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Laplace-Somé Blaise Abbo Method for Solving Nonlinear Coupled Burger's Equations

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Abstract. Burger's equations, an extension of fluid dynamics equations, are typically solved by several numerical methods. In this article, the laplace-Somé Blaise Abbo method is used to solve nonlinear Burger equations. This method is based on the combination of the laplace transform and the SBA method. After reminders of the laplace transform, the basic principles of the SBA method are described. The process of calculating the Laplace-SBA algorithm for determining the exact solution of a linear or nonlinear partial derivative equation is shown. Thus, three examples of PDE are solved by this method, which all lead to exact solutions. Our results suggest that this method can be extended to other more complex PDEs.

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Key Words and Phrases: Laplace-SBA Method, Coupled Burgers equation.

1. Introduction

Partial differential equations (PDEs) describe the reality of numerous physical and natural phenomena. Burger's equations are such partial differential equations issues from fluid mechanics. It finds its application in various field of applied mathematics, such as the modeling of gas dynamics, accoutis or road traffic. However, to translate the realty of physical and natural phenomena encountered, this PDE takes the from of a coupled system (PDES). Several methods have been used to investigate these PDES like the iterative variational method (VIM) [7], the homotopy perturbation method (HPM) [4] , [8], [10], double Laplace transform method [6] and Adomian Pade Technique [3], [5], [9]. However, most solutions of urger's partial differential equations (coupled or not) by these methods are rarely exact. Or the Some Blaise Abbo (SBA) method is a powerful to solve non linear PDES [1], [2], [12], [13], [15], [16], [17], [18]. By determining the exact solutions. Even if the calculation of the integrals tuns out to be difficult by this method, the combination of this-ci with the method of transformation of Laplace makes it possible to overcome this difficulty. In this paper we use the Laplace-SBA method to construct the exact solution of coupled Burger's equations.

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2. The Laplace-SBA method

Suppose that we need to solve the following equation:

$$Au = f \tag{1}$$

In a real Hilbert space H, where $A : H \longrightarrow H$, a linear is or a nonlinear operator, $f \in H$ and u is the unknown function. Let's suppose that we can decompose operator A in the following form:

$$A = L + R + N \tag{2}$$

where $L + R$ is linear, N nonlinear, L inversible in the "sense", of Adomian with L^{-1} as inverse. Using that decoposition equation (2) can be rewritten as:

$$Lu + Ru + Nu = f \tag{3}$$

let's note $\mathcal{L}_t(u)$ the Laplace transform of the function u with respect to the variable t. Applying the Laplace transform to equation (3), we obtain:

$$\mathcal{L}_t\left(Lu\right) + \mathcal{L}_t\left(Ru\right) + \mathcal{L}_t\left(Nu\right) = \mathcal{L}_t\left(f\right) \tag{4}$$

We suppose that: $L(.) = \dfrac{\partial^m}{\partial t^m}(.)$. Using the properties of the Laplace transform of a derivative, we get:

$$\mathcal{L}_t\left[\frac{\partial^m u}{\partial t^m}\right] = s^m \mathcal{L}_t(u) - \sum_{k=0}^{m-1} s^k u^{(m-k-1)}(x,0) \tag{5}$$

Equation (4) gives

$$s^m \mathcal{L}_t(u) - \sum_{k=0}^{m-1} s^k u^{(m-k-1)}(x,0) + \mathcal{L}_t\left(Ru\right) + \mathcal{L}_t\left(Nu\right) = \mathcal{L}_t\left(f\right) \tag{6}$$

(6)is equivalent to

$$s^m \mathcal{L}_t(u) = -\mathcal{L}_t(Ru) - \mathcal{L}_t(Nu) + \sum_{k=0}^{m-1} s^k u^{(m-k-1)}(x,0) + \mathcal{L}_t(f) \tag{7}$$

