_5\}$ is the only $\gamma^*_{t_{pw}}$-set of P_7 since u_2 and u_6 are the directly observed vertices while u_1 and u_7 are the remotely observed vertices, that is, $O^S_V(P_7) = V(P_7)$, $O^S_E(P_7) = E(P_7)$ and the induced subgraph $\langle S \rangle$ has no isolated vertex. By Theorem 3.1(i), $f\gamma^*_{t_{pw}}(P_7) = 0$.

Case 2: Suppose that $n \equiv 1(\text{mod } 5)$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_n) = \frac{2n-2}{5}$. Let $n = 6$. Then $\gamma^*_{t_{pw}}(P_6) = \frac{2(6)-2}{5} = 2$. Clearly, $S = \{u_3, u_4\}$ is the only $\gamma^*_{t_{pw}}$-set of P_6 since u_2 and u_5 are the directly observed vertices while u_1 and u_6 are the remotely observed vertices, that is, $O^S_V(P_6) = V(P_6)$ and $O^S_E(P_6) = E(P_6)$ and the induced subgraph $\langle S\rangle$ has no isolated vertex. By Theorem 3.1(i), $f\gamma^*_{t_{pw}}(P_6) = 0$. Now, suppose that $n > 6$. Let $p = \frac{n-1}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of P_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1\}$
$R_1 = \{u_2, u_3, u_4, u_5, u_6\}$
$R_2 = \{u_7, u_8, u_9, u_{10}, u_{11}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 2, 7, 12, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_{i+1}, u_{i+2} form a total dr-power dominating set since u_i and u_{i+3} are directly observed vertices while u_1 and u_{i+4} are remotely observed vertices for all $i = 2, 7, 12, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_{i+1}, u_{i+2} : i = 2, 7, 12, \ldots, n-9, n-4\} \\ &= \{u_3, u_4, u_8, u_9, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\} \end{aligned}$$

where $|R| = 2p = \frac{2n-2}{5}$, $O^R_V(P_n) = V(P_n)$, $O^R_E(P_n) = E(P_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.2, R is a $\gamma^*_{t_{pw}}$-set of P_n. Note that the set R contains pairs of adjacent vertices in each R_j's except in R_0 and the distance between the last choosen vertex in R_j and the first choosen vertex in R_{j+1} is always 4. Let T be a $\gamma^*_{t_{pw}}$-set of P_n different from R. Consider the following subcases:

Subcase 1: Let $u_2, u_3 \in T$. Then the next vertex to be chosen must be of distance 4 from u_3, that is, the vertex u_7, together with u_8, must be in T. Thus, $u_i, u_{i+1} \in T$ for all $i = 2, 7, \ldots, n-9, n-4$. It follows that $T = \{u_2, u_3, u_7, u_8, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\}$ and $|T| = |R|$. Since u_{n-3} is the last vertex in T, the vertex u_{n-2} is a directly observed vertex and u_{n-1} is a remotely observed vertex. Hence, u_n is neither a directly or a remotely observed vertex. Thus, $O^T_V(P_n) \neq V(P_n)$, a contradiction, that is, T is not a $\gamma^*_{t_{pw}}$-set of P_n. Hence, it is not possible to start with the vertex u_2 to form a $\gamma^*_{t_{pw}}$-set T. Similarly, it is not possible to start with the vertex u_1.

Subcase 2: Suppose that $u_3, u_4 \in T$. Now, replace $u_8 \in R$ by u_7 to form T. Then the next vertex to be chosen must be u_8, that is, the vertex u_i, u_{i+1} must be in T for all $i = 7, 12, \ldots, n-9, n-4$. Then $T = \{u_3, u_4, u_7, u_8, u_{13}, u_{14}, \ldots, u_{n-4}, u_{n-3}\}$ and $|T| = |R|$. Since u_{n-3} is the last vertex in T, by previous subcase, T is not a $\gamma^*_{t_{pw}}$-set of P_n. Since u_8 is arbitrarily replaced from R, we cannot replace the vertex u_{i+1} in R, where $i = 7, 12, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of P_n.

Thus, either of the subcases, it is not possible to form another $\gamma^*_{t_{pw}}$-set T of P_n which is different from R. Therefore, R is a unique $\gamma^*_{t_{pw}}$-set of P_n. By Theorem 3.1(i), $f\gamma^*_{t_{pw}}(P_n) = 0$.

Case 3: Suppose that $n \neq 7$ and $n \equiv 2(\text{mod } 5)$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_n) = \frac{2n+1}{5}$. Let $n = 12$. Then $\gamma^*_{t_{pw}}(P_{12}) = \frac{2(12)+1}{5} = 5$. Clearly, $S_1 = \{u_3, u_4, u_5, u_9, u_{10}\}$ and $S_2 = \{u_3, u_4, u_8, u_9, u_{10}\}$ are the only $\gamma^*_{t_{pw}}$-sets of P_{12}. Since $u_5 \in S_1$ and $u_5 \notin S_2$, by Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_{12}) = 1$. Now, suppose that $n > 12$. Let $p = \frac{n-2}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of P_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1, u_2\}$
$R_1 = \{u_3, u_4, u_5, u_6, u_7\}$
$R_2 = \{u_8, u_9, u_{10}, u_{11}, u_{12}\}$
$R_3 = \{u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 3, 8, 13, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_3, u_{i+1}, u_{i+2} form a total *dr*-power dominating set since u_2, u_i and u_{i+3} are directly observed vertices while u_1 and u_{i+4} are remotely observed vertices for all $i = 3, 8, 13, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_3, u_{i+1}, u_{i+2} : i = 3, 8, 13, \ldots, n-9, n-4\} \\ &= \{u_3, u_4, u_5, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\} \end{aligned}$$

where $|R| = 2p+1 = 2\left(\frac{n-2}{5}\right) + 1 = \frac{2n+1}{5}$, $O^R_V(P_n) = V(P_n)$, $O^R_E(P_n) = E(P_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.2, R is a $\gamma^*_{t_{pw}}$-set of P_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of P_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of P_n different from R. Note that in forming R, there are three vertices in R_1 such that the induced subgraph is a graph P_3 and two adjacent vertices in the other R_j's with $j > 1$. Thus, T_k can be formed by getting 3 vertices in any one of the R_j's where $j > 1$ such that the induced subgraph is a graph P_3 and two adjacent vertices in the other R_l's where $l \neq j$ and $l \neq 0$. Consider the following subcases.

Subcase 1: Choose 3 vertices in R_2 to form another $\gamma^*_{t_{pw}}$ set, say T_1, that is, replaced $u_5 \in R_1$ in R by $u_8 \in R_2$. It follows that $T_1 = \{u_3, u_4, u_8, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$, where $|T_1| = |R|$, $O^{T_1}_V(P_n) = V(P_n)$, $O^{T_1}_E(P_n) = E(P_n)$, and the induced subgraph $\langle T_1\rangle$ has no isolated vertex, that is, T_1 is a $\gamma^*_{t_{pw}}$-set of P_n. Clearly, $u_5 \notin T_1$.

