EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 1, 2021, 112-125
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Global Hop Domination Numbers of Graphs

Gemma P. Salasalan [1,*], Sergio R. Canoy, Jr.[1]

[1] *Department of Mathematics and Statistics, College of Science and Mathematics, Center for Graph Theory, Algebra and Analysis, Premier Research Institute of Science and Mathematics, MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines*

Abstract. A set $S \subseteq V(G)$ is a hop dominating set of G if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. It is a global hop dominating set of G if it is a hop dominating set of both G and the complement $\overline{G}$ of G. The minimum cardinality of a global hop dominating set of G, denoted by $\gamma_{gh}(G)$, is called the global hop domination number of G. In this paper, we study the concept of global hop domination in graphs resulting from some binary operations.

2020 Mathematics Subject Classifications: 05C69

Key Words and Phrases: Hop domination, global hop domination, join, corona, lexicographic product, and Cartesian product

1. Introduction

Domination is a well-studied topic in Graph Theory. From the standard concept, many other variations of domination have been investigated by researchers. Connected, total, independent, and global domination are among the numerous well-known variants of the standard domination concept. Other variants may be found in the two books authored by Haynes et al. (see [5] and [6]).

Recently, Natarajan and Ayyaswamy [10] introduced and studied the concept of hop domination in a graph. In another study, Ayyaswamy et al. [1] investigated the same concept and gave bounds of the hop domination number of some graphs. Henning and Rad [7] also studied the concept and answered a question posed by Ayyaswamy and Natarajan in [10]. They showed that the hop dominating set problem is NP-complete for planar bipartite graphs and planar chordal graphs. Hop domination and some of its variants are studied in [3], [8], [9], and [11]. In this paper, we study another variation of hop domination called global hop domination. This is obviously the analogue to global domination studied in [2] and [4].

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i1.3916

Email addresses: `gemma.salasalan@g.msuiit.edu.ph` (G. Salasalan), `sergio.canoy@g.msuiit.edu.ph` (S. Canoy, Jr.)

Let $G = (V(G), E(G))$ be a simple graph. The distance between two vertices u and v of G, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting u and v. Any path connecting u and v of length $d_G(u, v)$ is called a *u-v geodesic*. The *open neighbourhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighbourhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *open neighbourhood* of a subset S of $V(G)$ is the set $N_G(S) = \cup_{v\in S} N_G(v)$ and its *closed neighbourhood* is the set $N_G[S] = N_G(S) \cup S$. The *degree* of v, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. The *minimum degree* of G is $\delta(G) = \min\{deg_G(v) : v \in V(G)\}$ and its *maximum degree* is $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. The *open hop neighbourhood* of vertex v of G is the set $N_G(v, 2) = \{w \in V(G) : d_G(v, w) = 2\}$. A set $S \subseteq V(G)$ is a *dominating set* (resp. *total dominating set*) of G if $N_G[S] = V(G)$ (resp. $N_G(S) = V(G)$). The smallest cardinality of a dominating (resp. total dominating) set of G, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is called the *domination number* (resp. *total domination number*) of G. A dominating (resp. total dominating) set of G with cardinality $\gamma(G)$ (resp. $\gamma_t(G)$), is called a *γ-set* (resp. γ_t-set) of G. It should be noted that only graphs without isolated vertices admit total dominating sets.

A set $S \subseteq V(G)$ is a *hop dominating set* of G if for each $x \in V(G) \setminus S$, there exists $z \in S$ such that $d_G(x, z) = 2$. The smallest cardinality of a hop dominating set of G, denoted by $\gamma_h(G)$, is called the *hop domination number* of G. A hop dominating set of G with cardinality $\gamma_h(G)$ is called a *γ_h-set* of G. A set $S \subseteq V(G)$ is a *global hop dominating set* of G if it is a hop dominating set of G and $\overline{G}$. The smallest cardinality of a global hop dominating set of G, denoted by $\gamma_{gh}(G)$, is called the *global hop domination number* of G. A global hop dominating set of G with cardinality $\gamma_{gh}(G)$ is called a *γ_{gh}-set* of G.

A set $D \subseteq V(G)$ is a *pointwise non-dominating set* of G if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $v \notin N_G(u)$. The smallest cardinality of a pointwise non-dominating set of G, denoted by $pnd(G)$, is called the *pointwise non-domination number* of G. A dominating set S which is also a pointwise non-dominating set of G is called a *dominating pointwise non-dominating set* of G. The smallest cardinality of a dominating pointwise non-dominating set of G will be denoted by $\gamma_{pnd}(G)$. Any pointwise non-dominating (resp. dominating pointwise non-dominating) set of G with cardinality $pnd(G)$ (resp. $\gamma_{pnd}(G)$), is called a *pnd-set* (resp. *γ_{pnd}-set*) of G. These concepts and parameters have been defined and used in [3] and [9].

2. Results

It is worth mentioning here that every graph G admits a global hop dominating set. Indeed, the vertex set $V(G)$ of G is a global hop dominating set. Further, we have

Remark 1. *$1 \leq \gamma_{gh}(G) \leq |V(G)|$ for any graph G. Moreover, $\gamma_{gh}(G) = 1$ if and only if $G = K_1$.*

Theorem 1. *Let G be a non-trivial graph. Then $\gamma_{gh}(G) = 2$ if and only if there exist distinct vertices x and y of G satisfying the following conditions:*

(*i*) $N_G(x, 2) \cap N_G(y, 2) = \varnothing$ *and* $V(G) \setminus \{x, y\} = N_G(x, 2) \cup N_G(y, 2)$*;*

(*ii*) $N_G(x, 2) = N_G(y) \setminus \{x\}$ *and* $N_G(y, 2) = N_G(x) \setminus \{y\}$*; and*

(*iii*) *if* $xy \in E(G)$*, then* $N_G(x) \setminus N_G(w) \neq \varnothing$ *for each* $w \in N_G(x, 2)$ *and* $N_G(y) \setminus N_G(v) \neq \varnothing$ *for each* $v \in N_G(y, 2)$*.*

Proof. Suppose $\gamma_{gh}(G) = 2$. Let $S = \{x, y\}$ be γ_{gh}-set of G. Suppose there exists $z \in N_G(x, 2) \cap N_G(y, 2)$. Then $xz, yz \in E(\overline{G})$. This implies that $d_{\overline{G}}(x, z) \neq 2$ and $d_{\overline{G}}(y, z) \neq 2$. Hence, S is not hop dominating set of $\overline{G}$, a contradiction. Thus, $N_G(x, 2) \cap N_G(y, 2) = \varnothing$. Further, $V(G) \setminus \{x, y\} = N_G(x, 2) \cup N_G(y, 2)$ because S is a hop dominating of set G. This shows that (*i*) holds.