Using the successive approximations, we get:

$$s^m \mathcal{L}_t\left(u^k\right) = -\mathcal{L}_t\left(Ru^k\right) - \mathcal{L}_t\left(Nu^{k-1}\right) + \sum_{j=0}^{m-1} s^j u^{(m-j-1)}(x,0) + \mathcal{L}_t\left(f\right) \tag{8}$$

According to the SBA method, we suppose that the solution of (8) has the following form:

$$u = \lim_{k \longrightarrow +\infty} u^k \tag{9}$$

Where

$$u^k = \sum_{k=0}^{+\infty} u_n^k \quad ; \quad k \geq 1 \tag{10}$$

let's denote $g_{k-1} = Nu^{k-1}$. Substituting u^k into (8), we have:

$$s^m \sum_{n\geq 0} \mathcal{L}_t\left(u_n^k\right) = -\sum_{n\geq 0} \mathcal{L}_t\left(Ru_n^k\right) - \mathcal{L}_t(g_{k-1}) + \sum_{j=0}^{m-1} s^j u^{(m-j-1)}(x,0) + \mathcal{L}_t(f) \tag{11}$$

and for every $k \geq 1$, we get u_n^k for $n \geq 0$, through the following Laplace- SBA algorithm:

$$\begin{cases} s^m \mathcal{L}_t\left(u_0^k\right) = -\mathcal{L}_t\left(g_{k-1}\right) + \sum_{j=0}^{m-1} s^j u^{(m-j-1)}(x,0) + \mathcal{L}_t(f); \quad k \geq 1 \\ s^m \mathcal{L}_t\left(u_{n+1}^k\right) = -\mathcal{L}_t\left(Ru_n^{k-1}\right); \quad n \geq 0 \end{cases} \tag{12}$$

(12) is equivalent to

$$\begin{cases} \mathcal{L}_t\left(u_0^k\right) = -\frac{1}{s^m}\mathcal{L}_t(g_{k-1}) + \frac{1}{s^m}\sum_{j=0}^{m-1} s^k u^{(m-j-1)}(x,0) + \mathcal{L}_t\left(\frac{f}{s^m}\right); \quad k \geq 1 \\ \mathcal{L}_t\left(u_{n+1}^k\right) = -\frac{1}{s^m}\mathcal{L}_t\left(Ru_n^{k-1}\right); \quad n \geq 0 \end{cases} \tag{13}$$

Applying the inverse Laplace transform $\mathcal{L}_t^{-1}$ to (13), we obtain:

$$\begin{cases} (u_0^k) = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t(g_{k-1})\right] + \mathcal{L}_t^{-1}\left[\frac{1}{s^m}\sum_{j=0}^{m-1} s^k u^{(m-j-1)}(x,0)\right] + \mathcal{L}_t^{-1}\left(\frac{f}{s^m}\right); \quad k \geq 1 \\ (u_{n+1}^k) = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t\left(Ru_n^{k-1}\right)\right]; \quad n \geq 0 \end{cases} \tag{14}$$

The SBA principle needs that, for $k = 1$, we must choose u^0 like $Nu^0 = 0$ and for $k > 1$, we must verify that $Nu^{k-1} = 0$.
For $k = 1$, we get:

$$\begin{cases} u_0^1 = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t(g_0)\right] + \mathcal{L}_t^{-1}\left[\frac{1}{s^m}\sum_{j=0}^{m-1} s^k u^{(m-j-1)}(x,0)\right] + \mathcal{L}_t^{-1}\left(\frac{f}{s^m}\right); \\ (u_{n+1}^1) = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t\left(Ru_n^1\right)\right]; \quad n \geq 0 \end{cases} \tag{15}$$

If the series $\left(\sum_{n\geq 0}(u_n^1)\right)$ converges, then $u^1 = \sum_{n\geq 0}(u_n^1)$
For $k = 1$, we get:

$$\begin{cases} u_0^2 = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t(g_1)\right] + \mathcal{L}_t^{-1}\left[\frac{1}{s^m}\sum_{j=0}^{m-1} s^k u^{(m-j-1)}(x,0)\right] + \mathcal{L}_t^{-1}\left(\frac{1}{s^m}f\right); \\ u_{n+1}^2 = -\mathcal{L}_t^{-1}\left[\frac{1}{s^m}\mathcal{L}_t\left(Ru_n^2\right)\right]; \quad n \geq 0 \end{cases} \tag{16}$$

If the series $\left(\sum_{n\geq 0}(u_n^2)\right)$ converges, then $u^2 = \sum_{n\geq 0}(u_n^2)$. The converge of the series has been proved in [6].