Subcase 2: Choose 3 vertices in R_3 to form another $\gamma^*_{t_{pw}}$ set, say T_2, that is, replaced $u_{10} \in R_2$ in T_1 by $u_{13} \in R_3$. It follows that $T_2 = \{u_3, u_4, u_8, u_9, u_{13}, u_{14}, u_{15}, u_{19}, u_{20}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$, where $|T_2| = |R|$, $O_V^{T_2}(P_n) = V(P_n)$, $O_E^{T_2}(P_n) = E(P_n)$, and the induced subgraph $\langle T_2 \rangle$ has no isolated vertex, that is, T_2 is a $\gamma^*_{t_{pw}}$-set of P_n. Clearly, $u_5 \notin T_2$.

Continuing in this manner and in any subcase, $u_5 \notin T_k$ for all $k = 1, 2, \ldots, m$, and so, the vertex u_5 is contained in $\gamma^*_{t_{pw}}$-set R only. By Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_n) = 1$.

Case 4: Suppose that $n \equiv 3(\text{mod } 5)$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_n) = \frac{2n+4}{5}$. Suppose that $n = 8$. Then $\gamma^*_{t_{pw}}(P_8) = \frac{2(8)+4}{5} = 4$. Clearly, $S_1 = \{u_1, u_2, u_5, u_6\}$, $S_2 = \{u_1, u_2, u_6, u_7\}$, $S_3 = \{u_2, u_3, u_5, u_6\}$, $S_4 = \{u_2, u_3, u_6, u_7\}$, $S_5 = \{u_2, u_3, u_7, u_8\}$, $S_6 = \{u_3, u_4, u_5, u_6\}$, $S_7 = \{u_3, u_4, u_6, u_7\}$, and $S_8 = \{u_3, u_4, u_7, u_8\}$ are the $\gamma^*_{t_{pw}}$-sets of P_8. Clearly, for $i = 1, 2, \ldots, 8$, no subset $\{u_i\}$ is contained in exactly one of the S_l's, for $l = 1, 2, \ldots, 8$, that is, $f\gamma^*_{t_{pw}}(S_l) > 1$. Clearly, $\{u_1, u_5\}$ is forcing subset for S_1 since $\{u_1, u_5\} \nsubseteq S_l$ for all $l \neq 1$. Thus, $f\gamma^*_{t_{pw}}(S_1) = 2 = f\gamma^*_{t_{pw}}(P_8)$. Now, suppose that $n > 8$. Since $f\gamma^*_{t_{pw}}(P_8) = 2$, $f\gamma^*_{t_{pw}}(P_n) \geq 2$. Let $p = \frac{n-3}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of P_n into $p+1$ disjoint subsets R_j

$$
\begin{aligned}
R_0 &= \{u_1, u_2, u_3\} \\
R_1 &= \{u_4, u_5, u_6, u_7, u_8\} \\
R_2 &= \{u_9, u_{10}, u_{11}, u_{12}, u_{13}\} \\
R_3 &= \{u_{14}, u_{15}, u_{16}, u_{17}, u_{18}\} \\
&\vdots \\
R_{p-1} &= \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\} \\
R_p &= \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}
\end{aligned}
$$

Let $i = 4, 9, 14, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} \rangle$, the vertices $u_1, u_2, u_{i+1}, u_{i+2}$ form a total dr-power dominating set since u_3, u_i and u_{i+3} are directly observed vertices while u_{i+4} are remotely observed vertices for all $i = 4, 9, 14, \ldots, n-9, n-4$. Let the set

$$
\begin{aligned}
R &= \{u_1, u_2, u_{i+1}, u_{i+2} : i = 4, 9, 14, \ldots, n-9, n-4\} \\
&= \{u_1, u_2, u_5, u_6, u_{10}, u_{11}, u_{15}, u_{16}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}
\end{aligned}
$$

where $|R| = 2p + 2 = 2\left(\frac{n-3}{5}\right) + 2 = \frac{2n+4}{5}$, $O_V^R(P_n) = V(P_n)$, $O_E^R(P_n) = E(P_n)$, and the induced subgraph $\langle R \rangle$ has no isolated vertex. By Theorem 2.2, R is a $\gamma^*_{t_{pw}}$-set of P_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of P_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of P_n different from R. Consider the following subcases:

Subcase 1: T_k, say T_1 and T_2, can be formed by replacing u_1, u_2 in R by either u_2, u_3 or u_3, u_4. It follows that $T_1 = \{u_2, u_3, u_5, u_6, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ and the set $T_2 = \{u_3, u_4, u_5, u_6, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ are $\gamma^*_{t_{pw}}$-sets of P_n. Clearly,

$\{u_1, u_5\} \nsubseteq T_1$ and $\{u_1, u_5\} \nsubseteq T_2$.

Subcase 2: T_k, say T_3, can be formed by replacing u_5 in R by u_7. Thus, the set $T_3 = \{u_1, u_2, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ is a $\gamma^*_{t_{pw}}$-set of P_n. By the previous subcase, T_k can be formed by replacing u_1, u_2 in T_3 by either u_2, u_3 or u_3, u_4. It follows that $T_4 = \{u_2, u_3, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ and the set $T_5 = \{u_3, u_4, u_6, u_7, u_{10}, u_{11}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ are $\gamma^*_{t_{pw}}$-sets of P_n. Clearly, $\{u_1, u_5\} \nsubseteq T_3$, $\{u_1, u_5\} \nsubseteq T_4$ and $\{u_1, u_5\} \nsubseteq T_5$.

Subcase 3: Suppose that $\{u_1, u_5\}$ is a subset in one of the T_k's with $k > 5$, say T_6, that is, u_2 must be in T_6 and either u_4 or u_6 is in T_6. Suppose that $u_4 \in T_6$. Then the next pair to be choosen must be u_9 and u_{10} since the distance between u_5 and u_9 is 4, that is, u_i and u_{i+1} must be in T_6. Thus, $T_6 = \{u_1, u_2, u_4, u_5, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-4}, u_{n-3}\}$. Since u_{n-3} is the last vertex in T_6, by subcase 1 of case 2, T_6 is not a $\gamma^*_{t_{pw}}$-set of P_n. Now, suppose that $u_6 \in T_6$. Then the first four vertices u_1, u_2, u_5, u_6 in T_6 are the same with R. Replace $u_{10} \in R$ by u_9 to form T_6. Then the next vertex to be chosen must be u_{10}, that is, the vertex u_i, u_{i+1} must be in T_6 for all $i = 9, \ldots, n-9, n-4$. Then $T_6 = \{u_1, u_2, u_5, u_6, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-4}, u_{n-3}\}$. Since u_{n-3} is the last vertex in T_6, by subcase 1 of case 2, T_6 is not a $\gamma^*_{t_{pw}}$-set of P_n. Since u_{10} is arbitrarily replaced from R, we cannot replace the vertex u_{i+1} in R, where $i = 9, 14, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of P_n. Hence, $\{u_1, u_5\}$ is not a subset of T_k for all $k > 5$.

Therefore, in any subcase, $\{u_1, u_5\} \nsubseteq T_k$ for all $k = 1, 2, \ldots, m$, that is, $\{u_1, u_5\}$ is a forcing subset for R. Thus, $f\gamma^*_{t_{pw}}(R) = 2 = f\gamma^*_{t_{pw}}(P_n)$.

