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Forcing Total dr-Power Domination Number of Graphs Under Some Binary Operations

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Abstract. In this paper, the total dr-power domination number of graphs such as complete bipartite graph, generalized fan and generalized wheel are obtained. The forcing total dr-power domination number of graphs resulting from some binary operations such as join, corona and lexicographic product of graphs were determined.

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1. Introduction

Let $G = (V, E)$ be a graph representing the electrical power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. Some measurement devices must be placed at selected locations so that all the state variables of the system can be measured in order to monitor the power system. A Phase Measurement Unit (PMU) is a measurement device placed on a vertex and has the ability to measure the state of the vertex and the edges connected to the vertex. The vertices and edges that are measured by PMU's are said to be observed. In this study, it is necessary that each vertex with PMU is adjacent to another vertex with PMU also. But because of the high cost value of a PMU, it is desirable to minimize their number while maintaining the ability to monitor the entire power system.

The graphs considered in this paper are simple, connected, undirected and without loops or multiple edges.

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v in G is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For $X \subseteq V(G)$, the *open neighborhood* of X is the set $N(X) = \cup_{v \in X} N_G(v)$ and its *closed neighborhood* is the set $N[X] = N(X) \cup X$.

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A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N[S] = V(G)$ (resp. $N(S) = V(G)$). The *domination number* $\gamma(G)$ (resp. *total domination number* $\gamma_t(G)$) of G is the minimum cardinality of a dominating set (resp. total dominating set). If S is a dominating set (resp. a total dominating set) with $|S| = \gamma(G)$ (resp. $|S| = \gamma_t(G)$), then we call S a γ-set (resp. a γ_t-set) of G.

Let $G = (V, E)$ be a simple graph. Let $P \subseteq V(G)$. An edge $e = uv$ of G is directly observed by P if $u \in P$ or $v \in P$. A vertex u of G is directly observed if u is incident to a directly observed edge. An edge $e' = xy$ is remotely observed by P if $x, y \notin P$ and x, y are directly observed vertices or at least one of x and y is incident to k edges where $k - 1$ of these edges are directly observed by P. A non-directly observed vertex u of G which is incident to a remotely observed edge is called remotely observed vertex. Let $O_V^P(G)$ be the set of all directly and remotely observed vertices and $O_E^P(G)$ be the set of all directly and remotely observed edges. Then $P \subseteq V(G)$ is a dr-power dominating set (dr-pds) of G if $O_V^P(G) = V(G)$ and $O_E^P(G) = E(G)$. The minimum cardinality of a dr-power dominating set is called the *dr-power domination number of* G and is denoted by $\gamma_{pw}^*(G)$. A subset P of $V(G)$ with cardinality $\gamma_{pw}^*(G)$ is called a γ_{pw}^**-set of* G. A dr-power dominating set D is said to be a total dr-power dominating set(tdr-pds) if the induced subgraph $\langle D\rangle$ has no isolated vertex. The minimum cardinality of a total dr-power dominating set (tdr-pds) is called the *total dr-power domination number of* G and is denoted by $\gamma_{t_{pw}}^*(G)$. A subset T of $V(G)$ with cardinality $\gamma_{t_{pw}}^*(G)$ is called a $\gamma_{t_{pw}}^*$*-set of* G. Moreover, there exists a connected graph G such that $\gamma_{t_{pw}}^*(G) \leq \gamma_t(G)$.

Let S be a $\gamma_{t_{pw}}^*$-set of a graph G. A subset D of S is said to be a *forcing subset* for S if S is the unique $\gamma_{t_{pw}}^*$-set containing D. The *forcing total dr-power domination number of* S is given by $f\gamma_{t_{pw}}^*(S) = \min\{|D| : D \text{ is a forcing subset for } S\}$. The *forcing total dr-power domination number of* G is given by

$$f\gamma_{t_{pw}}^*(G) = \min\{f\gamma_{t_{pw}}^*(S) : S \text{ is a } \gamma_{t_{pw}}^*\text{-set of } G\}.$$

The *join* of two graphs G and H, denoted by $G + H$ is the graph with vertex set

$$V(G + H) = V(G) \cup V(H)$$

and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$

The *corona* $G \circ H$ of two graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H, and then forming the join $\langle\{v\}\rangle + H^v = v + H^v$, where H^v is a copy of H, for each $v \in V(G)$.

