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On Spectral-Equipartite Graphs and Eccentricity-Equipartite Graphs

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Abstract. Let $G = (V, E)$ be a graph of order $2n$. If $A \subseteq V$ and $\langle A \rangle \cong \langle V \backslash A \rangle$, then A is said to be isospectral. If for every n-element subset A of V we have $\langle A \rangle \cong \langle V \backslash A \rangle$, then we say that G is spectral-equipartite. In [1], Igor Shparlinski communicated with Bibak et al., proposing a full characterization of spectral-equipartite graphs. In this paper, we gave a characterization of disconnected spectral-equipartite graphs. Moreover, we introduced the concept eccentricity-equipartite graphs.

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1. Introduction

Let $G = (V, E)$ be a graph. The *distance* between vertices u and v in G, denoted by $d(u, v)$, is the length of the shortest path connecting u and v. If u and v is not connected, then we define $d(u, v)$ to be 0. The *eccentricity* of a vertex is its distance to a farthest vertex. G is said to be *k-regular* if every vertex of G has the same degree which is k. G is said to be *weakly equipartite* if every partition of V into two equal sets A and B, we have $\langle A \rangle \cong \langle B \rangle$. In addition, if there is an automorphism mapping A onto B, then we say that G is *equipartite*. The degree sequence of G is a non-decreasing sequence of degrees of the vertices of G. G is *degree-equipartite* if for every n-element subset A of V, the degree sequences of $\langle A \rangle$ and $\langle V \backslash A \rangle$ are the same.

The adjacency matrix $M = [a_{ij}]$ of G is the square matrix of order $4n^2$ given by $a_{ij} = 1$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise.

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The *spectrum* of G is the collection of all eigenvalues of all its adjacency matrix. Two graphs that have the same spectrum are said to be *cospectral* or *isospectral*. G is said to be *spectral-equipartite* if for every n-element subset A of V, the induced subgraph of A and $V\backslash A$ are *isospectral*.

Ferrero et al. in [5], mentioned the eccentricity sequence of a graph G as the non-decreasing sequence of eccentricities of the vertices of G. A graph $G = (V, E)$ of order $2n$ is said to be *eccentricity-equipartite* if for every n-element subset A of V, the induced subgraph of A and $V\backslash A$ have the same eccentricity sequence.

Here after please refer to [6] for the other concepts.

Over the past years, various applications spectral graph theory in many fields were discovered. In particular, spectral graph theory have important applications in chemistry, physics, computer science, and common real world problems. For instance, in computer science, the largest eigenvalue λ_1 plays a significant role in simulating virus proliferation in computer networks. Also mentioned in [3], Wang et al. in [10], claimed that the epidemic threshold in spreading viruses is proportional to $1/\lambda_1$.

Furthermore, spectral graph theory was also applied in connection with the famous 'traveling salesman problem'. This is mentioned by Cvetković et al. in [4].

Grünbaum et al. in [7] characterized equipartite graphs. They also presented a problem regarding the characterization of degree-equipartite graphs. Motivated by this problem, Bibak and Haghighi [1] published a paper that contains the characterization of degree-equipartite graphs. Moreover, they introduced a new type of graph called the spectral-equipartite graph. This new type was suggested by Igor Shparlinski, who also asked for its full characterization.

The latest study on equipartite graphs is by Shirdareh Haghighi et al. [9], which characterizes equipartite graphs in terms of their Laplacian spectra.

2. Known Results

2.1. Weakly-Equipartite Graphs

Theorem 1 is due to Grünbaum et al. [7] in their study on equipartite graphs.

Theorem 1. *([7],Theorem 13)* A graph G is weakly equipartite if and only if it is one of the following graphs: $2nK_1$, nK_2, $2C_4$, $K_{n,n}\backslash nK_2$, and $2K_n$, or one of their complements: K_{2n}, $K_{2n}\backslash nK_2$, $K_8\backslash 2C_4$, $2K_n + nK_2$, and $K_{n,n}$.

2.2. Degree-Equipartite Graphs

The following theorem is due to Bibak et al. [1] in their study on degree-equipartite graphs.

Theorem 2. *([1],Theorem 10)* A graph G of order $2n$ is degree-equipartite if and only if it is one of the following graphs: $2nK_1$, nK_2, $2C_4$, $K_{n,n}\backslash nK_2$, and $2K_n$, or one of their complements: K_{2n}, $K_{2n}\backslash nK_2$, $K_8\backslash 2C_4$, $2K_n + nK_2$, and $K_{n,n}$.