Now let $z \in N_G(x, 2)$. Then $z \notin S$ and $xz \in V(\overline{G})$. Since S is hop dominating set of $\overline{G}$, it follows that $z \in N_{\overline{G}}(y, 2)$. This implies that $z \in N_G(y) \setminus \{x\}$. On the other hand, if $u \in N_G(y) \setminus \{x\}$, then $u \in N_G(x, 2)$ since S is a hop dominating of set G. Therefore, $N_G(x, 2) = N_G(y) \setminus \{x\}$. Similarly, $N_G(y, 2) = N_G(x) \setminus \{y\}$, showing that (*ii*) holds.

Next, suppose that $xy \in E(G)$ and let $w \in N_G(x, 2)$. Then $w \notin S$ and $xw \in E(\overline{G})$. Since S is a hop dominating set of $\overline{G}$, $w \in N_{\overline{G}}(y, 2)$. Hence, there exists $z \in V(G) \setminus S$ such that $z \in N_{\overline{G}}(w) \cap N_{\overline{G}}(y)$. It follows that $z \in N_G(x) \setminus N_G(w)$, i.e., $N_G(x) \setminus N_G(w) \neq \varnothing$. Similarly, $N_G(y) \setminus N_G(v) \neq \varnothing$ for each $v \in N_G(y, 2)$, showing that (*iii*) holds.

Conversely, suppose that there exist distinct vertices x and y of G satisfying conditions (*i*), (*ii*), and (*iii*). Let $S = \{x, y\}$. By (*i*), S is a hop dominating set of G. Let $v \in V(\overline{G}) \setminus S$. Assume, without loss of generality, that $v \in N_G(x, 2)$. Then $v \in N_G(y) \setminus \{x\}$ by (*ii*). Suppose $xy \notin E(G)$. Then $xy, xv \in E(\overline{G})$. Thus, $d_{\overline{G}}(y, v) = 2$. Next, suppose that $xy \in E(G)$. Then by (*iii*), there exists $z \in N_G(x) \setminus N_G(v)$. Hence, $z \in N_{\overline{G}}(v) \cap N_{\overline{G}}(y)$, i.e., $d_{\overline{G}}(y, v) = 2$. Therefore, S is a global hop dominating set of G. Accordingly, $\gamma_{gh}(G) = 2$. □

Theorem 2. *Let* G *be a graph of order* $n \geq 2$*. Then* $\gamma_{gh}(G) = n$ *if and only if one of the following statements holds:*

(*i*) *Every component of* G *is complete.*

(*ii*) *For each* $v \in V(G)$*,* $V(G) \setminus N_G(v)$ *is an independent set and* $N_G(v) = N_G(a)$ *for each* $a \in V(G) \setminus N_G(v)$*.*

Proof. Suppose $\gamma_{gh}(G) = n$. Suppose first that G is disconnected and suppose that G has a component C which is not complete. Then there exist distinct vertices $x, y \in V(C)$ such that $d_G(x, y) = d_C(x, y) = 2$. Let $S = V(G) \setminus \{x\}$. Then S is a hop dominating set of G. Let $z \in C$ such that $[x, z, y]$ is an x-y geodesic in G. Let C' be a component of G with $C' \neq C$ and pick any $w \in C'$. Then $[x, w, z]$ is an x-z geodesic in $\overline{G}$. It follows that $d_{\overline{G}}(x, z) = 2$. Thus, S is a hop dominating set of $\overline{G}$, showing that S is a global hop dominating set of G. Therefore, $\gamma_{gh}(G) \leq |S| = n - 1$, a contradiction. Accordingly, every component of G is complete.

Next, suppose that G is connected. Suppose further that $\overline{G}$ is connected. Then, clearly, $G \neq K_n$. Let $u, v \in V(G)$ such that $d_G(u, v) = 2$ and let $[u, p, v]$ be a u-v

geodesic in G. Then $S^* = V(G) \setminus \{u\}$ is a hop dominating set of G. Since $up \notin E(\overline{G})$, it follows that $d_{\overline{G}}(u,p) \geq 2$. It follows that there exists $q \in S$ such that $d_{\overline{G}}(u,q) = 2$. This shows that S^* is hop dominating set of $\overline{G}$. Thus, S^* is a global hop dominating set of G and $\gamma_{gh}(G) \leq |S^*| = n-1$, a contradiction. Therefore $\overline{G}$ is disconnected. Since $\gamma_{gh}(\overline{G}) = \gamma_{gh}(G) = n$, this would imply that every component of $\overline{G}$ is complete (as in the first case applied to $\overline{G}$). Let $v \in V(G) = V(\overline{G})$ and suppose there exist distinct vertices $a, b \in V(G) \setminus N_G(v)$ such that $ab \in E(G)$. Then $[a, v, b]$ is an a-b geodesic in $\overline{G}$, implying that $S_a = V(G) \setminus \{a\}$ is a hop dominating set of $\overline{G}$. Now, since $a \in V(G) \setminus N_G(v)$, it follows that $d_G(a,v) \geq 2$. This implies that there exists $w \in S_a$ such that $d_G(a,w) = 2$, showing that S_a is also a hop dominating set of G. Hence, $\gamma_{gh}(G) \leq |S_a| = n-1$, a contradiction. Therefore, $V(G) \setminus N_G(v)$ is an independent set. Let $a \in V(G) \setminus N_G(v)$. Let C_v be the component of $\overline{G}$ with $v \in C_v$. Since $a \in N_{\overline{G}}(v)$ and C_v is complete, $N_G(a) = N_G(v)$, that is, $az \in E(G)$ for every $z \in N_G(v)$. This shows that (ii) holds.

For the converse, suppose first that (i) holds. Then, clearly, $S = V(G)$ is the only hop dominating set of G. It follows that S is the only global hop dominating set of G. Thus, $\gamma_{gh}(G) = n$. Next, suppose that (ii) holds. Then every component of $\overline{G}$ is complete. Since $V(G) = V(\overline{G})$ is the only hop dominating set of $\overline{G}$, it follows that $V(G)$ is the only global hop dominating set of G. Therefore,$\gamma_{gh}(G) = n$. □

The next result is a consequence of Theorem 2.

Corollary 1. *$\gamma_{gh}(K_n) = \gamma_{gh}(K_{1,n-1}) = n$ for all integer $n \geq 2$.*

A set $S \subseteq V(G)$ is a *pairwise non-dominating* set of G if for each $v \in V(G) \setminus S$, there exists vertex $w \in S \cap N_G(v)$ such that $N_G(\{w, v\}) \neq V(G)$. A set $S \subseteq V(G)$ is a *pairwise and pointwise non-dominating* (*ppnd*) set of G if it is both a pairwise non-dominating and pointwise non-dominating set of G. The minimum cardinality of a *ppnd* set of G is denoted by $\gamma_{ppnd}(G)$. Any pairwise and pointwise non-dominating set of G with cardinality equal to $\gamma_{ppnd}(G)$ is called a γ_{ppnd}*-set* of G.