Case 5: Suppose that $n \equiv 4 (\text{mod } 5)$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_n) = \frac{2n+2}{5}$. Let $n = 9$. Then $\gamma^*_{t_{pw}}(P_9) = \frac{2(9)+2}{5} = 4$. Clearly, $S_1 = \{u_1, u_2, u_6, u_7\}$, $S_2 = \{u_2, u_3, u_6, u_7\}$, $S_3 = \{u_2, u_3, u_7, u_8\}$, $S_4 = \{u_3, u_4, u_7, u_8\}$, and $S_5 = \{u_3, u_4, u_8, u_9\}$ are the only $\gamma^*_{t_{pw}}$-sets of P_9. Since $u_1 \in S_1$ and $u_1 \notin S_l$ for $l = 2, 3, 4, 5$, by Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_9) = 1$. Now, suppose that $n > 9$. Let $p = \frac{n-4}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of P_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1, u_2, u_3, u_4\}$
$R_1 = \{u_5, u_6, u_7, u_8, u_9\}$
$R_2 = \{u_{10}, u_{11}, u_{12}, u_{13}, u_{14}\}$
$R_3 = \{u_{15}, u_{16}, u_{17}, u_{18}, u_{13}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 5, 10, 15, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4} \rangle$, the vertices $u_1, u_2, u_{i+1}, u_{i+2}$ form a total *dr*-power dominating set since u_3, u_i and u_{i+3}

are directly observed vertices while u_4 and u_{i+4} are remotely observed vertices for all $i = 5, 10, 15, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_1, u_2, u_{i+1}, u_{i+2} : i = 5, 10, 15, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_6, u_7, u_{11}, u_{12}, u_{16}, u_{17}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\} \end{aligned}$$

where $|R| = 2p + 2 = 2\left(\frac{n-4}{5}\right) + 2 = \frac{2n+2}{5}$, $O_V^R(P_n) = V(P_n)$, $O_E^R(P_n) = E(P_n)$, and the induced subgraph $\langle R \rangle$ has no isolated vertex. By Theorem 2.2, R is a $\gamma^*_{t_{pw}}$-set of P_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of P_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of P_n different from R. Consider the following subcases:

Subcase 1: T_k, say T_1 and T_2, can be formed by replacing u_1, u_2 in R by either u_2, u_3 or u_3, u_4. It follows that $T_1 = \{u_2, u_3, u_6, u_7, u_{11}, u_{12}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ and the set $T_2 = \{u_3, u_4, u_6, u_7, u_{11}, u_{12}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ are $\gamma^*_{t_{pw}}$-sets of P_n. Clearly, $u_1 \notin T_1$ and $u_1 \notin T_2$.

Subcase 2: Let $k = 3$ and let $u_1, u_2 \in T_3$. Now, replace $u_6 \in R$ by u_5 to form T_3. Then the next vertex to be chosen must be u_6, that is, the vertex u_i, u_{i+1} must be in T_3 for all $i = 5, 10, 15, \ldots, n-9, n-4$. Then $T_3 = \{u_1, u_2, u_5, u_6, u_{10}, u_{11}, \ldots, u_{n-4}, u_{n-3}\}$ such that $|T_3| = |R|$. Since u_{n-3} is the last vertex in T_3, by subcase 1 of case 2, T_3 is not a $\gamma^*_{t_{pw}}$-set of P_n. Since u_6 is arbitrarily replaced from R, we cannot replace the vertex u_{i+1} in R, where $i = 10, 15, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of P_n. Thus, $u_1 \notin T_k$ for all $k \geq 3$.

Therefore, in any subcase, $u_1 \notin T_k$ for all $k = 1, 2, \ldots, m$ and so, the vertex u_1 is contained in $\gamma^*_{t_{pw}}$-set R only. By Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_n) = 1$.

Case 6: Suppose that $n \equiv 0 (\text{mod } 5)$. By Theorem 2.2, $\gamma^*_{t_{pw}}(P_n) = \frac{2n}{5}$. Let $n = 5$. Then $\gamma^*_{t_{pw}}(P_5) = \frac{2(5)}{5} = 2$. Clearly, $S_1 = \{u_2, u_3\}$ and $S_2 = \{u_3, u_4\}$ are the only $\gamma^*_{t_{pw}}$-sets of P_5. Since $u_2 \in S_1$ and $u_2 \notin S_2$, by Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_5) = 1$. Now, suppose that $n > 5$. Let $p = \frac{n}{5}$ and $j = 1, 2, \ldots, p-1, p$. Group the vertices of P_n into p disjoint subsets R_j

$R_1 = \{u_1, u_2, u_3, u_4, u_5\}$
$R_2 = \{u_6, u_7, u_8, u_9, u_{10}\}$
$R_3 = \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 1, 6, 11, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_{i+1}, u_{i+2} form a total dr-power dominating set since u_i and u_{i+3} are directly observed vertices while u_{i+4} is a remotely observed vertex for all $i = 1, 6, 11, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_{i+1}, u_{i+2} : i = 1, 6, 11, \ldots, n-9, n-4\} \\ &= \{u_2, u_3, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\} \end{aligned}$$

,where $|R| = 2p = \frac{2n}{5}$, $O_V^R(P_n) = V(P_n)$, $O_E^R(P_n) = E(P_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.2, R is a $\gamma^*_{t_{pw}}$-set of P_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of P_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of P_n different from R. Consider the following subcases:

Subcase 1: Let $k = 1$. T_1 can be formed by replacing u_2 in R by u_4. It follows that $T_1 = \{u_3, u_4, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-8}, u_{n-7}, u_{n-3}, u_{n-2}\}$ is a $\gamma^*_{t_{pw}}$-set of P_n. Clearly, $u_2 \notin T_1$.

Subcase 2: Let $k = 2$ and let $u_2, u_3 \in T_2$. Now, replace $u_7 \in R$ by u_6 to form T_2. Then the next vertex to be chosen must be u_7, that is, the vertices u_i, u_{i+1} must be in T_2 for all $i = 6, 11, 16, \ldots, n-9, n-4$. Then $T_2 = \{u_2, u_3, u_6, u_7, u_{11}, u_{12}, \ldots, u_{n-4}, u_{n-3}\}$ such that $|T_2| = |R|$. Since u_{n-3} is the last vertex in T_2, by subcase 1 of case 2, T_3 is not a $\gamma^*_{t_{pw}}$-set of P_n. Since u_7 is arbitrarily replaced from R, we cannot replace the vertex u_{i+1} in R, where $i = 6, 11, 16, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of P_n. Thus, it is not possible to start with vertex u_2 to form T_k for all $k \geq 2$, that is, $u_2 \notin T_k$ for all $k \geq 2$.

Therefore, in any subcase, $u_2 \notin T_k$ for all $k = 1, 2, \ldots, m$, and so, the vertex u_2 is contained in $\gamma^*_{t_{pw}}$-set R only. By Theorem 3.1(ii), $f\gamma^*_{t_{pw}}(P_n) = 1$.