2.3. Spectra of Graphs

The following theorems and lemmas are due to the different studies involving the spectra of graphs.

Theorem 3. *([8], Theorem 2.1).* Let G be a simple undirected graph and let A be its adjacency matrix. Let H be a graph isomorphic to G and let B be the adjacency matrix of H. Then, G and H have the same spectrum.

The next theorem is a consequence of Theorem 3 and the definition of isospectral graphs.

Theorem 4. If two graphs are isomorphic, then they are isospectral.

Lemma 1. *([9], Lemma 2.5).* If G is a non-complete regular graph such that every two non-adjacent vertices of G form a vertex cut, then G is a cycle.

Lemma 2 can be verified easily as a direct consequence of *[6] (F33, page 679).*

Lemma 2. If H is a proper subgraph of G, then $\lambda_1(H) < \lambda_1(G)$.

Lemma 3. *([6], F6, page 674).* The spectrum of a graph is the union of the spectra of its connected components.

Lemma 4. *([2], 1.4.1 and 1.4.2).* Let $m, n \in \mathbb{N}$. The spectrum of a complete graph K_n is $\{-1^{n-1}, n-1\}$ and the spectrum of a complete bipartite graph $K_{m,n}$ is $\{\pm\sqrt{mn}, 0^{m+n-2}\}$.

Lemma 5 is a direct consequence of Lemma 3 and Lemma 4.

Lemma 5. Let $n \in \mathbb{N}$. The spectrum of an empty graph nK_1 is $\{0^n\}$.

Theorem 5. *([1], Problem 1, page 891).* Every spectral-equipartite graph is regular.

3. Main Results

This section presents the main results of the study.

3.1. Characterization of disconnected Spectral-Equipartite Graphs

The following results lead to the characterization of disconnected spectral-equipartite graphs.

This section also shows that the complement of a disconnected spectral-equipartite graph is also spectral-equipartite.

Theorem 6. Every weakly-equipartite graph is spectral-equipartite.

Proof. Let $G = (V, E)$ be a weakly-equipartite graph of order $2n$ and let A be an n element subset of V. Then, $\langle A \rangle$ and $\langle V \backslash A \rangle$ are isomorphic. Thus, by Theorem 4, $\langle A \rangle$ and $\langle V \backslash A \rangle$ are isospectral. This shows that G is spectral-equipartite. □

Theorem 7. Every degree-equipartite graph is spectral-equipartite.

Proof. Let G be a degree-equipartite graph. By Theorem 1 and Theorem 2, every weakly-equipartite graph is degree-equipartite, and every degree-equipartite graph is weakly-equipartite. Hence, by Theorem 6, G is spectral-equipartite. □

Remark 1. There exists a disconnected k-regular (with $k > 1$) spectral-equipartite graph.

To see this, the following are disconnected k-regular (with $k > 1$) spectral-equipartite graph: $2nK_1$; nK_2; $2C_4$; and $2K_n$.

Lemma 6. If $a_1, a_2, \ldots, a_n \in \mathbb{N}$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 4$, then $(a_1 + a_2 + \cdots + a_{n-1}) - 2\,(n-1) \geq a_n$ for all positive integer $n \geq 3$.

Proof. We use induction. For $n = 3$, we have $a_1 \geq a_2 \geq a_3 \geq 4$, that is $a_1 \geq a_3$ and $a_2 \geq a_3$. Since $a_3 \geq 4$, $a_2 - 4 \geq 0$. Thus, $a_1 + a_2 - 4 \geq a_3$, that is $a_1 + a_2 - 2(3-1) \geq a_3$. Hence the assertion holds for $n = 3$. Now, let $k \geq 3$ and assume that the assertion holds for k. Then, $(a_1 + a_2 + \cdots + a_{k-1}) - 2\,(k-1) \geq a_k \geq a_{k+1}$. Since $a_k \geq a_{k+1} \geq 4$, $a_k - 2 \geq 0$. Thus, $(a_1 + a_2 + \cdots + a_k) - 2k \geq a_{k+1}$. Thus, the assertion also holds for $k+1$. This shows the lemma. □

Lemma 7. If $a_1, a_2, \ldots, a_n \in \mathbb{N}$ with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 4$, then there exists $r \in \mathbb{N}$ such that $(a_1 + a_2 + \cdots + a_r) - 2\,(r) \geq a_{r+1} + a_{r+2} + \cdots + a_n$ and $(a_1 + a_2 + \cdots + a_{r-1}) - 2\,(r-1) < a_r + a_{r+1} + a_{r+2} \cdots + a_n$.