Remark 2. *A pairwise non-dominating set of G is a dominating set of G.*

Theorem 3. *Let G be any graph of order n. Then $1 \leq ppnd(G) \leq n$. Moreover,*

- (i) *$\gamma_{ppnd}(G) = 1$ if and only if $G = K_1$,*
- (ii) *$\gamma_{ppnd}(G) = 2$ if and only if one of the following statements holds:*
  - (a) $G = K_2$
  - (b) $G = \overline{K}_2$
  - (c) *There exist non-adjacent vertices $x, y \in V(G)$ such that $N_G(x) \cap N_G(y) = \varnothing$ and $N_G[x] \cup N_G[y] = V(G)$.*
  - (d) *There exist adjacent vertices $x, y \in V(G)$ such that $N_G(x) \cap N_G(y) = \varnothing$, $N_G(x) \cup N_G(y) = V(G)$, and for each $v \in N_G(x) \setminus \{y\}$ and $w \in N_G(y) \setminus \{x\}$, there exist $p \in N_G(y) \setminus N_G(v)$ and $q \in N_G(x) \setminus N_G(w)$.*

(iii) $\gamma_{ppnd}(G) = n$ *if and only if* $G = \overline{K}_n$ *or* G *is connected such that* $N_G(\{u, v\}) = V(G)$ *for each pair of adjacent vertices* $u, v \in V(G)$.

Proof. Clearly, by definition, a pairwise and pointwise non-dominating set of G is nonempty. Thus, $ppnd(G) \geq 1$. Also, since $V(G)$ is a pairwise and pointwise non-dominating set of G, it follows that $\gamma_{ppnd}(G) \leq n$.

(i) Next, suppose that $\gamma_{ppnd}(G) = 1$, say $S = \{v\}$ is a γ_{ppnd}-set of G. If such a vertex outside S exists, then this would require two distinct vertices from S to satisfy the property of S. This forces us to conclude that $G = K_1$. Further, since $\gamma_{ppnd}(K_1) = 1$, (i) holds.

(ii) Suppose now that $\gamma_{ppnd}(G) = 2$, say $S = \{x, y\}$ is a γ_{ppnd}-set of G. If $n = 2$, then $G = K_2$ or $G = \overline{K}_2$. Suppose $n \geq 3$ and assume first that $xy \notin E(G)$. Since S is a *ppnd* set of G, $N_G(x) \cap N_G(y) = \varnothing$ and $N_G[x] \cup N_G[y] = V(G)$. Hence, (c) holds. Suppose $xy \in E(G)$. Again, since S a *ppnd* set of G, $N_G(x) \cap N_G(y) = \varnothing$ and $N_G(x) \cup N_G(y) = V(G)$. Let $v \in N_G(x) \setminus \{y\}$. Since $N_G(\{x, v\}) \neq V(G)$, there exists $p \in V(G) \setminus N_G(\{x, v\})$. Since $N_G(x) \cap N_G(y) = \varnothing$, it follows that $p \in N_G(y) \setminus N_G(v)$. Similarly, for each $w \in N_G(y) \setminus \{x\}$, there exists $q \in N_G(x) \setminus N_G(w)$, showing that (d) holds.

For the converse, suppose first that $G = K_2$ or $G = \overline{K}_2$. Then, clearly, $\gamma_{ppnd}(G) = 2$. Next, suppose that (c) holds. Let $S = \{x, y\}$ and let $v \in V(G) \setminus S$. By assumption, we may assume that $v \in N_G(x) \setminus N_G(y)$. Since $y \in V(G) \setminus N_G(\{x, v\})$, $N_G(\{x, v\}) \neq V(G)$. Thus, S is a *ppnd* set of G. Since $G \neq K_1$, it follows that S is a γ_{ppnd}-set, i.e., $\gamma_{ppnd}(G) = |S| = 2$. Finally, suppose that (d) holds. Let $S' = \{x, y\}$ and let $v \in V(G) \setminus S$. Assume, without loss of generality, that $v \in N_G(x)$. By assumption, there exists $p \in N_G(y) \setminus N_G(v)$. This implies that $p \notin N_G(\{x, v\})$. Therefore, S is a γ_{ppnd}-set of G, implying that $\gamma_{ppnd}(G) = 2$. This proves statement (ii).

(iii) Suppose $\gamma_{ppnd}(G) = n$. Suppose first that G is disconnected. Suppose further that $G \neq \overline{K}_n$. Then G has a non-trivial component C. Hence, there exist distinct vertices $x, y \in V(C)$ such that $xy \in V(G)$. Let $S_x = V(G) \setminus \{x\}$. Then $y \in S \cap N_G(x)$. Since G is disconnected, $N_G(x, y) \neq V(G)$ and there exists $w \in S \setminus N_G(x)$. Hence, S is a *ppnd* set of G and $\gamma_{ppnd}(G) \leq |S| = n - 1$, a contradiction. Therefore, $G = \overline{K}_n$.

Next, suppose that G is connected. Suppose there exist distinct adjacent vertices $u, v \in V(G)$ such that $N_G(\{u, v\}) \neq V(G)$, say $w \in V(G) \setminus N_G(\{u, v\})$. Let $S_u = V(G) \setminus \{u\}$. Then $v, w \in S$, $uw \notin E(G)$, $uv \in E(G)$, and $N_G(\{u, v\}) \neq V(G)$. This implies that S is a pairwise and pointwise non-dominating set of G. Hence, $\gamma_{ppnd}(G) \leq |S| = n - 1$, a contradiction. Therefore, $N_G(\{u, v\}) = V(G)$ for each pair of adjacent vertices $u, v \in V(G)$.

For the converse, suppose first that $G = \overline{K}_n$. Then, clearly, $S = V(G)$ is the only pairwise and pointwise non-dominating set of G. Thus, $\gamma_{ppnd}(G) = n$. Next, suppose that G is connected and satisfies the condition that $N_G(\{u, v\}) = V(G)$ for each pair of adjacent vertices $u, v \in V(G)$. Let S be a γ_{ppnd}-set and suppose that there exists $w \in V(G) \setminus S$. Then there exists $q \in S \cap N_G(w)$ such that $N_G(\{q, w\}) \neq V(G)$, contrary to our assumption. Therefore, $S = V(G)$ and $\gamma_{ppnd}(G) = n$. □

Theorem 4. *Let G and H be any two graphs. A set $S \subseteq V(G+H)$ is a global hop dominating set of $G+H$ if and only if $S = S_G \cup S_H$ and S_G and S_H are pairwise and pointwise non-dominating sets of G and H, respectively.*

Proof. Suppose S is a global hop dominating set of $G+H$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Since S is a hop dominating set of $G+H$, $S_G \neq \varnothing$ and $S_H \neq \varnothing$. Let $v \in V(G) \setminus S_G$. Since S is a hop dominating set of $G+H$, there exists $u \in S_G$ such that $d_{G+H}(u,v) = 2$. This implies that $uv \notin E(G)$. Now, since S is also a hop dominating set of $\overline{G+H} = \overline{G} \cup \overline{H}$, there exists $w \in S_G$ such that $d_{\overline{G+H}}(v,w) = d_{\overline{G}}(v,w) = 2$. This implies that $vw \in E(G)$ and there exists $z \in V(\overline{G})$ such that $z \in N_{\overline{G}}(v) \cap N_{\overline{G}}(w)$. Thus, $z \notin N_G(\{v,w\})$, showing that $N_G(\{v,w\}) \neq V(G)$. Therefore, S_G is a pairwise and pointwise non-dominating set of G. Similarly, S_H is a pairwise and pointwise non-dominating set of H.