The *lexicographic product (composition)* $G[H]$ of two graphs G and H is the graph with $V(G[H]) = V(G) \times V(H)$, and $(u,u')(v,v') \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $u'v' \in E(H)$.

Amos [1] studied total domination. The relation between forcing and domination concepts was investigated by Chartrand et al. [5] and they defined "forcing domination number". The following concepts: total dr-power domination [6], forcing domination number of graphs under some binary operations [7], forcing total domination number and forcing connected domination number under the lexicographic product of graphs [8], forcing independent domination number of a graph [4], and A-differential of graphs [3] was studied by Canoy, et al. The total dr-power domination number of some special graphs such as paths and cycles was studied by Armada [2].

2. Known Results

This section contains known results involving total dr-power domination, dr-power domination, total domination numbers of a graph G that are very useful in proving the main results of this paper.

Remark 2.1. [6] For a graph G without isolated vertices,

$$\gamma^*_{pw}(G) \leq \gamma^*_{t_{pw}}(G) \leq \gamma_t(G).$$

Proposition 2.2. [1] *The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is given by*

$$\gamma_t(C_n) = \gamma_t(P_n) = \begin{cases} \frac{n}{2}, & n \equiv 0(mod\ 4), \\ \frac{n+2}{2}, & n \equiv 2(mod\ 4), \\ \frac{n+1}{2}, & otherwise. \end{cases}$$

Theorem 2.3. [6] Let G and H be any graphs. Then $P \subseteq V(G+H)$ is a total dr-power dominating set of $G+H$ if and only if it satisfies one of the following conditions:

(i) $P \subseteq V(G)$ and is a total dominating set, provided that G is a graph with no isolated vertex;

(ii) $P \subseteq V(H)$ and is a total dominating set, provided that H is a graph with no isolated vertex; or

(iii) $P = P_1 \cup P_2$, where $\varnothing \neq P_1 \subseteq V(G)$ and $\varnothing \neq P_2 \subseteq V(G)$.

Corollary 2.4. [6] Let G and H be any graphs. Then

$$\gamma^*_{t_{pw}}(G+H) = 2.$$

Theorem 2.5. [6] Let G be a nontrivial connected graph and H be a graph with no isolated vertex. Then $P \subseteq V(G \circ H)$ is a total dr-power dominating set if and only if

$$P = A \bigcup \left(\bigcup_{v \in A} B_v \right) \bigcup \left(\bigcup_{u \notin A} D_u \right)$$

where $A \subseteq V(G)$, $B_v \subseteq V(H^v)$ for each $v \in A$ and $B_v \neq \varnothing$ for each $v \notin N_G(A)$, and $D_u \subseteq V(H^u)$ is a total dominating set of H^u for each $u \notin A$.

Corollary 2.6. [6] Let G be a nontrivial connected graph of order m and H be any graph with no isolated vertex. Then $\gamma^*_{t_{pw}}(G \circ H) = m$.

Theorem 2.7. [6] Let G and H be nontrivial connected graphs. Then $P = \bigcup_{x \in S} (\{x\} \times T_x)$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for all $x \in S$, is a total dr-power dominating set of $G[H]$ if and only if S is a dominating set of G, and T_x is a total dominating set of H for each $x \in S \backslash N(S)$.

Corollary 2.8. [6] Let G and H be nontrivial connected graphs. Then P is a total dr-power dominating set of $G[H]$ if and only if P is a total dominating set of $G[H]$. Moreover,

$$\gamma^*_{t_{pw}}(G[H]) = \gamma_t(G).$$

Theorem 2.9. [2] Let G be a graph. Then

(*i*) $f\gamma^*_{t_{pw}}(G) = 0$ if and only if G has a unique $\gamma^*_{t_{pw}}$-set.

(*ii*) $f\gamma^*_{t_{pw}}(G) = 1$ if and only if G has at least two $\gamma^*_{t_{pw}}$-sets and there exists a vertex v which is contained in exactly one $\gamma^*_{t_{pw}}$-set of G.