Repeating this same process for $k \geq 3$. If the series $\left(\sum_{n\geq 0}(u_n^k)\right)$ converges, then $u^k = \sum_{n\geq 0}(u_n^k)$.
Therefore, $u = \lim_{k\longrightarrow+\infty} u^k$ is the solution of the equation (1).

3. Applications

To illustrate the powerfull, the simplicity and efficiency of Laplace-SBA method in solving non linear coupled Burger's equations [6]. Here, we consider three systems of these equations.

3.1. Example 1

Let's consider the following system of Burger's equations [2] [3]

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \\ \frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + 2v(x,t)\frac{\partial v(x,t)}{\partial x} - \frac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \end{cases} \tag{17}$$

with initial conditions

$$\begin{cases} u(x,0) = \sin x \\ v(x,0) = \sin x \end{cases} \tag{18}$$

3.1.1. Résolution by Laplace-SBA method

According to the Laplace-SBA Method, we suppose that:

$$\begin{cases} N_1\left(u(x,t)v(x,t)\right) = 2u(x,t)\dfrac{\partial u(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \\ \\ N_2\left(u(x,t)v(x,t)\right) = 2v(x,t)\dfrac{\partial v(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \end{cases} \quad (19)$$

Then, the system (17) gives

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \dfrac{\partial^2 u(x,t)}{\partial x^2} + N_1\left(u(x,t)v(x,t)\right) \\ \\ \dfrac{\partial v(x,t)}{\partial t} = \dfrac{\partial^2 v(x,t)}{\partial x^2} + N_2\left(u(x,t)v(x,t)\right) \end{cases} \quad (20)$$

Applying the Laplace transform to (20). we obtain:

$$\begin{cases} \mathcal{L}_t\left(u(x,t)\right) = \frac{1}{s}u(x,0) + \frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right) + \frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right) \\ \\ \mathcal{L}_t\left(v(x,t)\right) = \frac{1}{s}v(x,0) + \frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v(x,t)}{\partial x^2}\right) + \frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right) \end{cases} \quad (21)$$

From (21), we have:

$$\begin{cases} u(x,t) = u(x,0)\mathcal{L}_t^{-1}\left(\frac{1}{s}\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right)\right) \\ \\ v(x,t) = v(x,0)\mathcal{L}_t^{-1}\left(\frac{1}{s}\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right)\right) \end{cases} \quad (22)$$

(22) equivalent to:

$$\begin{cases} u(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right)\right) \\ \\ v(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right)\right) \end{cases} \quad (23)$$

Using the successive approximations, we get:

$$\begin{cases} u^k(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u^k(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{k-1}(x,t)v^{k-1}(x,t)\right)\right)\right) \\ \\ v^k(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v^k(x,t)}{\partial x^2}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u^{k-1}(x,t)v^{k-1}(x,t)\right)\right)\right) \end{cases} \quad k \geq 1 \tag{24}$$

According to the SBA method, we suppose that the solution of (17) has the following form:

$$u(x,t) = \lim_{k \longrightarrow +\infty} u^k(x,t) \tag{25}$$

where

$$u^k(x,t) = \sum_{n=0}^{+\infty} u_n^k(x,t); \qquad k \geq 1 \tag{26}$$

and, for every $k \geq 1$, we get $u_n^k(x,t)$ for $n \geq 0$, through the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^k(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{k-1}(x,t)v^{k-1}(x,t)\right)\right)\right) & k \geq 1 \\ u_{n+1}^k(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u_n^k(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \\ \\ \begin{cases} v_0^k(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u^{k-1}(x,t)v^{k-1}(x,t)\right)\right)\right) & k \geq 1 \\ v_{n+1}^k(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v_n^k(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \end{cases} \tag{27}$$