For the converse, suppose that $S = S_G \cup S_H$ and S_G and S_H are pairwise and pointwise non-dominating sets of G and H, respectively. Let $v \in V(G+H) \setminus S$. Suppose, without loss of generality, that $v \in V(G) \setminus S_G$. Since S_G is a pairwise and pointwise non-dominating set of G, there exist $u, w \in S_G \subseteq S$ such that $uv \notin E(G)$, $wv \in E(G)$, and $N_G(\{w,v\}) \neq V(G)$. It follows that $d_{G+H}(u,v) = 2$ and $d_{\overline{G+H}}(w,v) = d_{\overline{G}}(w,v) = 2$. Thus, S is a global dominating set of $G+H$. □

The next result is immediate from Theorem 4 and Theorem 3(iii).

Corollary 2. *Let G and H be any two graphs. Then $\gamma_{gh}(G+H) = \gamma_{ppnd}(G) + \gamma_{ppnd}(H)$. In particular,*

(i) $\gamma_{gh}(K_n + H) = n + \gamma_{ppnd}(H)$ for all integer $n \geq 1$, and

(ii) $\gamma_{gh}(K_{m,n}) = m + n$ for all positive integers m and n.

The *corona* of graphs G and H, denoted by $G \circ H$, is the graph obtained from G by taking a copy H^v of H and forming the join $\langle v \rangle + H^v = v + H^v$ for each $v \in V(G)$.

Theorem 5. *Let G be a connected non-trivial graph and let H be any graph. A set $C \subseteq V(G \circ H)$ is a global hop dominating set of $G \circ H$ if and only if $C = A \cup (\cup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfy the following properties:*

(i) For each $w \in V(G) \setminus A$, there exists $x_w \in A$ with $d_G(w, x_w) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $S_y \neq \varnothing$.

(ii) S_v is a dominating set of H^v for each $v \in N_G(A) \setminus A$.

(iii) S_v is a pointwise non-dominating set of H^v for each $v \in A \setminus N_G(A)$.

(iv) S_v is a dominating pointwise non-dominating set of H^v for each $v \in V(G) \setminus N_G[A]$.

Proof. Suppose C is a global hop dominating set of $G \circ H$ and let $A = C \cap V(G)$. Let $S_v = C \cap V(H^v)$ for each $v \in V(G)$. Then $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and $C = A \cup (\cup_{v \in V(G)} S_v)$. Now, since C is a hop dominating set of G, (i) holds. Next, let $v \in V(G)$ and consider the following cases:

Case 1: $v \in N_G(A) \setminus A$

Let $x \in V(H^v) \setminus S_v$. Since C is hop dominating set of $\overline{G \circ H}$, there exists $y \in C$ such that $d_{\overline{G \circ H}}(x, y) = 2$. Since $v \notin A$ and $V(\overline{G \circ H}) \setminus V(v + H^v) \subseteq N_{\overline{G \circ H}}(x)$, it follows that $y \in S_v$. Thus, $y \in S_v \cap N_{H^v}(x)$, showing that S_v is a dominating set of H^v. Therefore, (ii) holds.

Case 2: $v \in A \setminus N_G(A)$

Let $w \in A \setminus N_G(A)$ and let $q \in V(H^v) \setminus S_v$. Since C is a hop dominating set of $G \circ H$, there exists $u \in C$ such that $d_{G \circ H}(q, u) = 2$. By assumption, $u \notin A$. Thus, $u \in S_v$ and $qu \notin E(H^v)$. Therefore S_v is a pointwise non-dominating set of H^v, showing that (iii) holds.

Case 3: $v \in V(G) \setminus N_G[A]$

Since $v \notin A$ and C is a hop dominating set of G, similar arguments in *Case* 1 will show that S_v is a dominating set of H^v. Further, since $v \notin N_G(A)$, the arguments in *Case* 2 can be used to show that S_v is a pointwise non-dominating set of H^v, showing that (iv) holds.

For the converse, suppose that C has the given form and satisfies properties (i), (ii), (iii), and (iv). Next, let $z \in V(G \circ H) \setminus C = V(\overline{G \circ H}) \setminus C$ and let $v \in V(G)$ such that $z \in V(v + H^v)$. Consider the following cases:

Case 1. $z = v$

Then there exists $h \in C$ such that $d_{G \circ H}(z, h) = 2$, by (i). Now, from the assumption that (ii) and (iv) hold, it follows that $S_z \neq \varnothing$. Pick any $p \in S_z$ and $y \in V(H^w)$, where $w \in V(G) \cap N_G(z)$. Then $zy, yp \in E(\overline{G \circ H})$; hence, $d_{\overline{G \circ H}}(z, p) = 2$.

Case 2. $z \neq v$

Then $z \in V(H^v) \setminus S_v$. If $v \in N_G(A)$, then $d_{G \circ H}(z, a) = 2$ for $a \in A \cap N_G(v)$. If $v \notin N_G(A)$, then there exists $b \in S_v \subset C$ such that $d_{G \circ H}(z, b) = 2$ by (iii) and (iv).

Next, suppose first that $v \in A$. Pick any $w \in V(G) \setminus \{v\}$ and let $p \in V(H^w)$. Then $p \in N_{\overline{G \circ H}}(z) \cap N_{\overline{G \circ H}}(v)$. Thus, $d_{\overline{G \circ H}}(z, v) = 2$. Suppose now that $v \notin A$. By (ii) and (iv), S_v is a dominating set of H^v. It follows that there exists $q \in S_v \cap N_{H^v}(z)$. Pick any $u \in V(G) \setminus \{v\}$. Then $u \in N_{\overline{G \circ H}}(z) \cap N_{\overline{G \circ H}}(q)$. Hence, there exists $q \in C$ such that $d_{\overline{G \circ H}}(z, q) = 2$.