Corollary 2.10. [2] Let G be a connected graph. Then

$$0 \leq f\gamma^*_{t_{pw}}(G) \leq \gamma^*_{t_{pw}}(G).$$

Theorem 2.11. [2] Let G be a nontrivial graph. Then $f\gamma^*_{t_{pw}}(G) = \gamma^*_{t_{pw}}(G)$ if and only if for every $\gamma^*_{t_{pw}}$-set P of G and for each $v \in P$, there exists $u \in V(G)\backslash P$ such that $[P\backslash\{v\}] \cup \{u\}$ is a $\gamma^*_{t_{pw}}$-set of G.

3. Forcing Total dr-Power Domination Number of the Join of Graphs

This section contains the total dr-power domination number of the complete bipartite graphs, generalized fan graphs, generalized wheel graphs, $P_n + P_m$, $P_n + C_m$ and $C_n + C_m$ and their forcing total dr-power domination numbers.

Corollary 3.1. *Let G and H be any graphs. Then $R \subseteq V(G+H)$ is a $\gamma^*_{t_{pw}}$-set of $G+H$ if and only if at least one of the following holds:*

(*i*) *R is a γ_t-set of G and $|R| = 2$,*

(*ii*) *R is a γ_t-set of H and $|R| = 2$,*

(*iii*) *$|R \cap V(G)| = 1$ and $|R \cap V(H)| = 1$.*

Theorem 3.2. *For any graphs G and H,*

$$f\gamma^*_{t_{pw}}(G+H) = \begin{cases} 0, & \text{if } G \text{ and } H \text{ are both trivial,} \\ 1, & \text{if } G \text{ is trivial and (i) } H \text{ has an isolated vertex or} \\ & \text{(ii) } \gamma_t(H) > 2 \text{ or (iii) } \gamma_t(H) = 2 \text{ and there exists} \\ & \text{a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } H, \\ & \text{or if } H \text{ is trivial and (i) } G \text{ has an isolated vertex or} \\ & \text{(ii) } \gamma_t(G) > 2 \text{ or (iii) } \gamma_t(G) = 2 \text{ and there exists} \\ & \text{a vertex in } H \text{ which is not in any } \gamma_t\text{-set of } G, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. Consider the following cases:

Case 1: G and H are both trivial graphs.
$\{x, y\}$ such that $x \in V(G)$ and $y \in V(H)$ is the only γ_t-set of $G+H$ by Corollary 3.1. Thus, $f\gamma^*_{t_{pw}}(G+H) = 0$ by Theorem 2.9(i).

Case 2: G is trivial and (i) H has an isolated vertex or (ii) $\gamma_t(H) > 2$ or (iii) $\gamma_t(H) = 2$ and H contains a vertex which is not in any γ_t-set of H
Let $V(G) = \{x\}$. Suppose that H has an isolated vertex, say w. By Corollary 3.1, $R_w = \{x, w\}$ is the only $\gamma^*_{t_{pw}}$-set of $G+H$ containing w. If $\gamma_t(H) > 2$, then by Corollary 3.1, for each $u \in V(H)$, $R_u = \{x, u\}$ is the only $\gamma^*_{t_{pw}}$-set of $G+H$ containing u. If $\gamma_t(H) = 2$ and there exists a vertex, say v, in H which is not in any γ_t-set of H, then by Corollary 3.1, $R_v = \{x, v\}$ is the only $\gamma^*_{t_{pw}}$-set of $G+H$ containing v. Thus, in any of the three cases, there always exists a vertex which is contained in exactly one $\gamma^*_{t_{pw}}$-set of $G+H$. By Theorem 2.9(ii), $f\gamma^*_{t_{pw}}(G+H) = 1$. Similarly, if H is trivial and (i) G has an isolated vertex or (ii) $\gamma_t(G) > 2$ or (iii) $\gamma_t(G) = 2$ and there exists a vertex in G which is not in any γ_t-set of G, then $f\gamma^*_{t_{pw}}(G+H) = 1$.