Proof. Let $S = \{k \in \mathbb{N} : (a_1 + a_2 + \cdots + a_k) - 2\,(k) \geq a_{k+1} + a_{k+2} + \cdots + a_n\}$. By Lemma 6, $n - 1 \in S$, that is, $S \neq \varnothing$. By the Well-ordering Principle, S contains a least element, say r. If $r \in S$, then $(a_1 + a_2 + \cdots + a_r) - 2\,(r) \geq a_{r+1} + a_{r+2} + \cdots + a_n$. Since r is the least element of S, $r - 1 \notin S$. Hence, $(a_1 + a_2 + \cdots + a_{r-1}) - 2\,(r-1) < a_r + a_{r+1} + a_{r+2} \cdots + a_n$. □

The following remark follows from Lemma 7.

Remark 2. Let $n \geq 3$ and $A = \{G_1, G_2, \ldots, G_n\}$ be the set of all components of a disconnected k-regular ($k > 1$) graph. Then there exists a set $B = \{G_{i_1}, G_{i_2}, \ldots, G_{i_r}\}$ ($r < n$) subset of A such that $|V(G_{i_1})\backslash\{u_1\} \cup V(G_{i_2})\backslash\{u_2\} \cup \cdots \cup V(G_{i_r})\backslash\{u_r\}| \geq |V(G_{i_{r+1}}) \cup V(G_{i_{r+2}}) \cup \cdots \cup V(G_{i_n}) \cup \{u_1, u_2, \ldots, u_r\}\,|$, and $|V(G_{i_1})\backslash\{u_1\} \cup V(G_{i_2})\backslash\{u_2\} \cup \cdots \cup V(G_{i_{r-1}})\backslash\{u_{r-1}\}| < |V(G_{i_r}) \cup V(G_{i_{r+1}}) \cup V(G_{i_{r+2}}) \cup \cdots \cup V(G_{i_n}) \cup \{u_1, u_2, \ldots, u_{r-1}\}\,|$.

Lemma 8. A disconnected k-regular ($k > 1$) spectral-equipartite graph cannot have more than two components.

Proof. Supposed $G = (V, E)$ has more than two components, say $A = G_1 \cup G_2 \cup \ldots \cup G_n$ ($n > 2$), where G_i is a component for $i = 1, 2, \ldots, n$. By Remark 2, Then there exists a set $B = \{G_{i_1}, G_{i_2}, \ldots, G_{i_r}\}$ ($r < n$) subset of A such that $|V(G_{i_1})\backslash\{u_1\} \cup$

$V(G_{i_2})\backslash\{u_2\}\cup\cdots\cup V(G_{i_r})\backslash\{u_r\}| \geq |V(G_{i_{r+1}})\cup V(G_{i_{r+2}})\cup\cdots\cup V(G_{i_n})\cup\{u_1,u_2,\ldots,u_r\}|$, and $|V(G_{i_1})\backslash\{u_1\}\cup V(G_{i_2})\backslash\{u_2\}\cup\cdots\cup V(G_{i_{r-1}})\backslash\{u_{r-1}\}| < |V(G_{i_r})\cup V(G_{i_{r+1}})\cup V(G_{i_{r+2}})\cup\cdots\cup V(G_{i_n})\cup\{u_1,u_2,\ldots,u_{r-1}\}|$. Partition V as follows: (1) Remove from $V(G_{i_j})$ a non-empty set of vertices A_j from $V(G_{i_j})$ to form $V'_j = V(G_{i_j})\backslash A_j$ for $j = 1,2,\ldots,r$ such that $|V'_1\cup V'_2\cup\cdots\cup V'_r| = |V(G_{i_{r+1}})\cup V(G_{i_{r+2}})\cup\cdots(G_{i_n})\cup A_1\cup A_2\cup\cdots\cup A_r|$. (2) Let $H_1 = \bigcup_{j=1}^{r} V'_j$ and $H_2 = \left(\bigcup_{j=r+1}^{n} V(G_{i_j})\right)\cup\left(\bigcup_{j=1}^{r} A_i\right)$. Then $\langle H_2\rangle$ has k-regular components while $\langle H_1\rangle$ does not have. Thus, by Lemma 2 and Lemma 3 spec$(\langle H_1\rangle) \neq$ spec$(\langle H_2\rangle)$. This shows the lemma. □

Lemma 9. Let G be a disconnected k-regular (with $k > 1$) graph of order $2n$. If G is a spectral-equipartite graph, then it has only two components which are both of order n.