Accordingly, C is a hop dominating set of $G \circ H$ and $\overline{G \circ H}$, showing that C is a global hop dominating set of $G \circ H$. □

Corollary 3. *Let G be a connected non-trivial graph and let H be any graph. Then $\gamma_{gh}(G \circ H) = |V(G)|$.*

Proof. Let $A = V(G)$ and set $S_v = \varnothing$ for each $v \in V(G)$. Then $C = A = A \cup (\cup_{v \in V(G)} S_v)$ is a global hop dominating set of G by Theorem 5. Hence, $\gamma_{gh}(G \circ H) \leq |C| = |V(G)|$.

Next, let C_0 be a γ_{gh}-set of $G \circ H$. Then $C_0 = A_0 \cup (\cup_{v \in V(G)} R_v)$, where $A_0 \subseteq V(G)$ and $R_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfy conditions (i), (ii), (iii), and (iv) of Theorem 5. Since C_0 is a γ_{gh}-set of $G \circ H$, it follows that $R_v = \varnothing$ for all $v \in D_1 = A_0 \cap N_G(A_0)$. From conditions (ii), (iii), and (iv), we find that $|R_v| \geq 1$ for each $v \in D_2 = V(G) \setminus D_1$. Thus, $\gamma_{gh}(G \circ H) = |C_0| = |A_0| + \sum_{v \in D_2} |R_v| \geq |A_0| + |D_2| = |V(G)| + (|A_0| - |D_1|) \geq |V(G)|$. Therefore, $\gamma_{gh}(G \circ H) = |V(G)|$. □

The *lexicographic product* of graphs G and H, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v,a)(u,b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that every non-empty subset C of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S}[\{x\} \times T_x]$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

Theorem 6. *Let G and H be connected non-trivial graphs. A subset $C = \cup_{x \in S}[\{x\} \times T_x]$ of $V(G[H])$ is a global hop dominating set of $G[H]$ if and only if the each following conditions holds:*

(i) *S is both a dominating and a hop dominating set of G.*

(ii) *T_x is a pointwise non-dominating set of H for each $x \in S$ with $|N_G(x,2) \cap S| = 0$.*

(iii) *T_x is a dominating set of H for each $x \in S$ with $S \cap N_G(x) = \varnothing$ or $[V(G) \setminus N_G(x)] \cap [V(G) \setminus N_G(y)] = \varnothing$ for each $y \in S \cap N_G(x)$. If, in addition, $N_G[x] = V(G)$, then T_x is a pairwise non-dominating set of H.*

(iv) *For each $z \in V(G) \setminus S$, there exists $y \in S \cap N_G(z)$ such that $[V(G) \setminus N_G(z)] \cap [V(G) \setminus N_G(y)] \neq \varnothing$.*

Proof. Suppose C is a global hop dominating set of $G[H]$. Let $u \in V(G) \setminus S$ and pick any $a \in V(H)$. Since C is a hop dominating set of $G[H]$ and $(u,a) \notin C$, there exists $(y,b) \in C$ such that $d_{G[H]}((u,a)(y,b)) = 2$. This implies that $y \in S$ and $d_G(u,y) = 2$. Also, since C is a hop dominating set of $\overline{G[H]}$ and $(u,a) \notin C$, there exists $(z,c) \in C$ such that $d_{\overline{G[H]}}((u,a)(z,c)) = 2$. It follows that $z \in S$ and $d_G(u,z) = 1$. Hence, S is both a dominating and a hop dominating set of G, showing that (i) holds.

Let $x \in S$. Suppose that $|N_G(x,2) \cap S| = 0$. Then T_x is a pointwise non-dominating set of H. Hence, (ii) holds. Suppose now that $S \cap N_G(x) = \varnothing$ or $[V(G) \setminus N_G(x)] \cap [V(G) \setminus N_G(y)] = \varnothing$ for each $y \in S \cap N_G(x)$. Let $p \in V(H) \setminus T_x$. Since $(x,p) \in V(\overline{G[H]}) \setminus C$ an C is a hop dominating set of $\overline{G[H]}$, there exists $(w,q) \in C$ such that $d_{\overline{G[H]}}((x,p)(w,q)) = 2$, that is, $d_{G[H]}((x,p)(w,q)) = 1$. If $S \cap N_G(x) = \varnothing$, then $w = x$ and $q \in T_x \cap N_H(p)$, implying that T_x is a dominating set of H. Suppose $S \cap N_G(x) \neq \varnothing$. Suppose further that $w \neq x$. Then $w \in S \cap N_G(x)$. By assumption, $[V(G) \setminus N_G(x)] \cap [V(G) \setminus N_G(w)] = \varnothing$. Let $[(x,p),(u,t),(w,q)]$ be an (x,p)-(w,q) geodesic in $\overline{G[H]}$. Suppose $u \neq x$. Since $xu \in E(\overline{G})$, $u \in V(G) \setminus N_G(x)$. The assumption would now imply that $u \notin V(G) \setminus N_G(w)$. Thus, $u \in N_G(w)$, a contradiction. Hence, $u = x$. This, however, is not possible because $xw \in E(G)$. Therefore, $w = x$, implying that $q \in T_x \cap N_H(p)$. Hence, T_x is a dominating set of H. Finally, suppose that $N_G[x] = V(G)$. Then $u = x$ and $t \in N_{\overline{H}}(p) \cap N_{\overline{H}}(q)$. It follows that $t \notin N_H[\{p,q\}]$. Thus, T_x is a pairwise non-dominating set of H. Therefore, (iii) holds.

Now let $z \in V(G) \setminus S$. Choose any $b \in V(H)$. Since C is a hop dominating set of $\overline{G[H]}$, there exists $(y,c) \in C$ such that $d_{\overline{G[H]}}((z,b)(y,c)) = 2$, that is, $d_{G[H]}((z,b)(y,c)) = 1$. Hence, $y \in S \cap N_G(z)$. Let $[(z,b),(s,d),(y,c)]$ be a (z,b)-(y,c) geodesic in $\overline{G[H]}$. Then $s \in [V(G) \setminus N_G(z)] \cap [V(G) \setminus N_G(y)]$, showing that (iv) holds.

For the converse, suppose that C satisfies properties (i), (ii), (iii), and (iv). By (i) and (ii), C is a hop dominating set of $G[H]$. Let $(v,a) \in V(\overline{G[H]}) \setminus C$ and consider the following cases:

Case 1. $v \notin S$

By (iv), let $y \in S \cap N_G(v)$ and let $u \in [V(G) \setminus N_G(v)] \cap [V(G) \setminus N_G(y)] = \varnothing$. Let $p \in T_y$. Then $(y,p) \in C$ and $[(v,a),(u,a),(y,p)]$ is a (v,a)-(y,p) geodesic in $\overline{G[H]}$. Thus, $d_{\overline{G[H]}}((v,a)(y,p)) = 2$.