For $k = 1$, we have the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^1(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^0(x,t)v^0(x,t)\right)\right)\right) \\ u_{n+1}^1(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u_n^1(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \\ \\ \begin{cases} v_0^1(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u^0(x,t)v^0(x,t)\right)\right)\right) \\ v_{n+1}^1(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v_n^1(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \end{cases} \tag{28}$$

Let's suppose that one can find u^0 and v^0 as $N_1\left(u^0(x,t)v^0(x,t)\right) = 0$ and $N_2\left(u^0(x,t)v^0(x,t)\right) = 0$, we remark that, taking $u^0(x,t) = v^0(x,t) = 0$ we obtain $N_1\left(u^0(x,t)v^0(x,t)\right) = N_2\left(u^0(x,t)v^0(x,t)\right) =$

0

For $k = 1$, we have the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^1(x,t) = \sin x \\ u_{n+1}^1(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u_n^1(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} & (a) \\ \begin{cases} v_0^1(x,t) = \sin x \\ v_{n+1}^1(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v_n^1(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} & (b) \end{cases} \tag{29}$$

From $(29)_{(a)}$, we get:

$$\begin{cases} u_0^1(x,t) &= \sin x \\ u_1^1(x,t) &= -t\sin x \\ u_2^1(x,t) &= \frac{(-t)^2}{2!}\sin x \\ \vdots \\ u_n^1(x,t) &= \frac{(-t)^n}{n!}\sin x \end{cases} \tag{30}$$

From $(29)_{(a)}$, we get:

$$\begin{cases} v_0^1(x,t) &= \sin x \\ v_1^1(x,t) &= -t\sin x \\ u_2^1(x,t) &= \frac{(-t)^2}{2!}\sin x \\ \vdots \\ v_n^1(x,t) &= \frac{(-t)^n}{n!}\sin x \end{cases} \tag{31}$$

From (30) and (31), we obtain:

$$\begin{cases} u^1(x,t) = \sum_{n=0}^{+\infty} u_n^1(x,t) = \left(\sum_{n=0}^{+\infty}\frac{(-t)^n}{n!}\right)\sin x = e^{-t}\sin x \\ v^1(x,t) = \sum_{n=0}^{+\infty} v_n^1(x,t) = \left(\sum_{n=0}^{+\infty}\frac{(-t)^n}{n!}\right)\sin x = e^{-t}\sin x \end{cases} \tag{32}$$

For $k=2$, we have the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^2(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^1(x,t)v^1(x,t)\right)\right)\right) \\ u_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \\ \begin{cases} v_0^2(x,t) = \sin x + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u^1(x,t)v^1(x,t)\right)\right)\right) \\ v_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \end{cases} \tag{33}$$

We remark that:

$$\begin{cases} N_1\left(u^1(x,t)v^1(x,t)\right) = 2u^1(x,t)\dfrac{\partial u^1(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u^1(x,t)v^1(x,t)\right) \\ \qquad = 2\left(e^{-t}\sin x\right)\left(\dfrac{\partial}{\partial x}\left(e^{-t}\sin x\right)\right) - \dfrac{\partial}{\partial x}\left(e^{-2t}\sin^2 x\right) \\ \qquad = 2\left(\cos x \sin x\right)e^{-2t} - 2\left(\cos x \sin x\right)e^{-2t} = 0 \\ N_1\left(u^1(x,t)v^1(x,t)\right) = 2u^1(x,t)\dfrac{\partial u^1(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u^1(x,t)v^1(x,t)\right) \\ \qquad = 2\left(e^{-t}\sin x\right)\left(\dfrac{\partial}{\partial x}\left(e^{-t}\sin x\right)\right) - \dfrac{\partial}{\partial x}\left(e^{-2t}\sin^2 x\right) \\ \qquad = 2\left(\cos x \sin x\right)e^{-2t} - 2\left(\cos x \sin x\right)e^{-2t} = 0 \end{cases} \tag{34}$$

and (33) becomes:

$$\begin{cases} \begin{cases} u_0^2(x,t) = \sin x \\ u_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad \forall n \geq 0 \end{cases} \\ \begin{cases} v_0^1(x,t) = \sin x \\ v_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad \forall n \geq 0 \end{cases} \end{cases} \tag{35}$$