Proof. Let G be a disconnected k-regular ($k > 1$) graph of order $2n$ and V be the vertex set of G. Suppose G is a spectral-equipartite graph. By Lemma 8 and by the definition of disconnected graphs, G has exactly two components. Next, we will prove that the two components of G are both of order n. Let G_1 and G_2 be the components of G. Suppose to the contrary $|V(G_1)| \neq |V(G_2)|$. Without loss of generality, assume that $|V(G_1)| < |V(G_2)|$. Let $|V(G_2)| - |V(G_1)| = m$. Partition V into two sets A and B with $\langle A\rangle = \langle V(G_2)\backslash V(mK_1)\rangle$ and $\langle B\rangle = G_1\cup mK_1$. By Lemma 2 and Lemma 3, the spectrum of $\langle A\rangle$ does not contain k while the spectrum of $\langle B\rangle$ does. Hence, $Spec(\langle A\rangle) \neq Spec(\langle B\rangle)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and G is not spectral-equipartite. This proves that the two components of G have an equal number of vertices which is n. This shows the lemma. □

Theorem 8. Let G be a disconnected k-regular (with $k > 1$) graph of order $2n$. If G is a spectral-equipartite graph, then $G = 2K_n$ or $G = 2C_4$.

Proof. By Lemma 9, the two components of G, say G_1 and G_2, has n vertices each. Suppose to the contrary that G_1 is not a complete graph nor a cycle. By Lemma 1, we can find two non-adjacent vertices, say x and y, in G_1 where $G_1\setminus\{x,y\}$ is connected. Let pq be an in G_2. We can partition $V(G)$ into two sets, A and B, with n vertices each such that $\langle A\rangle = \langle V(G_1)\setminus\{x,y\}\rangle\cup(p,q)$ and $\langle B\rangle = \langle V(G_2)\setminus\{p,q\}\rangle\cup\langle\{x\}\rangle\cup\langle\{y\}\rangle$. By Lemma 3, Lemma 4, and Lemma 5, we have $Spec(\langle A\rangle) = Spec(\langle V(G_1)\setminus\{x,y\}\rangle)\cup\{1,-1\}$ and $Spec(\langle B\rangle) = Spec(\langle V(G_2)\setminus\{p,q\}\rangle)\cup\{0\}\cup\{0\}$.

Consider the following cases:

Case 1. Both G_1 and G_2 are not bipartite

If both G_1 and G_2 are not bipartite, then $Spec(\langle V(G_2)\setminus\{p,q\}\rangle)$ cannot have 1 and -1 at the same time as elements. Hence, $Spec(\langle A\rangle) \neq Spec(\langle B\rangle)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and G is not spectral-equipartite.

Case 2. Only one between G_1 and G_2 is bipartite

Suppose that G_1 is bipartite and G_2 is not. Hence, using the same argument in Case 1 above, if G_2 is not bipartite, then $Spec(\langle V(G_2)\setminus\{p,q\}\rangle)$ cannot have 1 and -1 at the same time as elements. Hence, $Spec(\langle A\rangle) \neq Spec(\langle B\rangle)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and G is not spectral-equipartite.

Suppose that G_2 is bipartite and G_1 is not. By Lemma 2, $\lambda_1\left(\langle V\left(G_2\right)\setminus\{p,q\}\rangle\right)>1$ since $K_2\subset\langle V\left(G_2\right)\setminus\{p,q\}\rangle$. Since G_1 is not bipartite, only one between λ_1 and $-\lambda_1$ may exist as an eigenvalue of $\langle V\left(G_1\right)\setminus\{x,y\}\rangle$. Thus, $Spec\left(\langle A\rangle\right)\neq Spec\left(\langle B\rangle\right)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and G is not spectral-equipartite.

Case 3. Both G_1 and G_2 are bipartite

If both G_1 and G_2 are bipartite, then we have the following subcases:

Subcase 1. Both G_1 and G_2 are not complete bipartite graphs.