Case 2. $v \in S$

Suppose $S \cap N_G(v) \neq \varnothing$ and $[V(G) \setminus N_G(v)] \cap [V(G) \setminus N_G(y)] \neq \varnothing$ for some $y \in S \cap N_G(v)$. Choose any $q \in T_y$ and let $w \in [V(G) \setminus N_G(v)] \cap [V(G) \setminus N_G(y)]$. Then $(y,q) \in C$ and $[(v,a),(w,a),(y,q)]$ is a (v,a)-(y,q) geodesic in $\overline{G[H]}$. Thus, $d_{\overline{G[H]}}((v,a)(y,q)) = 2$. Next, suppose that $S \cap N_G(v) = \varnothing$ or $[V(G) \setminus N_G(v)] \cap [V(G) \setminus N_G(y)] = \varnothing$ for all $y \in S \cap N_G(v)$. Suppose $N_G[v] = V(G)$. Then T_v is a pairwise non-dominating set of H by (iii). Hence, there exists $d \in T_v \cap N_H(a)$ such that $N_H(\{a,d\}) \neq V(H)$. This implies that $(v,d) \in C$ and there exists $t \in V(H) \setminus N_H(\{a,d\})$. Hence, $[(v,a),(v,t),(v,d)]$ is a (v,a)-(v,d) geodesic in $\overline{G[H]}$, that is, $d_{\overline{G[H]}}((v,a),(v,d)) = 2$. Suppose $N_G[v] \neq V(G)$. By (iii), T_v is a dominating set of H. Again, let $d \in T_v \cap N_H(a)$ and pick $w \in V(H) \setminus N_G[v]$. Then $(v,d) \in C$ and $[(v,a),(w,a),(v,d)]$ is a (v,a)-(v,d) geodesic in $\overline{G[H]}$. Thus, $d_{\overline{G[H]}}((v,a),(v,d)) = 2$.

Therefore, C is a hop dominating set of $\overline{G[H]}$. Accordingly, C is a global hop dominating set of $G[H]$. □

A set $S \subseteq V(G)$ is said to be *dominating complement-neighborhood intersecting* (*dcni*) (resp. *total dominating complement-neighborhood intersecting* (*tdcni*)) set of a graph G if for each $v \in V(G) \setminus S$ (resp. for each $v \in S$), there exists $w \in S \cap N_G(v)$ such that $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w)) \neq \varnothing$. Let

$$\gamma^h_{cni}(G) = \min\{|S| : S \text{ is a } dcni \text{ hop dominating set of } G\}, \text{ and}$$

$$\gamma_{tcni}(G) = \min\{|S| : S \text{ is a } tdcni \text{ set of } G\}.$$

Any *dcni* hop dominating set of G with cardinality $\gamma^h_{cni}(G)$ is called a γ^h_{cni}-set of G and any *tdcni* set of G with cardinality $\gamma_{tcni}(G)$ is called a γ_{tcni}-set of G.

Observe that for any graph G, the vertex set $V(G)$ is a dominating complement-neighborhood intersecting and hop dominating set of G. Also, if G_1 is the graph obtained from the cycle $C_4 = [a,b,c,d,a]$ by adding the edges av and bw, then $S = \{a,b\}$ is a *dcni* hop dominating set of G_1.

Proposition 1. *Let G be graph without isolated vertices.*

(i) *If G is disconnected, then G admits a tdcni set.*

(ii) *If G admits a tdcni set, then $3 \leq \gamma_{tcni}(G) \leq |V(G)|$.*

(iii) *If $\gamma_t(G) \neq 2$, then G admits a tdcni set. If, in addition, G has at most one vertex of degree one, then $\gamma_{tcni}(G) \leq |V(G)| - 1$.*

Proof. (i) Suppose G is disconnected and let $S = V(G)$. Let $v \in S$. Since G has no isolated vertices, there exists $w \in S \cap N_G(v)$. Let C_1 and C_2 be distinct components of G with $w, v \in C_1$. Pick any $z \in C_2$. Then $z \in (V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w))$. Hence, $S = V(G)$ is a *tdcni* set of G.

(ii) Suppose G admits a *tcnid* set. Since a *tdcni* set is a total dominating set, it follows that $2 \leq \gamma_{tcni}(G) \leq n$. Suppose $\gamma_{tcni}(G) = 2$, say $S = \{x, y\}$ is a γ_{tcni}-set of G. Since S is a dominating set, $V(G) \setminus S \subseteq N_G(\{x, y\})$. Hence, $(V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(y)) = \varnothing$, contrary to the assumption that S is a *tdcni* set. Thus, $3 \geq \gamma_{tcni}(G)$.

(iii) Suppose $\gamma_t(G) \neq 2$. Let $v \in V(G)$ and let $w \in V(G) \cap N_G(v)$. By assumption, $N_G(\{v, w\} \neq V(G)$. This implies that there exists $y \in (V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w))$, showing that $V(G)$ is a *tdcni* set of G. Suppose further that G has at most one vertex of degree one. Let $v \in V(G)$ such that $\delta(G) = deg_G(v)$ and let $S = V(G) \setminus \{v\}$. Note that if $deg_G(v) = 1$, then $deg_G(w) \geq 2$ for all $w \in V(G) \setminus \{v\}$. Let $u \in S \cap N_G(v)$. Since $\gamma(G) \neq 2$, $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w)) \neq \varnothing$. Let $z \in S$. Since $deg_G(z) \geq 2$, there exists $y \in S \cap N_G(z)$. Again, since $\gamma(G) \neq 2$, $(V(G) \setminus N_G(z)) \cap (V(G) \setminus N_G(y)) \neq \varnothing$. This implies that S is a *tdcni* set and $\gamma_{tcni}(G) \leq |S| = |V(G)| - 1$. □

Corollary 4. *Let G and H be non-trivial connected graphs.*

(i) *If $\gamma(G) = 1$, then $\gamma_{gh}(G[H]) \leq \gamma^h_{cni}(G).\gamma_{ppnd}(H)$.*

(ii) *If $\gamma(G) \neq 1$, then $\gamma_{gh}(G[H]) \leq \gamma^h_{cni}(G).\gamma_{pnd}(H)$.*

Proof. Let S be a γ^h_{cni}-set of G. Let D_1 and D_2 be, respectively, a γ_{ppnd}-set and γ_{pnd}-set of H. Set $T_x = D$ for each $x \in S$ and $R_x = D_2$. If $\gamma(G) = 1$, then $C_1 = \cup_{x \in S}[\{x\} \times T_x] = S \times D_1$ is a global hop dominating set of $G[H]$ by Theorem 6. Hence, $\gamma_{gh}(G[H]) \leq |C_1| = |S||D_1| = \gamma^h_{cni}(G).\gamma_{ppnd}(H)$, proving that (i) holds. If $\gamma(G) \neq 1$, then $C_2 = \cup_{x \in S}[\{x\} \times R_x] = S \times D_2$ is a global hop dominating set of $G[H]$ by Theorem 6. Hence, $\gamma_{gh}(G[H]) \leq |C_2| = |S||D_2| = \gamma^h_{cni}(G).\gamma_{pnd}(H)$, showing that (ii) holds. □