We remark that (35)is the same algorithm that (29). Thus

$$u^2(x,t) = v^2(x,t) = e^{-t}\sin x \tag{36}$$

Using the same the procedure for $k \geq 3$, we get:

$$u^1(x,t) = v^1(x,t) = u^2(x,t) = v^2(x,t) = u^3(x,t) = v^3(x,t) = \cdots = u^k(x,t) = v^k(x,t) = e^{-t}\sin x \tag{37}$$

and the exact solution of (17) is:

$$\begin{cases} u(x,t) = \lim\limits_{k\longrightarrow+\infty} u^k(x,t) = e^{-t}\sin x \\ \\ v(x,t) = \lim\limits_{k\longrightarrow+\infty} v^k(x,t) = e^{-t}\sin x \end{cases} \tag{38}$$

3.2. Example 2

Now, we consider the following system of Burger's equations [10]

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \dfrac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = 2t^2x^3 + t^2 - 2t + x^2 \\ \\ \dfrac{\partial v(x,t)}{\partial t} - \dfrac{\partial^2 v(x,t)}{\partial x^2} + v(x,t)\dfrac{\partial v(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = \dfrac{1}{x} - 2\dfrac{t}{x^3} - \dfrac{t^2}{x^3} - t^2 \end{cases} \tag{39}$$

with initial conditions

$$\begin{cases} u(x,0) = 0 \\ v(x,0) = 0 \end{cases} \tag{40}$$

3.2.1. Résolution by Laplace-SBA method

Let's denote:

$$\begin{cases} N_1\left(u(x,t)v(x,t)\right) = u(x,t)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \\ \\ N_2\left(u(x,t)v(x,t)\right) = v(x,t)\dfrac{\partial v(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \end{cases} \tag{41}$$

the system (41) can be rewritten as follows

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = x^2 - 2t + 2x^3t^2 + t^2 + \dfrac{\partial^2 u(x,t)}{\partial x^2} - N_1\left(u(x,t)v(x,t)\right) \\ \\ \dfrac{\partial v(x,t)}{\partial t} = \dfrac{1}{x} - 2\dfrac{t}{x^3} - \dfrac{t^2}{x^3} - t^2 + \dfrac{\partial^2 v(x,t)}{\partial x^2} - N_2\left(u(x,t)v(x,t)\right) \end{cases} \tag{42}$$

Applying the Laplace transform to (42). we obtain:

$$\begin{cases} \mathcal{L}_t\left(u(x,t)\right) = \left(\frac{1}{s^2}x^2 + \frac{4}{s^4}x^3t^2 - \frac{2}{s^3} + \frac{2}{s^4}\right) + \frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right) \\ \mathcal{L}_t\left(v(x,t)\right) = \frac{1}{s^2x} - \frac{2}{s^3x^3} - \frac{2}{s^4x^3} - \frac{2}{s^4} + \frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right) - \frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right) \end{cases} \tag{43}$$

From (43), we have:

$$\begin{cases} u(x,t) = tx^2 - t^2 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)\right) - \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right)\right) + \frac{2}{3}t^3x^3 + \frac{1}{3}t^3 \\ v(x,t) = \frac{t}{x} - \frac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)\right) - \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right)\right) - \frac{1}{3}\frac{t^3}{x^3} - \frac{1}{3}t^3 \end{cases} \tag{44}$$