If both G_1 and G_2 are not complete bipartite graphs, then we will partition G into two sets A and B with n vertices each such that $\langle A\rangle=\langle V\left(G_1\right)\setminus\{x,y\}\rangle\cup(p,q)$ and $\langle B\rangle=\langle V\left(G_2\right)\setminus\{p,q\}\rangle\cup\langle\{x\}\rangle\cup\langle\{y\}\rangle$. Since G_1 is not a complete bipartite graph, we can have two non-adjacent vertices, x, and y, to belong to different partite sets. Hence, they do not have a common neighbor. Thus, for $\langle V\left(G_1\right)\setminus\{x,y\}\rangle$, there are $2k$ vertices with degree $k-1$ and $(n-2)-2k$ vertices of degree k. On the other hand, since pq is an edge, p and q must belong to different partite sets, so they do not have a common neighbor. Hence, for $\langle V\left(G_2\right)\setminus\{p,q\}\rangle$, there are $2k-2$ vertices of degree $k-1$ and $(n-2)-(2k-2)=n-2k$ vertices with degree k. Clearly, $\langle V\left(G_2\right)\setminus\{p,q\}\rangle$ contains one more edge when compared to $\langle V\left(G_1\right)\setminus\{x,y\}\rangle$. With $2k-2<2k$ for vertices with degree $k-1$ and $n-2k>(n-2)-2k$ for vertices with degree k, we could say that $\langle V\left(G_1\right)\setminus\{x,y\}\rangle$ is isomorphic to some proper subgraph of $\langle V\left(G_2\right)\setminus\{p,q\}\rangle$. Thus, by Lemma 2 $\lambda_1\left(\langle V\left(G_2\right)\setminus\{p,q\}\rangle\right)>\lambda_1\left(\langle V\left(G_1\right)\setminus\{x,y\}\rangle\right)$. Since K_2 is a proper subset of $\langle V\left(G_2\right)\setminus\{p,q\}\rangle$, then $\lambda_1\left(\langle V\left(G_2\right)\setminus\{p,q\}\rangle\right)>1$. Hence, $Spec\left(\langle A\rangle\right)\neq Spec\left(\langle B\rangle\right)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, so G is not spectral-equipartite.

Subcase 2. Both G_1 and G_2 are complete bipartite graphs.

If both G_1 and G_2 are complete bipartite graphs, then we will use a different partitioning of the graph G into two sets, A and B, with n vertices each and consider the following subsubcases:

Subsubcase 1. $n/2$ is even.

If $n/2$ is even, then we will partition $V(G)$ into two sets, A and B, with n vertices each such that $\langle A\rangle=\left(\frac{n}{2}-1\right)K_1\cup K_{\frac{n}{4},\frac{n+4}{4}}$, and $\langle B\rangle=K_{1,\frac{n}{2}}\cup K_{\frac{n}{4},\frac{n+4}{4}}$. Thus,by Lemma 3 and Lemma 4, we have $spec\,\langle(A)\rangle=\left\{0^{\frac{n}{2}-1}\right\}\cup\left\{\pm\sqrt{\frac{n}{4}\left(\frac{n}{4}+1\right)},0^{\frac{n}{2}-1}\right\}$ and $Spec\left(\langle B\rangle\right)=\left\{\pm\sqrt{\frac{n}{2}},0^{\frac{n}{2}-1}\right\}\cup\left\{\pm\sqrt{\frac{n}{4}\left(\frac{n}{4}-1\right)},0^{\frac{n}{2}-3}\right\}$. Hence, $Spec\left(\langle A\rangle\right)\neq Spec\left(\langle B\rangle\right)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and so G is not spectral-equipartite.

Subsubcase 2. $n/2$ is odd.

If $n/2$ is odd, then we will partition $V(G)$ into two sets, A and B, with n vertices each such that $\langle A\rangle=\left(\frac{n}{2}-1\right)K_1\cup K_{\frac{n+2}{4},\frac{n+2}{4}}$, and $\langle B\rangle=K_{1,\frac{n}{2}}\cup K_{\frac{n-2}{4},\frac{n-2}{4}}$. Thus, by Lemma 3, Lemma 4, and Lemma 5, $spec\,\langle(A)\rangle=\left\{0^{\frac{n}{2}-1}\right\}\cup\left\{\pm\frac{n+2}{4},0^{\frac{n}{2}-1}\right\}$ and $spec\,\langle(B)\rangle=\left\{\pm\sqrt{\frac{n}{2}},0^{\frac{n}{2}-1}\right\}\cup\left\{\pm\frac{n-2}{4},0^{\frac{n}{2}-3}\right\}$. Hence, $Spec\left(\langle A\rangle\right)\neq Spec\left(\langle B\rangle\right)$. Thus, $\langle A\rangle$ and $\langle B\rangle$ are not isospectral, and so G is not spectral-equipartite.