□

Theorem 3.5. *Let n be a positive integer with $n \geq 5$. Then*

$$f\gamma^*_{t_{pw}}(C_n) = \begin{cases} 4, & n \equiv 2(mod\ 5) \\ 2, & otherwise. \end{cases}$$

Proof. Let the cycle $C_n = [u_1, u_2, \ldots, u_n, u_1]$. Note that and $deg(u_i) = 2$ for all $i = 1, 2, 3, \ldots, n-1, n$. By definition of total dr-power dominating set, a choosen vertex in a $\gamma^*_{t_{pw}}$-set, say D, of C_n must always have an adjacent vertex in D, say u_i and u_{i+1} and by Theorem 2.3, the distance of the choosen vertex u_{i+1} must be of at most 4 to the next choosen vertex in D since if the next vertex to be choosen is of distance 5, say $u_1, u_2 \in D$ and choose u_7 to be the next vertex, then u_3 and u_6 are directly observed vertices while u_4 and u_5 are remotely observed vertices but the edge u_4u_5 is neither directly nor remotely observed edge which is a contradiction. Note that u_i is contained in $\gamma^*_{t_{pw}}$-sets containing the pairs of sets $\{u_{i-1}, u_i\}$ and $\{u_i, u_{i+1}\}$ for all $i = 1, 2, \ldots n$, and so, the set $\{u_i\}$ is not a forcing subset of any $\gamma^*_{t_{pw}}$-set of C_n, that is, $f\gamma^*_{t_{pw}}(C_n) \geq 2$. Now, consider the following cases:

Case 1: Suppose that $n \equiv 2(\text{mod } 5)$. By Theorem 2.3, $\gamma^*_{t_{pw}}(C_n) = \frac{2n+6}{5}$. Suppose that $n = 7$. Then $\gamma^*_{t_{pw}}(C_7) = \frac{2(7)+6}{5} = 4$. Clearly, $S_1 = \{u_1, u_2, u_3, u_4\}$, $S_2 = \{u_1, u_2, u_4, u_5\}$, $S_3 = \{u_1, u_2, u_5, u_6\}$, $S_4 = \{u_1, u_2, u_6, u_7\}$, $S_5 = \{u_2, u_3, u_4, u_5\}$, $S_6 = \{u_2, u_3, u_5, u_6\}$, $S_7 = \{u_2, u_3, u_6, u_7\}$, $S_8 = \{u_3, u_4, u_5, u_6\}$, $S_9 = \{u_3, u_4, u_6, u_7\}$, $S_{10} = \{u_4, u_5, u_6, u_7\}$, $S_{11} = \{u_7, u_1, u_2, u_3\}$, $S_{12} = \{u_7, u_1, u_3, u_4\}$, $S_{13} = \{u_7, u_1, u_4, u_5\}$ and $S_{14} = \{u_7, u_1, u_5, u_6\}$ are the only $\gamma^*_{t_{pw}}$-sets of C_7. Note that $f\gamma^*_{t_{pw}}(C_7) \geq 2$ and each pair of vertices in C_7 is contained in more than one $\gamma^*_{t_{pw}}$-sets of C_7, that is, $f\gamma^*_{t_{pw}}(C_7) \geq 3$. Note that each of the vertices u_1, u_2, u_3, and u_4 in S_1 can be replaced by u_5, u_7, u_5 and u_7, respectively to form another $\gamma^*_{t_{pw}}$-sets of C_7 which are S_5, S_{12}, S_2, and S_{11}, respectively, that is, $f\gamma^*_{t_{pw}}(S_1) = 4$. Clearly, for all $l = 1, 2, \ldots, 14$ and for every $\gamma^*_{t_{pw}}$-set S_l of C_7 and for each $u_i \in S_l$, there exists $u_j \in V(C_7)\backslash S_l$ and $j \neq i$ such that $[S_l\backslash\{u_i\}] \cup \{u_j\}$ is a $\gamma^*_{t_{pw}}$-set of C_7. By Theorem 3.3, $f\gamma^*_{t_{pw}}(C_7) = \gamma^*_{t_{pw}}(C_7) = 4$. Now, suppose that $n > 7$. Since $f\gamma^*_{t_{pw}}(C_7) = 4$, $f\gamma^*_{t_{pw}}(C_n) \geq 4$. Let $p = \frac{n-2}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of C_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1, u_2\}$
$R_1 = \{u_3, u_4, u_5, u_6, u_7\}$
$R_2 = \{u_8, u_9, u_{10}, u_{11}, u_{12}\}$
$R_3 = \{u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 3, 8, 13, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_1, u_2, u_i, u_{i+1} form a total dr-power dominating set since u_{i+2} and u_{i+4} are directly observed vertices while u_{i+3} is a remotely observed vertex for all $i = 3, 8, 13, \ldots, n-9, n-4$.

Let the set

$$\begin{aligned} R &= \{u_1, u_2, u_i, u_{i+1} : i = 3, 8, 13, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_3, u_4, u_8, u_9, u_{13}, u_{14}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\} \end{aligned}$$

,where $|R| = 2p + 2 = 2(\frac{n-2}{5}) + 2 = \frac{2n+6}{5}$, $O_V^R(C_n) = V(C_n)$, $O_E^R(C_n) = E(C_n)$, and the induced subgraph $\langle R \rangle$ has no isolated vertex. By Theorem 2.3, R is a $\gamma^*_{t_{pw}}$-set of C_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of C_n where m is a positive integer. Let $k = 1, 2, \ldots, m+1$ and T_k be a $\gamma^*_{t_{pw}}$-set of C_n and one of the T_k's is equal to R. Note that in forming R, there are four vertices taken both from R_0 and R_1 such that the induced subgraph is a graph P_4 and two adjacent vertices in the other R_j's where $j > 1$. Now, T_k can be formed by starting all the vertices of S_l for $l = 1, 2, \ldots, 14$ and replacing u_7 by u_n in S_{11}, S_{12}, S_{13}, and S_{14}, that is, $T_k = S_l \cup H$ for some set H. If T_k starts with $S_{11} = \{u_n, u_1, u_2, u_3\}$, say T_1, note that S_{11} is the only set in the S_l's that ends with u_3 and its induced subgraph is a graph P_4, then the next pairs of vertices must be choosen are $u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-5}, u_{n-4}$, that is, $T_1 = \{u_n, u_1, u_2, u_3, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_1| = |R|$, $O_V^{T_1}(C_n) = V(C_n)$, $O_E^{T_1}(C_n) = E(C_n)$, and the induced subgraph $\langle T_1 \rangle$ has no isolated vertex, that is, T_1 is a $\gamma^*_{t_{pw}}$-set of C_n. Note also that if T_k starts with $S_{14} = \{u_n, u_1, u_5, u_6\}$, say T_2, then the next pairs of vertices must be choosen are $u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-5}, u_{n-4}$ such that induced subgraph of $\{u_5, u_6, u_7, u_8\} \subseteq T_2$ is a graph P_4, that is, $T_2 = \{u_n, u_1, u_5, u_6, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_2| = |R|$, $O_V^{T_2}(C_n) = V(C_n)$, $O_E^{T_2}(C_n) = E(C_n)$, and the induced subgraph $\langle T_2 \rangle$ has no isolated vertex, that is, T_2 is a $\gamma^*_{t_{pw}}$-set of C_n. Clearly, $T_1 \backslash S_{11} = T_2 \backslash S_{14}$. Now, since none of the other S_l's ended with a unique vertex, the set of vertices in $T_k \backslash S_l$ must be contained in T_r for some $r \neq k$. Hence, $T_k \backslash S_l$ is not a forcing subset for T_k. Therefore, either the sets S_l or T_k must be the forcing subset for T_k for some $l = 1, 2, \ldots, 14$ and for some $k = 1, 2, \ldots, m+1$. Now, if T_k starts with S_1, then let $k = 3$ and $\{u_1, u_2, u_3, u_4\} \subseteq T_3$ such that T_3 is a $\gamma^*_{t_{pw}}$-set of C_n different from R. Replace $u_8 \in R$ by u_7 to form T_3. Then the next vertex to be chosen must be u_8, that is, the vertex u_{i+4}, u_i must be in T_3 for all $i = 3, 8, 13, \ldots, n-9, n-4$. Then $T_3 = \{u_1, u_2, u_3, u_4, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_3| = |R|$. Since $u_1 \in T_3$ and u_{n-4} is the last vertex in T_3, the vertices u_n and u_{n-3} are directly observed vertices while u_{n-1} and u_{n-2} are remotely observed vertices. Hence, by definition, the edge $u_{n-1}u_{n-2}$ is neither a directly or a remotely observed edge. Thus, $O_E^{T_3}(C_n) \neq E(C_n)$, a contradiction, that is, T_3 is not a $\gamma^*_{t_{pw}}$-set of C_n. Since u_8 is arbitrarily replaced from R, we cannot replace the vertex u_i in R, where $i = 8, 13, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of C_n. Therefore, only the $\gamma^*_{t_{pw}}$-set R starts with S_1 and so, $\{u_1, u_2, u_3, u_4\} \nsubseteq T_k$ for all $T_k \neq R$. Hence, $\{u_1, u_2, u_3, u_4\}$ is a forcing subset for R. Therefore, $f\gamma^*_{t_{pw}}(R) = 4 = f\gamma^*_{t_{pw}}(C_n)$.