Let's denote

$$\begin{cases} \tilde{N_1}\left(u(x,t)v(x,t)\right) = -\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u(x,t)v(x,t)\right)\right)\right) + \frac{2}{3}t^3x^3 + \frac{1}{3}t^3 \\ \tilde{N_2}\left(u(x,t)v(x,t)\right) = -\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u(x,t)v(x,t)\right)\right)\right) - \frac{1}{3}\frac{t^3}{x^3} - \frac{1}{3}t^3 \end{cases} \tag{45}$$

From (44) and (45), gives:

$$\begin{cases} u(x,t) = tx^2 - t^2 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)\right) + \tilde{N_1}\left(u(x,t)v(x,t)\right) \\ v(x,t) = \frac{t}{x} - \frac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)\right) + \tilde{N_2}\left(u(x,t)v(x,t)\right) \end{cases} \tag{46}$$

Applying successive approximations to (46), we have:

$$\begin{cases} u^k(x,t) = tx^2 - t^2 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u^k(x,t)}{\partial x^2}\right)\right) + \tilde{N_1}\left(u^{k-1}(x,t)v^{k-1}(x,t)\right) \\ v^k(x,t) = \frac{t}{x} - \frac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v^k(x,t)}{\partial x^2}\right)\right) + \tilde{N_2}\left(u^{k-1}(x,t)v^{k-1}(x,t)\right) \end{cases} \tag{47}$$

We look for the solution of (47) in the following form:

$$\begin{cases} u^k(x,t) = \sum_{n=0}^{+\infty} u_n^k(x,t); & k \geq 1 \\ v^k(x,t) = \sum_{n=0}^{+\infty} v_n^k(x,t); & k \geq 1 \end{cases} \tag{48}$$

From (47), we obtain the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^k(x,t) = tx^2 - t^2 + \tilde{N}_1\left(u^{k-1}(x,t)v^{k-1}(x,t)\right) & k \geq 1 \\ u_{n+1}^k(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u_n^k(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \\ \begin{cases} v_0^k(x,t) = \frac{t}{x} + \tilde{N}_2\left(u^{k-1}(x,t)v^{k-1}(x,t)\right) & k \geq 1 \\ v_{n+1}^k(x,t) = -\frac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v_n^k(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \end{cases} \tag{49}$$

For $k = 1$, we have the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^1(x,t) = tx^2 - t^2 + \tilde{N}_1\left(u^{0}(x,t)v^{0}(x,t)\right) & k \geq 1 \\ u_{n+1}^1(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 u_n^1(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \\ \begin{cases} v_0^1(x,t) = \frac{t}{x} + \tilde{N}_2\left(u^{0}(x,t)v^{0}(x,t)\right) & k \geq 1 \\ v_{n+1}^1(x,t) = -\frac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{\partial^2 v_n^1(x,t)}{\partial x^2}\right)\right) & ; \quad n \geq 0 \end{cases} \end{cases} \tag{50}$$

Let's suppose that one can find u^0 and v^0 as $\tilde{N}_1\left(u^0(x,t)v^0(x,t)\right) = 0$ and $\tilde{N}_2\left(u^0(x,t)v^0(x,t)\right) = 0$, we obtain the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^1(x,t) = tx^2 - t^2 \\ u_1^1(x,t) = t^2 \\ \vdots \\ u_n^1(x,t) = 0 \quad \forall \ n \geq 2 \end{cases} \\ \\ \begin{cases} v_0^1(x,t) = \dfrac{t}{x} \\ v_1^1(x,t) = -\dfrac{t^2}{x^3} + \dfrac{t^2}{x^3} = 0 \\ \vdots \\ v_n^1(x,t) = 0 \quad \forall \ n \geq 2 \end{cases} \end{cases} \tag{51}$$

Therefore, we get:

$$\begin{cases} u^1(x,t) = u_0^1(x,t) + u_1^1(x,t) + \cdots = tx^2 \\ v^1(x,t) = v_0^1(x,t) + v_1^1(x,t) + \cdots = \dfrac{t}{x} \end{cases} \tag{52}$$