By Cases 1,2, and 3, we have shown that if G_1 is neither a complete graph nor a cycle, then G will not be a spectral-equipartite graph. Thus, G_1 must either be a complete graph or a cycle. If G_1 is complete, then G_2 having the same order and size as G_1 is also complete. Whence, $G = 2K_n$.

If G_1 is a cycle C_n, then $k = 2$ and G_2 is also C_n. Suppose $n > 4$. Then, we choose two non-adjacent vertices, a and b, from G_1 that have a common neighbor and two adjacent vertices, c and d, from G_2. Now, we partition G into two sets, A and B, with n vertices each such that $\langle A\rangle = \langle V(G_2)\setminus\{c,d\}\rangle \cup \langle\{a,b\}\rangle$ and $\langle B\rangle = \langle V(G_1)\setminus\{a,b\}\rangle \cup \langle\{c,d\}\rangle$. Then $\langle A\rangle = P_{n-2} \cup K_1 \cup K_1$ and $\langle B\rangle = P_{n-3} \cup K_1 \cup P_2$. Clearly, P_{n-3}, P_2, and K_1 are proper subgraphs of P_{n-2}. By Lemma 2, $\lambda_1(P_{n-2}) > \lambda_1(P_{n-3}) > \lambda_1(P_2) > 0$. Hence, by Lemma 3, $Spec(\langle A\rangle) \neq Spec(\langle B\rangle)$. Therefore, $n \leq 4$. For $n = 1, 2$ or 3, we will have K_1, K_2, and K_3, respectively, and for $n = 4$, we have C_4. Thus, we have $G = 2K_1, 2K_2, 2K_3$, and $2C_4$. Therefore, if G is a disconnected k-regular spectral-equipartite graph of order 2n with $k > 1$, then $G = 2K_n$ or $G = 2C_4$. □

Theorem 9. Let G be a disconnected graph of order $2n$. G is spectral-equipartite if and only if it is one of the following graphs: $2nK_1$, nK_2, $2K_n$, and $2C_4$.

Proof. Suppose G is one of the graphs $2nK_1$, nK_2, $2K_n$, and $2C_4$. By Theorem 2, G is weakly-equipartite, and by Theorem 6, it must be spectral-equipartite.

Now suppose that G is a disconnected spectral-equipartite graph. By Theorem 5, G must be a k-regular graph of order $2n$. If $k = 0$, then the $2n$ vertices of G are isolated. Hence, we have $G = 2nK_1$. If $k = 1$, then G is just the union of the n copies of K_2. Hence, we have $G = nK_2$. If $k > 1$, then by Theorem 8, $G = 2K_n$ or $G = 2C_4$. □

Theorem 10. If G is a disconnected spectral-equipartite graph, then its complement is also spectral-equipartite.

Proof. Let G be a disconnected spectral-equipartite graph. By Theorem 9, G is one of the graphs $2nK_1$, nK_2, $2C_4$, and $2K_n$. Observe that the complements of these graphs are K_{2n}, $K_{2n}\backslash nK_2$, $K_8\backslash 2C_4$, and $K_{n,n}$. By Theorem 2, these graphs are weakly equipartite, and by Theorem 6, they must be spectral-equipartite. □

3.2. Eccentricity-Equipartite Graphs

This section gives some eccentricity-equipartite graphs.

Theorem 11. Every weakly-equipartite graph is eccentricity-equipartite.

Proof. Let G be a weakly-equipartite graph. Thus, every partition of $V(G)$ into two sets, A and B, with n vertices each, $\langle A\rangle$ and $\langle B\rangle$ are isomorphic. Hence, $\langle A\rangle$ and $\langle B\rangle$ have the same eccentricity sequence. Thus, G is eccentricity-equipartite. □

Corollary 1. Let $n \in \mathbb{N}$. The following graphs are eccentricity-equipartite: $2nK_1$; nK_2; $2C_4$; $K_{n,n}\backslash nK_2$; $2K_n$; K_{2n}; $K_{2n}\backslash nK_2$; $K_8\backslash 2C_4$; $2K_n + nK_2$ and $K_{n,n}$.

Proof. The statement immediately follows from Theorem 1 and Theorem 11. □

Theorem 12. Every degree-equipartite graph is eccentricity-equipartite.

Proof. The proof for this theorem will immediately follow from Theorem 2 and Corollary 1. □

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