Remark 3. *The bounds in Corollary 4 are sharp.*

To see this, let G_1 be the graph obtained from the cycle $C_4 = [a, b, c, d, a]$ by adding the edges av and bw, and let $H = P_3$. As pointed out earlier, $S = \{a, b\}$ is a *dcni* hop dominating set of G_1. In fact, $\gamma^h_{cni}(G_1) = |S| = 2$. Now, $\gamma_{pnd}(H) = 2$ by Theorem 3(iii). It can easily be verified that $\gamma_{gh}(G[H]) = 4 = \gamma^h_{cni}(G).\gamma_{pnd}(H)$. Also, $\gamma_{gh}(P_4[P_2]) = \gamma^h_{cni}(P_4).\gamma_{pnd}(P_2) = 2(2) = 4$ and $\gamma_{gh}(P_2[P_2]) = \gamma_{gh}(K_4) = \gamma^h_{cni}(P_4).\gamma_{ppnd}(P_2) = 2(2) = 4$.

The *Cartesian product* of graphs G and H, denoted by $G\Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ such that $(v, p)(u, q) \in E(G\Box H)$ if and only if $uv \in E(G)$ and $p = q \in E(H)]$ or $u = v$ and $pq \in E(G)$.

Theorem 7. *Let G and H be connected non-trivial graphs. A subset $C = \cup_{x\in S}[\{x\} \times T_x]$ of $V(G\Box H)$ is a global hop dominating set of $G\Box H$ if and only if the following conditions hold:*

(i) *For each $x \in V(G) \setminus S$ and for each $p \in V(H)$,*

- (a) *there exists $y \in S \cap N_G(x)$ such that $T_y \cap N_H(p) \neq \varnothing$ or there exists $z \in S \cap N_G(x, 2)$ such that $p \in T_z$, and*
- (b) *there exists $w \in S \cap N_G(x)$ such that $p \in T_w$ and $[N_H[p] \neq V(H)$ or $(V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(w)) \neq \varnothing]$.*

(ii) *For each $v \in S$ and for each $p \in V(H) \setminus T_v$, the following statements are satisfied:*

- (c) *$N_H(p, 2) \cap T_v \neq \varnothing$ or there exists $y \in S \cap N_G(v)$ such that $T_y \cap N_H(p) \neq \varnothing$, or there exists $z \in S \cap N_G(v, 2)$ such that $p \in T_z$.*
- (d) *$N_H(p) \cap T_v \neq \varnothing$ and $[V(G) \setminus N_G[v] \neq \varnothing$ or $|V(H)| \geq 3]$ or there exists $u \in S \cap N_G(v)$ such that $p \in T_u$ and $[N_H[p] \neq V(H)$ or $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(u)) \neq \varnothing]$.*

Proof. Suppose C is a global hop dominating set of $G\Box H$. Let $x \in V(G) \setminus S$ and let $p \in V(H)$. Since C is a hop dominating set of $G\Box H$ and $(x, p) \notin C$, there exists $(y, q) \in C$ such that $d_{G\Box H}((x, p)(y, q)) = 2$. Since $y \in S$, $x \neq y$. If $xy \in E(G)$, then $pq \in E(H)$. Hence, $q \in T_y \cap N_H(p)$. So suppose that $y \notin N_G(x)$. Since $d_{G\Box H}((x, p)(y, q)) = 2$, it follows that $y \in N_G(x, 2)$ and $p = q$. Hence, $p \in T_y$, showing that (a) holds. Now, since C is also a hop dominating set of $\overline{G\Box H}$, there exists $(w, t) \in C$ such that $d_{\overline{G\Box H}}((x, p)(w, t)) = 2$. It follows that $d_{G\Box H}((x, p)(w, t)) = 1$. This implies that $w \in S \cap N_G(x)$ and $p \in T_w$. Now, if $[(x, p), (z, s), (w, t)]$ is an (x, p)-(w, t) geodesic in $\overline{G\Box H}$, then $s \in V(H) \setminus N_H[p]$ or $z \in ((V(G) \setminus N_G(x) \cap (V(G) \setminus N_G(w))$. This shows that (b) holds.

Next, let $v \in S$ and let $p \in V(H) \setminus T_v$. Since C is a hop dominating set of $G\Box H$ and $(v, p) \notin C$, there exists $(y, q) \in C$ such that $d_{G\Box H}((v, p)(y, q)) = 2$. Suppose $y = v$. Then $d_H(p, q) = 2$ and so $q \in N_G(p, 2) \cap T_v$. Suppose $y \neq v$. If $d_G(y, v) = 1$, then $y \in S \cap N_G(v)$ and $d_H(p, q) = 1$, i.e. $q \in T_y \cap N_H(p)$. If $d_G(y, v) \neq 1$, then $d_G(y, v) = 2$, Hence, $y \in S \cap N_G(v, 2)$ and $p = q$, that is, $p \in T_y$. Thus, (b) holds.

On the other hand, since C is also a hop dominating set of $\overline{G\Box H}$ and $(v, p) \notin V(\overline{G\Box H}) \setminus C$, there exists $(u, t) \in C$ such that $d_{\overline{G\Box H}}((v, p)(u, t)) = 2$. Again, this would imply that $d_{G\Box H}((v,p)(u,t)) = 1$. If $u = v$, then $t \in N_H(p) \cap T_v$. Since $d_{\overline{G\Box H}}((v,p)(u,t)) = d_{\overline{G\Box H}}((v,p)(v,t)) = 2$, $V(G) \setminus N_G[v] \neq \varnothing$ or $|V(H)| \geq 3$. Suppose $u \neq v$. Then $u \in S \cap N_G(v)$ and $p \in T_u$. Since $d_{\overline{G\Box H}}((v,p)(u,t)) = 2$, $V(H) \setminus N_H(p) \neq \varnothing$ or $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(u)) \neq \varnothing$.

For the converse, suppose that C satisfies properties (i) and (ii). Let $(v,p) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. $v \notin S$

By the assumption that (a) of (i) holds, suppose first that there exists $y \in S \cap N_G(x)$ such that $T_y \cap N_H(p) \neq \varnothing$. Let $q \in T_y \cap N_H(p) \neq \varnothing$. Then $(y,q) \in C$ and $d_{G\Box H}((v,p)(y,q)) = d_G(v,y) + d_H(p,q) = 2$. Next, suppose that there exists $z \in S \cap N_G(v,2)$ such that $p \in T_z$. Then $(z,p) \in C$ and $d_{G\Box H}((v,p)(z,p)) = d_G(v,z) = 2$.