Case 2: Suppose that $n \equiv 0 (\mathrm{mod}\ 5)$. By Theorem 2.3, $\gamma^*_{t_{pw}}(C_n) = \frac{2n}{5}$. Suppose that $n = 5$. Then $\gamma^*_{t_{pw}}(C_5) = \frac{2(5)}{5} = 2$. Clearly, $S_1 = \{u_1, u_2\}$, $S_2 = \{u_2, u_3\}$, $S_3 = \{u_3, u_4\}$, $S_4 = \{u_4, u_5\}$, $S_5 = \{u_5, u_1\}$, are the only $\gamma^*_{t_{pw}}$-sets of C_5. Note that $f\gamma^*_{t_{pw}}(C_5) \geq 2$. It follows that

$2 \leq f\gamma^*_{t_{pw}}(C_5) \leq \gamma^*_{t_{pw}}(C_5) = 2$, that is, $f\gamma^*_{t_{pw}}(C_5) = 2$. Now, suppose that $n > 5$. Let $p = \frac{n}{5}$ and $j = 1, 2, \ldots, p-1, p$. Group the vertices of C_n into p disjoint subsets R_j

$R_1 = \{u_1, u_2, u_3, u_4, u_5\}$
$R_2 = \{u_6, u_7, u_8, u_9, u_{10}\}$
$R_3 = \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 1, 6, 11, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_i, u_{i+1} form a total *dr*-power dominating set since u_{i+2} and u_{i+4} are directly observed vertices while u_{i+3} is a remotely observed vertex for all $i = 1, 6, 11, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_i, u_{i+1} : i = 1, 6, 11, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_6, u_7, u_{11}, u_{12}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\} \end{aligned}$$

,where $|R| = 2p = 2(\frac{n}{5}) = \frac{2n}{5}$, $O^R_V(C_n) = V(C_n)$, $O^R_E(C_n) = E(C_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.3, R is a $\gamma^*_{t_{pw}}$-set of C_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of C_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of C_n different from R. Note that T_k can be formed by starting all the vertices of S_l for $l = 1, 2, \ldots, 5$ and replacing u_5 by u_n in S_5.

Now, if T_k starts with S_1, then let $k = 1$ and $\{u_1, u_2\} \subseteq T_1$. Replace $u_6 \in R$ by u_5 to form T_1. Then the next vertex to be choosen must be u_6, that is, the vertex u_{i+4}, u_i must be in T_1 for all $i = 1, 6, 11, \ldots, n-9, n-4$. It follows that $T_1 = \{u_1, u_2, u_5, u_6, u_{10}, u_{11}, u_{15}, u_{16}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_1| = |R|$. Since $u_1 \in T_1$ and u_{n-4} is the last vertex in T_1, by the same argument in Case 1, T_1 is not a $\gamma^*_{t_{pw}}$-set of C_n. Since u_6 is arbitrarily replaced from R, we cannot replace the vertex u_i in R, where $i = 6, 11, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of C_n. Therefore, only the $\gamma^*_{t_{pw}}$-set R starts with S_1 and so, $\{u_1, u_2\} \nsubseteq T_k$ for all $k = 1, 2, \ldots, m$. Hence, $\{u_1, u_2\}$ is a forcing subset for R. Therefore, $f\gamma^*_{t_{pw}}(R) = 2 = f\gamma^*_{t_{pw}}(C_n)$.

Case 3: Suppose that $n \equiv 1 (\text{mod } 5)$. By Theorem 2.3, $\gamma^*_{t_{pw}}(C_n) = \frac{2n+3}{5}$. Suppose that $n = 6$. Then $\gamma^*_{t_{pw}}(C_6) = \frac{2(6)+3}{5} = 3$. Clearly, $S_1 = \{u_1, u_2, u_3\}$, $S_2 = \{u_2, u_3, u_4\}$, $S_3 = \{u_3, u_4, u_5\}$, $S_4 = \{u_4, u_5, u_6\}$, $S_5 = \{u_5, u_6, u_1\}$, $S_6 = \{u_6, u_1, u_2\}$, are the only $\gamma^*_{t_{pw}}$-sets of C_6. Clearly, for $l = 1, 2, \ldots, 6$, $\{u_1, u_3\} \subseteq S_1$ and $\{u_1, u_3\} \nsubseteq S_l$ for all $l \neq 1$. Thus, $\{u_1, u_3\}$ is a forcing subset for S_1, that is, $f\gamma^*_{t_{pw}}(S_1) = 2 = f\gamma^*_{t_{pw}}(C_6)$. Now, suppose that $n > 6$. Let $p = \frac{n-1}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of C_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1\}$
$R_1 = \{u_2, u_3, u_4, u_5, u_6\}$
$R_2 = \{u_7, u_8, u_9, u_{10}, u_{11}\}$
$R_3 = \{u_{12}, u_{13}, u_{14}, u_{15}, u_{16}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 2, 7, 12, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_1, u_i, u_{i+1} form a total dr-power dominating set since u_{i+2} and u_{i+4} are directly observed vertices while u_{i+3} is a remotely observed vertex for all $i = 2, 7, 12, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_1, u_i, u_{i+1} : i = 2, 7, 12, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_3, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\} \end{aligned}$$

,where $|R| = 2p + 1 = 2(\frac{n-1}{5}) + 1 = \frac{2n+3}{5}$, $O_V^R(C_n) = V(C_n)$, $O_E^R(C_n) = E(C_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex,that is, R is a $\gamma^*_{t_{pw}}$-set of C_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of C_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of C_n different from R. Note that T_k can be formed by starting all the vertices of S_l for $l = 1, 2, \ldots, 5$ and replacing u_6 by u_n in S_5 and S_6 and also, T_k can be formed by having three vertices in any one or two of the R_j's where $j \geq 0$ such that the induced subgraph is a graph P_3 and two vertices in the other R_l's where $l \neq j$. Replacing $u_3 \in R$ by u_6 to form T_k, say T_1, that is, $T_1 = \{u_1, u_2, u_6, u_7, u_8, u_{12}, u_{13}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\}$ is a $\gamma^*_{t_{pw}}$-set of C_n. Clearly, $\{u_1, u_3\} \nsubseteq T_1$. Replacing $u_8 \in T_1$ by u_{11} to form T_k, say T_2, that is, $T_2 = \{u_1, u_2, u_6, u_7, u_{11}, u_{12}, u_{13}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\}$ is a $\gamma^*_{t_{pw}}$-set of C_n. Clearly, $\{u_1, u_3\} \nsubseteq T_2$. Continuing in this manner, $\{u_1, u_3\} \nsubseteq T_k$ for some k.