Case 3: G is trivial , $\gamma_t(H) = 2$ and every vertex $v \in V(H)$ is contained in a γ_t-set of H
Let $V(G) = \{x\}$. By Corollary 3.1, for each $u \in V(H)$, $R_u = \{x, u\}$ is a $\gamma^*_{t_{pw}}$-set of $G+H$. Also by assumption and Corollary 3.1, for all $v, w \in V(H)$ such that $v \neq w$, $R = \{v, w\}$ is a γ_t-set of H and a $\gamma^*_{t_{pw}}$-set of $G + H$. Clearly, no single element is contained in exactly one $\gamma^*_{t_{pw}}$-set of $G+H$, that is, $f\gamma^*_{t_{pw}}(G+H) \geq 2$. Consequently, by Corollary 2.10, $2 \leq f\gamma^*_{t_{pw}}(G+H) \leq \gamma^*_{t_{pw}}(G+H) = 2$. Therefore, $f\gamma^*_{t_{pw}}(G+H) = 2$. Similarly, if H is trivial , $\gamma_t(G) = 2$ and every vertex $v \in V(G)$ is contained in a γ_t-set of G, then $f\gamma^*_{t_{pw}}(G+H) = 2$.

Case 4: G and H are both nontrivial graphs.
By Corollary 3.1, for each $x \in V(G)$ and for each $u \in V(H)$, $R = \{x, u\}$ is a $\gamma^*_{t_{pw}}$-set of $G + H$. Thus, for each $u \in R$, there exists $u_y \in V(G+H) \backslash R$ such that $[R \backslash \{u\}] \cup \{u_y\}$ is a $\gamma^*_{t_{pw}}$-set of $G+H$. By Theorem 2.11, it follows that $f\gamma^*_{t_{pw}}(G+H) = \gamma^*_{t_{pw}}(G+H) = 2$. □

The next result is a direct consequence of Corollary 2.4 and Theorem 3.2.

Corollary 3.3. *For any graph H, the total dr-power domination number of the join K_1+H is given by $\gamma^*_{t_{pw}}(K_1 + H) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(K_1 + H) = \begin{cases} 0, & \text{if } H \text{ is trivial,} \\ 1, & \text{if } H \text{ has an isolated vertex, or } \gamma_t(H) > 2, \text{ or} \\ & \gamma_t(H) = 2 \text{ and there exists a vertex in } H \\ & \text{which is not in any } \gamma_t\text{-set of } H, \\ 2, & \text{otherwise.} \end{cases}$$

Corollary 3.4. *The total dr-power domination number of the complete bipartite $K_{n,m} = \overline{K}_n + \overline{K}_m$ such that $n, m \geq 1$, is given by $\gamma^*_{t_{pw}}(K_{n,m}) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(K_{n,m}) = \begin{cases} 0, & n = 1 \text{ and } m = 1, \\ 1, & n = 1 \text{ and } m \geq 2 \text{ or } m = 1 \text{ and } n \geq 2, \\ 2, & n \geq 2 \text{ and } m \geq 2. \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(K_{n,m}) = \gamma^*_{t_{pw}}(\overline{K}_n + \overline{K}_m) = 2$. If $n = 1$ and $m = 1$, then $\overline{K}_1 = K_1$ is trivial, and so by Corollary 3.3, $f\gamma^*_{t_{pw}}(K_{1,1}) = 0$. If $n = 1$ and $m \geq 2$, then $\overline{K}_m$ has an isolated vertex, and so by Corollary 3.3, $f\gamma^*_{t_{pw}}(K_{1,m}) = 1$. Similarly, if $m = 1$ and $n \geq 2$, then $f\gamma^*_{t_{pw}}(K_{n,1}) = 1$. If $n \geq 2$ and $m \geq 2$, then $\overline{K}_n$ and $\overline{K}_m$ are nontrivial graphs and so, by Theorem 3.2, $f\gamma^*_{t_{pw}}(K_{n,m}) = 2$. □