Since (b) of (i) also holds, suppose that there exists $w \in S \cap N_G(v)$ such that $p \in T_w$. Then $(w,p) \in C \cap N_{G\Box H}((v,p))$. If $N_H[p] \neq V(H)$, we may pick any $s \in V(H) \setminus N_H[p]$. Then $(w,s) \notin N_{G\Box H}((v,p)) \cup N_{G\Box H}((w,p))$. It follows that $[(v,p),(w,s),(w,p)]$ is a (v,p)-(w,p) geodesic in $\overline{G\Box H}$. Thus, $d_{G\Box H}((v,p)(w,p)) = 2$. Instead of $N_H(p) \neq V(H)$, suppose that $(V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(w)) \neq \varnothing]$, say $u \in (V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(w))$. Then $[(v,p),(u,p),(w,p)]$ is a (v,p)-(w,p) geodesic in $\overline{G\Box H}$, implying that $d_{G\Box H}((v,p)(w,p)) = 2$.

Case 2. $v \in S$

Utilizing (c) of (ii), suppose first that $N_H(p,2) \cap T_v \neq \varnothing$. Let $q \in N_H(p,2) \cap T_v$. Then $(v,q) \in C$ and $d_{G\Box H}((v,p)(v,q)) = d_H(p,q) = 2$. Suppose there exists $y \in S \cap N_G(v)$ such that $T_y \cap N_H(p) \neq \varnothing$. Then $(y,t) \in C$ and $d_{G\Box H}((v,p)(y,t)) = 2$, where $t \in T_y \cap N_H(p)$. If there exists $z \in S \cap N_G(v,2)$ such that $p \in T_z$, then $(z,p) \in C$ and $d_{G\Box H}((v,p)(z,p)) = 2$.

Now, using (d) of (ii), assume that $N_H(p) \cap T_v \neq \varnothing$, say $a \in N_H(p) \cap T_v$. Then $(v,a) \in C$. If there exists $w \in V(G) \setminus N_G[v]$, then $[(v,p),(w,p),(v,a)]$ is a (v,p)-(v,a) geodesic in $\overline{G\Box H}$. Thus, $d_{G\Box H}((v,p)(v,a)) = 2$. If $|V(H)| \geq 3$, then we may pick any $b \in V(H) \setminus \{a,p\}$. Let $z \in N_G(v)$. Then $[(v,p),(z,b),(v,a)]$ is a (v,p)-(v,a) geodesic in $\overline{G\Box H}$. Hence, $d_{G\Box H}((v,p)(v,a)) = 2$. Next, assume that there exists $u \in S \cap N_G(v)$ such that $p \in T_u$. Then $(u,p) \in C$. If $V(H) \setminus N_H(p)$, then $[(v,p),(u,l),(u,p)]$ is a (v,p)-(u,p) geodesic in $\overline{G\Box H}$. This implies that $d_{G\Box H}((v,p)(u,p)) = 2$. If there exists $z \in (V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(u))$, then $[(v,p),(z,p),(u,p)]$ is a (v,p)-(u,p) geodesic in $\overline{G\Box H}$, implying that $d_{G\Box H}((v,p)(u,p)) = 2$.

Therefore, C is a hop dominating set of $G\Box H$ and $\overline{G\Box H}$. Accordingly, C is a global hop dominating set of $G\Box H$. $\Box$

Corollary 5. *Let G and H be non-trivial connected graphs.*

(i) If $\gamma(H) = 1$, then $\gamma_{gh}(G\Box H) \leq |V(H)|.\gamma_{tcni}(G)$.

(ii) If $\gamma(H) \neq 1$, then $\gamma_{gh}(G\Box H) \leq |V(H)|.\gamma_t(G)$.

Proof. Let S be a γ_{tcni}-set of G and let $T_x = V(H)$ for all $x \in S$. Let $C = \cup_{x\in S}[\{x\} \times T_x] = S \times V(H)$. If $\gamma(H) = 1$, then C is a global hop dominating set of $G\Box H$ by Theorem 7. Thus, $\gamma_{gh}(G\Box H) \leq |C| = |V(H)|.\gamma_{tcni}(G)$.

Next, let S' be a γ_t-set of G and let $R_x = V(H)$ for all $x \in S'$. Let $C' = \cup_{x\in S}[\{x\} \times R_x] = S \times V(H)$. If $\gamma(H) \neq 1$, then C' is a global hop dominating set of $G\Box H$ by Theorem 7. This implies that $\gamma_{gh}(G\Box H) \leq |C'| = |V(H)|.\gamma_t(G)$. □

3. Conclusion

The global hop dominating sets in the join, corona, lexicographic product, and the Cartesian product of two graphs have been characterized. From these characterizations, we determined either the exact values or upper bounds of the global hop domination numbers of the corresponding graphs.

Acknowledgements

The authors would like to thank the referees for reviewing the paper and for pointing out some (minor) errors in the original manuscript. Also, we would like to thank the Department of Science and Technology - Accelerated Science and Technology Human Resource Development Program (DOST-ASTHRDP), Philippines for funding this research.

References

[1] S. Ayyaswamy, B. Krishnakumari, B. Natarjan, and Y. Venkatakrishnan. Bounds on the hop domination number of a tree. *Proceedings-Mathematical Sciences*, 125(4):449–455, 2015.

[2] S. Jr. Canoy, S.A. Canoy, and M. Cruzate. Global domination in a graph. *Advances and Applications in Discrete Mathematics*, 19(4):401–408, 2018.

[3] S. Jr. Canoy, R. Mollejon, and J.G. Canoy. Hop dominating sets in graphs under binary operations. *European Journal of Pure and Applied Mathematics*, 12(4):1455–1463, 2019.

[4] W. Desormeaux, P. Gibson, and T. Haynes. Bounds on the global domination number. *Quaestiones Mathematicae*, 38(4):563–572, 2015.

[5] T. Haynes, S. Hedetniemi, and P. Slater. *Domination in Graphs, Advanced Topics.* Marcell Dekker, New York, USA, 1998.

[6] T. Haynes, S. Hedetniemi, and P. Slater. *Fundamentals of Domination in Graphs.* Marcell Dekker, New York, USA, 1998.

[7] M. Henning and N. Rad. On 2-step and hop dominating sets in graphs. *Graphs and Combinatorics*, 33(4):913–927, 2017.

[8] G. Mahadevan and V. Vijayalakshmi. Clone hop domination number of a graph. *Journal of Discrete Mathematical Sciences and Cryptography*, 22(5):719–729, 2019.

[9] R. Mollejon and S. Jr. Canoy. Double hop dominating sets in graphs. *Discrete Mathematics, Algorithms and Applications*, 2020, https://doi.org/10.1142/S1793830921500579.

[10] C. Natarajan and S. Ayyaswamy. Hop domination in graphs II. *Versita*, 23(2):187–199, 2015.

[11] Y. Pabilona and H. Rara. Connected hop domination in graphs under some binary operations. *Asian-European Journal of Mathematics*, 11(5):1850075–1–1850075–11, 2018.