For $k = 2$, we have the following Laplace-SBA algorithm:

$$\begin{cases} \begin{cases} u_0^2(x,t) = tx^2 - t^2 + \tilde{N}_1\left(u^1(x,t)v^1(x,t)\right) \quad k \geq 1 \\ u_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\dfrac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 u_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \\ \\ \begin{cases} v_0^2(x,t) = \dfrac{t}{x} + \tilde{N}_2\left(u^1(x,t)v^1(x,t)\right) \quad k \geq 1 \\ v_{n+1}^2(x,t) = -\dfrac{t^2}{x^3} + \mathcal{L}_t^{-1}\left(\dfrac{1}{s}\mathcal{L}_t\left(\dfrac{\partial^2 v_n^2(x,t)}{\partial x^2}\right)\right) \quad ; \quad n \geq 0 \end{cases} \end{cases} \tag{53}$$

We remark that:

$$\begin{cases} \tilde{N}_1\left(u(x,t)v(x,t)\right) = -\frac{1}{6}t^3\left(4x^3 + 2\right) + \frac{2}{3}t^3x^3 + \frac{1}{3}t^3 = 0 \\ \\ \tilde{N}_2\left(u(x,t)v(x,t)\right) = \frac{1}{6}\frac{t^3}{x^3}\left(2x^3 + 2\right) - \frac{1}{3}\dfrac{t^3}{x^3} - \frac{1}{3}t^3 = 0 \end{cases} \tag{54}$$

and (53) becomes:

$$\begin{cases} \begin{cases} u_0^1(x,t) = tx^2 - t^2 \\ u_1^1(x,t) = t^2 \\ \vdots \\ u_n^1(x,t) = 0 \quad \forall \ n \geq 2 \end{cases} \\ \\ \begin{cases} v_0^1(x,t) = \dfrac{t}{x} \\ v_1^1(x,t) = -\dfrac{t^2}{x^3} + \dfrac{t^2}{x^3} = 0 \\ \vdots \\ v_n^1(x,t) = 0 \quad \forall \ n \geq 2 \end{cases} \end{cases} \tag{55}$$

We remark that (55) is the same algorithm that (51). Thus

$$\begin{cases} u^2(x,t) = tx^2 \\ v^2(x,t) = \dfrac{t}{x} \end{cases} \tag{56}$$

Using the same the procedure for $k \geq 3$, we get:

$$\begin{cases} u^2(x,t) = u^3(x,t) = \cdots = x^2 t \\ v^2(x,t) = v^3(x,t) = \cdots = \dfrac{t}{x} \end{cases} \tag{57}$$

Thus, the solution of example 2 is:

$$\begin{cases} u(x,t) = \lim\limits_{k \longrightarrow +\infty} u^k(x,t) = x^2 t \\ \\ v(x,t) = \lim\limits_{k \longrightarrow +\infty} v^k(x,t) = \dfrac{t}{x} \end{cases} \tag{58}$$

3.3. Example 3

Let's consider the following non homogeneous form of coupled Burger's equations [6]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial u(x,t)}{\partial x}\right) - 2u(x,t)\dfrac{\partial u(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = \left(-x^2-4\right)e^{-t} \\ \dfrac{\partial v(x,t)}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial v(x,t)}{\partial x}\right) - 2v(x,t)\dfrac{\partial v(x,t)}{\partial x} + \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) = \left(-x^2-4\right)e^{-t} \end{cases} \tag{59}$$

with initial conditions

$$\begin{cases} u(x,0) = x^2 \\ v(x,0) = x^2 \end{cases} \tag{60}$$

3.3.1. Résolution by Laplace-SBA method

Let's denote:

$$\begin{cases} N_1(u,v) = 2u(x,t)\dfrac{\partial u(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \\ N_2(u,v) = 2v(x,t)\dfrac{\partial v(x,t)}{\partial x} - \dfrac{\partial}{\partial x}\left(u(x,t)v(x,t)\right) \end{cases} \tag{61}$$

the system (59) can be rewritten as follows

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \left(-x^2-4\right)e^{-t} + \frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial u(x,t)}{\partial x}\right) + N_1(u,v) \\ \dfrac{\partial v(x,t)}{\partial t} = \left(-x^2-4\right)e^{-t} + \frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial v(x,t)}{\partial x}\right) + N_2(u,v) \end{cases} \tag{62}$$