Corollary 3.5. *The total dr-power domination number of the generalized fan $F_{n,m} = \overline{K}_n + P_m$, where $n \geq 1$ and $m \geq 2$, is given by $\gamma^*_{t_{pw}}(F_{n,m}) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(F_{n,m}) = \begin{cases} 1, & n = 1 \text{ and } m \geq 4, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(F_{n,m}) = \gamma^*_{t_{pw}}(\overline{K}_n + P_m) = 2$. Let $P_n = [u_1, u_2, \ldots, u_n]$. If $n = 1$ and $m = 4$, then by Proposition 2.2, $\gamma_t(P_4) = 2$ and P_4 has exactly one γ_t-set which is $\{u_2, u_3\}$, that is, u_1 is a vertex not in a γ_t-set of P_4. By Corollary 3.3, $\gamma^*_{t_{pw}}(F_{1,4}) = 1$. If $n = 1$ and $m > 4$, then by Proposition 2.2, $\gamma_t(P_m) > 2$. By Corollary 3.3, $\gamma^*_{t_{pw}}(F_{1,m}) = 1$. If $n = 1$ and $m < 4$, then by Proposition 2.2, $\gamma_t(P_2) = \gamma_t(P_3) = 2$ and so, $\{u_1, u_2\}$ is the γ_t-set of P_2 while $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are γ_t-sets of P_3. Clearly, for $m = 2, 3$, every vertex $u_i \in V(P_m)$ is contained in a γ_t-set of P_m. By Corollary 3.3, $\gamma^*_{t_{pw}}(F_{1,m}) = 2$. If $n \geq 2$ and $m \geq 2$, then $\overline{K}_n$ and P_m are nontrivial graphs. By Corollary 3.3, $f\gamma^*_{t_{pw}}(F_{n,m}) = 2$. □

Corollary 3.6. *The total dr-power domination number of the generalized wheel $W_{n,m} = \overline{K}_n + C_m$, where $n \geq 1$ and $m \geq 3$, is given by $\gamma^*_{t_{pw}}(W_{n,m}) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(W_{n,m}) = \begin{cases} 1, & n = 1 \text{ and } m \geq 5, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(W_{n,m}) = \gamma^*_{t_{pw}}(\overline{K}_n + C_m) = 2$. Let $C_n = [u_n, u_1, u_2, \ldots, u_n]$. If $n = 1$ and $m \geq 5$, then by Proposition 2.2, $\gamma_t(C_m) > 2$. By Corollary 3.3, $\gamma^*_{t_{pw}}(W_{1,m}) = 1$. If $n = 1$ and $m < 5$, then by Proposition 2.2, $\gamma_t(C_3) = \gamma_t(C_4) = 2$ and so, $\{u_1, u_2\}$, $\{u_2, u_3\}$ and $\{u_3, u_1\}$ are the γ_t-sets of C_3 while $\{u_1, u_2\}$, $\{u_2, u_3\}$, $\{u_3, u_4\}$ and $\{u_4, u_1\}$ are γ_t-sets of C_4. Clearly, for $m = 3, 4$, every vertex $u_i \in V(C_m)$ is contained in a γ_t-set of C_m. By Corollary 3.3, $\gamma^*_{t_{pw}}(W_{1,m}) = 2$. If $n \geq 2$ and $m \geq 3$, then $\overline{K}_n$ and C_m are nontrivial graphs. By Corollary 3.3, $f\gamma^*_{t_{pw}}(W_{n,m}) = 2$. □

Corollary 3.7. *The total dr-power domination number of the join $P_n + P_m$, where $n \geq 1$ and $m \geq 1$, is given by $\gamma^*_{t_{pw}}(P_n + P_m) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(P_n + P_m) = \begin{cases} 0, & n = 1 \text{ and } m = 1, \\ 1, & \text{either } n = 1 \text{ and } m \geq 4, \text{ or } m = 1 \text{ and } n \geq 4, \\ 2, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(P_n + P_m) = 2$. If $n = 1$ and $m = 1$, then $P_1 = K_1$ is a trivial graph and so, by Corollary 3.3, $\gamma^*_{t_{pw}}(P_1 + P_1) = 0$. If $n = 1$ and $m \geq 4$, then $f\gamma^*_{t_{pw}}(P_1 + P_m) = f\gamma^*_{t_{pw}}(F_{1,m}) = 1$ by Corollary 3.5. Similarly, if $m = 1$ and $n \geq 4$, $f\gamma^*_{t_{pw}}(P_n + P_1) = 1$. If $n = 1$ and either $m = 2$ or $m = 3$, then by Corollary 3.5,