Now, if T_k starts with S_1, then let $k = 3$ and $\{u_1, u_3\} \subseteq T_3$. Then u_2 must be in T_3. Replace $u_7 \in R$ by u_6 to form T_3. Then the next vertex to be choosen must be u_7, that is, the vertex u_{i+4}, u_i must be in T_3 for all $i = 2, 7, 12, \ldots, n-9, n-4$. Then $T_3 = \{u_1, u_2, u_3, u_6, u_7, u_{11}, u_{12}, u_{16}, u_{17}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_3| = |R|$. Since $u_1 \in T_3$ and u_{n-4} is the last vertex in T_3, by the same argument in Case 1, T_3 is not a $\gamma^*_{t_{pw}}$-set of C_n. Since u_7 is arbitrarily replaced from R, we cannot replace the vertex u_i in R, where $i = 7, 12, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of C_n. Therefore, only the $\gamma^*_{t_{pw}}$-set R starts with S_1 and so, $\{u_1, u_3\} \nsubseteq T_k$ for all $k = 1, 2, \ldots, m$. Hence, $\{u_1, u_3\}$ is a forcing subset for R. Therefore, $f\gamma^*_{t_{pw}}(R) = 2 = f\gamma^*_{t_{pw}}(C_n)$.

Case 4: Suppose that $n \equiv 3 (\text{mod } 5)$. By Theorem 2.3, $\gamma^*_{t_{pw}}(C_n) = \frac{2n+4}{5}$. Suppose that $n = 8$. Then $\gamma^*_{t_{pw}}(C_8) = \frac{2(8)+4}{5} = 4$. Clearly, $S_1 = \{u_1, u_2, u_4, u_5\}$, $S_2 = \{u_1, u_2, u_5, u_6\}$, $S_3 = \{u_1, u_2, u_6, u_7\}$, $S_4 = \{u_2, u_3, u_5, u_6\}$, $S_5 = \{u_2, u_3, u_6, u_7\}$, $S_6 = \{u_2, u_3, u_7, u_8\}$, $S_7 = \{u_3, u_4, u_6, u_7\}$, $S_8 = \{u_3, u_4, u_7, u_8\}$, $S_9 = \{u_8, u_1, u_3, u_4\}$, $S_{10} = \{u_4, u_5, u_7, u_8\}$, $S_{11} = \{u_8, u_1, u_4, u_5\}$, and $S_{12} = \{u_8, u_1, u_5, u_6\}$, are the only

$\gamma^*_{t_{pw}}$-sets of C_8. Clearly, for $l = 1, 2 \ldots, 12$, $\{u_2, u_4\} \subseteq S_1$ and $\{u_2, u_4\} \nsubseteq S_l$ for all $l \neq 1$. Thus, $\{u_2, u_4\}$ is a forcing subset for S_1, that is, $f\gamma^*_{t_{pw}}(S_1) = 2 = f\gamma^*_{t_{pw}}(C_8)$. Now, suppose that $n > 8$. Let $p = \frac{n-3}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of C_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1, u_2, u_3\}$
$R_1 = \{u_4, u_5, u_6, u_7, u_8\}$
$R_2 = \{u_9, u_{10}, u_{11}, u_{12}, u_{13}\}$
$R_3 = \{u_{14}, u_{15}, u_{16}, u_{17}, u_{18}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 4, 9, 14 \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_1, u_2, u_i, u_{i+1} form a total dr-power dominating set since u_3, u_{i+2} and u_{i+4} are directly observed vertices while u_{i+3} is a remotely observed vertex for all $i = 4, 9, 14, \ldots, n-9, n-4$. Let the set

$$\begin{aligned} R &= \{u_1, u_2, u_i, u_{i+1} : i = 4, 9, 14, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_4, u_5, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\} \end{aligned}$$

,where $|R| = 2p + 2 = 2(\frac{n-3}{5}) + 2 = \frac{2n+4}{5}$, $O^R_V(C_n) = V(C_n)$, $O^R_E(C_n) = E(C_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.3, R is a $\gamma^*_{t_{pw}}$-set of C_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of C_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of C_n different from R. Note that T_k can be formed by starting all the vertices of S_l for $l = 1, 2, \ldots, 12$ and replacing u_8 by u_n in S_9, S_{11} and S_{12}. Now, if T_k starts with S_1, then let $k = 1$ and $\{u_2, u_4\} \subseteq \{u_1, u_2, u_4, u_5\} \subseteq T_1$. Replace $u_9 \in R$ by u_8 to form T_1. Then the next vertex to be choosen must be u_9, that is, the vertex u_{i+4}, u_i must be in T_1 for all $i = 4, 9, 14, \ldots, n-9, n-4$. Then $T_1 = \{u_1, u_2, u_4, u_5, u_8, u_9, u_{13}, u_{14}, u_{18}, u_{19}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_1| = |R|$. Since $u_1 \in T_1$ and u_{n-4} is the last vertex in T_1, by the same argument in Case 1, T_1 is not a $\gamma^*_{t_{pw}}$-set of C_n. Since u_9 is arbitrarily replaced from R, we cannot replace the vertex u_i in R, where $i = 9, 14, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of C_n. Therefore, only the $\gamma^*_{t_{pw}}$-set R starts with S_1 and so, $\{u_2, u_4\} \nsubseteq T_k$ for all $k = 1, 2, \ldots, m$. Hence, $\{u_2, u_4\}$ is a forcing subset for R. Therefore, $f\gamma^*_{t_{pw}}(R) = 2 = f\gamma^*_{t_{pw}}(C_n)$.