$f\gamma^*_{t_{pw}}(P_1 + P_m) = f\gamma^*_{t_{pw}}(F_{1,m}) = 2$. Similarly, if $m = 1$ and either $n = 2$ or $n = 3$, then $f\gamma^*_{t_{pw}}(P_n + P_1) = 2$. If $n \geq 2$ and $m \geq 2$, then P_n and P_m are nontrivial graphs. By Corollary 3.3, $f\gamma^*_{t_{pw}}(P_n + P_m) = 2$. □

Corollary 3.8. *The total dr-power domination number of the join $P_n + C_m$, where $n \geq 1$ and $m \geq 3$, is given by $\gamma^*_{t_{pw}}(P_n + C_m) = 2$ and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(P_n + C_m) = \begin{cases} 1, & n = 1 \text{ and } m \geq 5 \\ 2, & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(P_n + C_m) = 2$. If $n = 1$ and $m \geq 5$, then by Corollary 3.6, $f\gamma^*_{t_{pw}}(P_1 + C_m) = f\gamma^*_{t_{pw}}(W_{1,m}) = 1$. If $n = 1$ and $m < 5$, then by Corollary 3.6, $f\gamma^*_{t_{pw}}(P_1 + C_m) = f\gamma^*_{t_{pw}}(W_{1,m}) = 2$. If $n \geq 2$ and $m \geq 2$, then P_n and C_m are nontrivial graphs. By Corollary 3.3, $f\gamma^*_{t_{pw}}(P_n + C_m) = 2$. □

Corollary 3.9. *The total dr-power domination number of the join $C_n + C_m$, where $n \geq 3$ and $m \geq 3$, and its forcing total dr-power domination number is given by*

$$f\gamma^*_{t_{pw}}(C_n + C_m) = \gamma^*_{t_{pw}}(C_n + C_m) = 2.$$

Proof. By Corollary 2.4, $\gamma^*_{t_{pw}}(C_n + C_m) = 2$. Note that the cycles C_n and C_m are nontrivial graphs. By Theorem 3.2, $f\gamma^*_{t_{pw}}(C_n + C_m) = 2$. □

4. Forcing Total dr-Power Domination Number of the Corona of Graphs

This section contains the forcing total dr-power domination number of the coronas $G \circ H$ and $K_m \circ H$ such that G is a nontrivial connected graph, K_m is a complete graph and H is any graph.

Theorem 4.1. *Let G be a nontrivial connected graph and let H be any graph. Then $R \subseteq V(G \circ H)$ is a $\gamma^*_{t_{pw}}$-set of $G \circ H$ if and only if $R = V(G)$. In particular,*

$$f\gamma^*_{t_{pw}}(G \circ H) = 0.$$

Proof. Suppose that $R \subseteq V(G \circ H)$ is a $\gamma^*_{t_{pw}}$-set of $G \circ H$. Note that $V(G)$ is a $\gamma^*_{t_{pw}}$-set of $G \circ H$ by Corollary 2.6. Suppose that $R \neq V(G)$ and let $A = R \cap V(G)$, that is, $|A| < |V(G)|$. Since R is a $\gamma^*_{t_{pw}}$-set of $G \circ H$, $R = A \cup \left(\bigcup_{v \in A} B_v\right) \cup \left(\bigcup_{u \notin A} D_u\right)$ as described in Theorem 2.5 where $|B_v| = 0$ for each $v \in A$ and $|D_u| = \gamma_t(H)$ for each $u \notin A$ or $u \in V(G) \backslash A$. Hence,

$$\begin{aligned} |R| &= |A| + \gamma_t(H)(|V(G)| - |A|) \\ &\geq |A| + 2|V(G)| - 2|A| \qquad \text{since } \gamma_t(H) \geq 2 \\ &\geq 2|V(G)| - |A| \\ &> |V(G)| = m \qquad\qquad \text{since } |V(G)| > |A| \end{aligned}$$

This is a contradiction since $|R| = m$ by Corollary 2.6. Thus, $R = V(G)$. The converse is clear. In particular, since $V(G)$ is the unique $\gamma^*_{t_{pw}}$-set of $G \circ H$, by Theorem 2.9(i), $f\gamma^*_{t_{pw}}(G \circ H) = 0$. □

The next result follows directly from Corollary 3.3 and Theorem 4.1. Note that $K_1 \circ H = K_1 + H$.

Corollary 4.2. *Let K_m be a complete graph of order $m \geq 1$ and let H be any graph. Then*

$$f\gamma^*_{t_{pw}}(K_m \circ H) = \begin{cases} 0, & \textit{if either } m = 1 \textit{ and } H \textit{ is trivial, or } m > 1, \\ 1, & \textit{if } m = 1 \textit{ and either (i) } H \textit{ has an isolated vertex, or} \\ & \textit{(ii) } \gamma_t(H) > 2, \textit{ or (iii) } \gamma_t(H) = 2 \textit{ and there exists} \\ & \textit{a vertex in } H \textit{ which is not in any } \gamma_t\textit{-set of } H, \\ 2, & \textit{if } m = 1, \gamma_t(H) = 2 \textit{ and every vertex in } H \\ & \textit{is contained in any } \gamma_t\textit{-set of } H. \end{cases}$$

5. Forcing Total dr-Power Domination Number of the Lexicographic Product of Graphs

This section contains the forcing total dr-power domination number of the graphs $G[H]$, $P_n[H]$ and $C_n[H]$ where G and H are nontrivial connected graphs.

Theorem 5.1. *Let G and H be nontrivial connected graphs. Then*

$$f\gamma^*_{t_{pw}}(G[H]) = \gamma_t(G).$$

Proof. Note that $\gamma^*_{t_{pw}}(G[H]) = \gamma_t(G)$ by Corollary 2.8. Now, suppose that $P = \bigcup_{u \in S}(\{u\} \times T_u)$, where S is a γ_t-set of G and $T_u \subseteq V(H)$. Consequently, $|P| = |S| = \gamma_t(G)$. By Theorem 2.7 and Corollary 2.8, P is a $\gamma^*_{t_{pw}}$-set of $G[H]$. Suppose that $f\gamma^*_{t_{pw}}(G[H]) = f\gamma^*_{t_{pw}}(P)$. Moreover, suppose that P has a forcing subset R with $|R| < |P|$, that is, $P = R \cup N$, where $N = \{(u,v) \in P : (u,v) \notin R\}$. Pick $(u,v) \in N$. Note that there always exists a vertex $(u,w) \in V(G[H]) \backslash P$ such that $w \neq v$ and $[P \backslash \{(u,v)\}] \cup \{(u,w)\} = Q$ is a $\gamma^*_{t_{pw}}$-set of $G[H]$ since all adjacent vertices (r,s)

of (u, v) with $u \neq r$ are adjacent vertices of (u, w) also. Thus, $Q = R \cup M$, where $M = [N \backslash \{(u, v)\}] \cup \{(u, w)\}$, $O_V^Q(G[H]) = V(G[H])$ and $O_E^Q(G[H]) = E(G[H])$, that is, Q is a $\gamma^*_{t_{pw}}$-set containing R, a contradiction. Thus,$|R| = |P|$ and P is the only forcing subset for P. Therefore, $f\gamma^*_{t_{pw}}(G[H]) = |P| = \gamma_t(G)$. □

The next result follows from Theorem 5.1 and Corollary 2.2.

Corollary 5.2. *Let H be a nontrivial connected graph and $n \geq 3$. Then*

$$f\gamma^*_{t_{pw}}(P_n[H]) = f\gamma^*_{t_{pw}}(C_n[H]) = \begin{cases} \frac{n}{2}, & n \equiv 0(mod\ 4), \\ \frac{n+2}{2}, & n \equiv 2(mod\ 4), \\ \frac{n+1}{2}, & otherwise. \end{cases}$$

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