Case 5: Suppose that $n \equiv 4 (\text{mod } 5)$. By Theorem 2.3, $\gamma^*_{t_{pw}}(C_n) = \frac{2n+2}{5}$. Suppose that $n = 9$. Then $\gamma^*_{t_{pw}}(C_9) = \frac{2(9)+2}{5} = 4$. Clearly, $S_1 = \{u_1, u_2, u_5, u_6\}$, $S_2 = \{u_1, u_2, u_6, u_7\}$, $S_3 = \{u_2, u_3, u_6, u_7\}$, $S_4 = \{u_2, u_3, u_7, u_8\}$, $S_5 = \{u_3, u_4, u_7, u_8\}$, $S_6 = \{u_3, u_4, u_8, u_9\}$, $S_7 = \{u_4, u_5, u_8, u_9\}$, $S_8 = \{u_4, u_5, u_9, u_1\}$, and $S_9 = \{u_5, u_6, u_9, u_1\}$ are the only $\gamma^*_{t_{pw}}$-sets of C_9. Clearly, for $l = 1, 2, \ldots, 9$, $\{u_2, u_5\} \subseteq S_1$ and $\{u_2, u_5\} \nsubseteq S_l$ for all $l \neq 1$. Thus, $\{u_2, u_5\}$ is a forcing subset for S_1, that is, $f\gamma^*_{t_{pw}}(S_1) = 2 = f\gamma^*_{t_{pw}}(C_9)$. Now, suppose that $n > 9$. Let $p = \frac{n-4}{5}$ and $j = 0, 1, 2, \ldots, p-1, p$. Group the vertices of

C_n into $p+1$ disjoint subsets R_j

$R_0 = \{u_1, u_2, u_3, u_4\}$
$R_1 = \{u_5, u_6, u_7, u_8, u_9\}$
$R_2 = \{u_{10}, u_{11}, u_{12}, u_{13}, u_{14}\}$
$R_3 = \{u_{15}, u_{16}, u_{17}, u_{18}, u_{19}\}$
$\vdots$
$R_{p-1} = \{u_{n-9}, u_{n-8}, u_{n-7}, u_{n-6}, u_{n-5}\}$
$R_p = \{u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n\}$

Let $i = 5, 10, 15, \ldots, n-4$. For every induced subgraph $\langle u_i, u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\rangle$, the vertices u_1, u_2, u_i, u_{i+1} form a total dr-power dominating set since u_3,u_4,u_{i+2} and u_{i+4} are directly observed vertices while u_{i+3} is a remotely observed vertex $\forall\ i = 5, 10, 15, \ldots, n-4$. Let the set

$$\begin{aligned} R &= \{u_1, u_2, u_i, u_{i+1} : i = 5, 10, 15, \ldots, n-9, n-4\} \\ &= \{u_1, u_2, u_5, u_6, u_{10}, u_{11}, u_{15}, u_{16}, \ldots, u_{n-9}, u_{n-8}, u_{n-4}, u_{n-3}\} \end{aligned}$$

,where $|R| = 2p+2 = 2(\frac{n-4}{5})+2 = \frac{2n+2}{5}$, $O_V^R(C_n) = V(C_n)$, $O_E^R(C_n) = E(C_n)$, and the induced subgraph $\langle R\rangle$ has no isolated vertex. By Theorem 2.3, R is a $\gamma^*_{t_{pw}}$-set of C_n. Let $m+1$ be the number of $\gamma^*_{t_{pw}}$-sets of C_n where m is a positive integer. Let $k = 1, 2, \ldots, m$ and T_k be a $\gamma^*_{t_{pw}}$-set of C_n different from R. Note that T_k can be formed by starting all the vertices of S_l for $l = 1, 2, \ldots, 9$ and replacing u_9 by u_n in S_8 and S_9. Now, if T_k starts with S_1, then let $k = 1$ and $\{u_2, u_5\} \subseteq \{u_1, u_2, u_5, u_6\} \subseteq T_1$. Replace $u_{10} \in R$ by u_9 to form T_1. Then the next vertex to be choosen must be u_{10}, that is, the vertex u_{i+4}, u_i must be in T_1 for all $i = 5, 10, 15, \ldots, n-9, n-4$. Then $T_1 = \{u_1, u_2, u_5, u_6, u_9, u_{10}, u_{14}, u_{15}, \ldots, u_{n-5}, u_{n-4}\}$ such that $|T_1| = |R|$. Since $u_1 \in T_1$ and u_{n-4} is the last vertex in T_1, by the same argument in Case 1, T_1 is not a $\gamma^*_{t_{pw}}$-set of C_n. Since u_{10} is arbitrarily replaced from R, we cannot replace the vertex u_i in R, where $i = 10, 15, \ldots, n-9, n-4$ to form another $\gamma^*_{t_{pw}}$-set of C_n. Therefore, only the $\gamma^*_{t_{pw}}$-set R starts with S_1 and so, $\{u_2, u_5\} \not\subseteq T_k$ for all $k = 1, 2, \ldots, m$. Hence, $\{u_2, u_5\}$ is a forcing subset for R. Therefore, $f\gamma^*_{t_{pw}}(R) = 2 = f\gamma^*_{t_{pw}}(C_n)$. □

Theorem 3.6. *Let n be a positive integer with $n \geq 2$. Then the total dr-power domination number of the complete graph K_n is given by $\gamma^*_{t_{pw}}(K_n) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(K_n) = \begin{cases} 0, & n = 2 \\ 2, & n > 2. \end{cases}$$

Proof. Let $V(K_n) = \{u_1, u_2, u_3, \ldots, u_n\}$. Clearly, each pair of vertices u_i, u_j such that $i \neq j$ in K_n forms a $\gamma^*_{t_{pw}}$-set of K_n, and so, $\gamma^*_{t_{pw}}(K_n) = 2$. If $n = 2$, then K_2 has exactly one $\gamma^*_{t_{pw}}$-set which is $V(K_2)$. By Theorem 3.1 (i), $f\gamma^*_{t_{pw}}(K_2) = 0$. Suppose that $n > 2$. Note that for all $i = 1, 2, \ldots n$, u_i is contained in $\gamma^*_{t_{pw}}$-sets $\{u_i, u_j\}$ and $\{u_i, u_k\}$ such that $i \neq j \neq k \neq i$ and so, the set $\{u_i\}$ is not a forcing subset for any $\gamma^*_{t_{pw}}$-set of K_n, that is,

$f\gamma^*_{t_{pw}}(K_n) \geq 2$. Consequently, by Corollary 3.2, $2 \leq f\gamma^*_{t_{pw}}(K_n) \leq \gamma^*_{t_{pw}}(K_n) = 2$. Therefore, $f\gamma^*_{t_{pw}}(K_n) = 2$ for all $n > 2$. □

Theorem 3.7. *Let n be a positive integer with $n \geq 2$. Then the total dr-power domination number of the fan graph $F_n = K_1 + P_n$ of order $n+1$ is given by $\gamma^*_{t_{pw}}(F_n) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(F_n) = \begin{cases} 2, & n = 2, 3, \\ 1, & n \geq 4. \end{cases}$$

Proof. By Corollary 2.6, $\gamma^*_{t_{pw}}(F_n) = \gamma^*_{t_{pw}}(K_1 + P_n) = 2$. Let $V(F_n) = \{v, u_1, u_2, u_3, \ldots, u_n\}$ such that $deg(v) = n$. Note that the $\gamma^*_{t_{pw}}$-sets of F_n are of the form $\{v, u_i\}$ for all $u_i \in V(F_n)$ and of the form $\{u_i, u_j\}$ such that $i \neq j$ and $\{u_i, u_j\}$ is a γ_t-set of P_n by Theorem 2.5 and Corollary 2.6. Consider the following cases:

Case 1: Suppose that either $n = 2$ or $n = 3$.
By Proposition 2.4, $\gamma_t(P_n) = 2$ for $n = 2, 3$. If $n = 2$, then $R_1 = \{v, u_1\}$, $R_2 = \{v, u_2\}$ and $R_3 = \{u_1, u_2\}$ are the $\gamma^*_{t_{pw}}$-sets of F_2. If $n = 3$, then $R_1 = \{v, u_1\}$, $R_2 = \{v, u_2\}$, $R_3 = \{v, u_3\}$, $R_4 = \{u_1, u_2\}$ and $R_5 = \{u_2, u_3\}$ are the $\gamma^*_{t_{pw}}$-sets of F_3. Clearly, for all $u_i \in V(F_n)$ and $n = 2, 3$, the singleton $\{u_i\}$, together with $\{v\}$, is not contained in exactly one $\gamma^*_{t_{pw}}$-set of F_n, that is, the sets $\{u_i\}$ and $\{v\}$ are not forcing subsets for any $\gamma^*_{t_{pw}}$-set of F_n. Thus, $f\gamma^*_{t_{pw}}(F_n) \geq 2$. Then $2 \leq f\gamma^*_{t_{pw}}(F_n) \leq \gamma^*_{t_{pw}}(F_n) = 2$. Therefore, $f\gamma^*_{t_{pw}}(F_n) = 2$ for $n = 2, 3$.

Case 2: Suppose that $n \geq 4$.
By Proposition 2.4, $\gamma_t(P_4) = 2$. If $n = 4$, then $R_1 = \{v, u_1\}$, $R_2 = \{v, u_2\}$, $R_3 = \{v, u_3\}$, $R_4 = \{v, u_4\}$, and $R_5 = \{u_2, u_3\}$ are the $\gamma^*_{t_{pw}}$-sets of F_4. Clearly, $\{u_1\} \subseteq R_1$ and $\{u_1\} \nsubseteq R_l$ for $l = 2, 3, 4, 5$, that is, $\{u_1\}$ is a forcing subset for R_1 and so, $f\gamma^*_{t_{pw}}(R_1) = 1 = f\gamma^*_{t_{pw}}(F_4)$. If $n > 4$, then $\gamma_t(P_n) > 2$ by Proposition 2.4 and so, the $\gamma^*_{t_{pw}}$-sets of F_n are of the form $\{v, u_i\}$ for all $u_i \in V(F_n)$. Clearly, $R = \{v, u_1\}$ is the only $\gamma^*_{t_{pw}}$-set of F_n containing u_1. Thus, $\{u_1\}$ is a forcing subset for R, that is, $f\gamma^*_{t_{pw}}(R) = 1 = f\gamma^*_{t_{pw}}(F_n)$ for $n > 4$. □

Theorem 3.8. *Let n be a positive integer with $n \geq 3$. Then the total dr-power domination number of the wheel graph $W_n = K_1 + C_n$ of order $n+1$ is given by $\gamma^*_{t_{pw}}(W_n) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(W_n) = \begin{cases} 2, & n = 3, 4 \\ 1, & n \geq 5. \end{cases}$$

Proof. By Corollary 2.6, $\gamma^*_{t_{pw}}(W_n) = \gamma^*_{t_{pw}}(K_1 + C_n) = 2$. Let $V(W_n) = \{v, u_1, u_2, u_3, \ldots, u_n\}$ such that $deg(v) = n$. Note that the $\gamma^*_{t_{pw}}$-sets of W_n are of the form $\{v, u_i\}$ for all $u_i \in V(W_n)$ and of the form $\{u_i, u_j\}$ such that $i \neq j$ and $\{u_i, u_j\}$ is a γ_t-set of C_n by Theorem 2.5 and Corollary 2.6. Consider the following cases:

Case 1: Suppose that either $n = 3$ or $n = 4$.
By Proposition 2.4, $\gamma_t(C_n) = 2$ for $n = 3, 4$. If $n = 3$, then $R_1 = \{v, u_1\}$, $R_2 = \{v, u_2\}$, $R_3 = \{v, u_3\}$, $R_4 = \{u_1, u_2\}$, $R_5 = \{u_2, u_3\}$ and $R_6 = \{u_1, u_3\}$ are the $\gamma^*_{t_{pw}}$-sets of W_3. If $n = 4$, then $R_1 = \{v, u_1\}$, $R_2 = \{v, u_2\}$, $R_3 = \{v, u_3\}$, $R_4 = \{v, u_4\}$, $R_5 = \{u_1, u_2\}$, $R_6 = \{u_2, u_3\}$, $R_7 = \{u_3, u_4\}$ and $R_8 = \{u_4, u_1\}$ are the $\gamma^*_{t_{pw}}$-sets of W_4. Clearly, for all $u_i \in V(W_n)$ and for $n = 3, 4$, the singleton $\{u_i\}$, together with $\{v\}$, is not contained in exactly one $\gamma^*_{t_{pw}}$-set of W_n, that is, the sets $\{u_i\}$ and $\{v\}$ are not forcing subsets for any $\gamma^*_{t_{pw}}$-set of W_n. Thus, $f\gamma^*_{t_{pw}}(W_n) \geq 2$. Then $2 \leq f\gamma^*_{t_{pw}}(W_n) \leq \gamma^*_{t_{pw}}(W_n) = 2$. Therefore, $f\gamma^*_{t_{pw}}(W_n) = 2$ for $n = 3, 4$.

Case 2: Suppose that $n \geq 5$.
Then $\gamma_t(C_n) > 2$ by Proposition 2.4 and so, the $\gamma^*_{t_{pw}}$-sets of W_n are of the form $\{v, u_i\}$ for all $u_i \in V(W_n)$. Clearly, $R = \{v, u_1\}$ is the only $\gamma^*_{t_{pw}}$-set of W_n containing u_1. Thus, $\{u_1\}$ is a forcing subset for R, that is, for all $n \geq 5$, $f\gamma^*_{t_{pw}}(R) = 1 = f\gamma^*_{t_{pw}}(W_n)$. □

Theorem 3.9. *Let n be a positive integer with $n \geq 1$. Then the total dr-power domination number of the star graph $S_n = K_1 + \overline{K}_n$ of order $n + 1$ is given by $\gamma^*_{t_{pw}}(S_n) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(S_n) = \begin{cases} 0, & n = 1 \\ 1, & n > 1. \end{cases}$$

Proof. By Corollary 2.6, $\gamma^*_{t_{pw}}(S_n) = \gamma^*_{t_{pw}}(K_1 + \overline{K}_n) = 2$. Let $V(S_n) = \{v, u_1, u_2, u_3, \ldots, u_n\}$ such that $deg(v) = n$. If $n = 1$, then $R = \{v, u_1\}$ is the only $\gamma^*_{t_{pw}}$-set of S_1. By Theorem 3.1(i), $f\gamma^*_{t_{pw}}(S_1) = 0$. If $n > 1$, then the $\gamma^*_{t_{pw}}$-sets of S_n are of the form $\{v, u_i\}$ for all $u_i \in V(S_n)$ by Theorem 2.5. Clearly, $R = \{v, u_1\}$ is the only $\gamma^*_{t_{pw}}$-set of S_n containing u_1. Thus, $\{u_1\}$ is a forcing subset for R, that is, $f\gamma^*_{t_{pw}}(R) = 1 = f\gamma^*_{t_{pw}}(S_n)$ for all $n > 1$. □

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