Applying the Laplace transform to the system (62) with respect to variable t, we have:

$$\begin{cases} s\mathcal{L}_t\left(u(x,t)\right) = u(x,0) + \mathcal{L}_t\left(\left(-x^2-4\right)e^{-t}\right) + \mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial u(x,t)}{\partial x}\right)\right) + \mathcal{L}_t\left(N_1(u,v)\right) \\ s\mathcal{L}_t\left(v(x,t)\right) = v(x,0) + \mathcal{L}_t\left(\left(-x^2-4\right)e^{-t}\right) + \mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial v(x,t)}{\partial x}\right)\right) + \mathcal{L}_t\left(N_2(u,v)\right) \end{cases} \tag{63}$$

(63) is equivalent to

$$\begin{cases} \mathcal{L}_t\left(u(x,t)\right) = \frac{1}{s}x^2 + \frac{1}{s}\left(-\frac{x^2+4}{s+1}\right) + \frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial u(x,t)}{\partial x}\right)\right) + \frac{1}{s}\mathcal{L}_t\left(N_1(u,v)\right) \\ \mathcal{L}_t\left(v(x,t)\right) = \frac{1}{s}x^2 + \frac{1}{s}\left(-\frac{x^2+4}{s+1}\right) + \frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\dfrac{\partial v(x,t)}{\partial x}\right)\right) + \frac{1}{s}\mathcal{L}_t\left(N_2(u,v)\right) \end{cases} \tag{64}$$

Using the inverse Laplace transform to (64), we get

$$\begin{cases} u(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}x^2\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\left(-\frac{x^2+4}{s+1}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u,v\right)\right)\right) \\ v(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}x^2\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\left(-\frac{x^2+4}{s+1}\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u,v\right)\right)\right) \end{cases} \tag{65}$$

From (65), we have:

$$\begin{cases} u(x,t) = e^{-t}x^2 + 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u,v\right)\right)\right) \\ v(x,t) = e^{-t}x^2 + 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u,v\right)\right)\right) \end{cases} \tag{66}$$

Applying the method of successive approximations to the system (66), we obtain:

$$\begin{cases} u^k(x,t) = e^{-t}x^2 + 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u^k(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{k-1},v^{k-1}\right)\right)\right) \\ v^k(x,t) = e^{-t}x^2 + 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v^k(x,t)}{\partial x}\right)\right)\right) + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_2\left(u^{k-1},v^{k-1}\right)\right)\right) \end{cases} \tag{67}$$

We look for the solution of (67)in the following form:

$$\begin{cases} u^k(x,t) = \sum_{n=0}^{+\infty} u_n^k(x,t); \qquad k \geq 1 \\ v^k(x,t) = \sum_{n=0}^{+\infty} v_n^k(x,t); \qquad k \geq 1 \end{cases} \tag{68}$$

From (67), we obtain the following Laplace-SBA algorithm:

$$\begin{cases}\begin{cases}u_0^k(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{k-1},v^{k-1}\right)\right)\right)\\ u_1^k(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_0^k(x,t)}{\partial x}\right)\right)\right)\\ u_{n+1}^k(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_n^k(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\\ \begin{cases}v_0^k(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{k-1},v^{k-1}\right)\right)\right)\\ v_1^k(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0^k(x,t)}{\partial x}\right)\right)\right)\\ v_{n+1}^k(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_n^k(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\end{cases}\quad k\geq 1 \tag{69}$$

Foor $k=1$, we have the following Laplace-SBA algorithm:

$$\begin{cases}\begin{cases}u_0^1(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{0},v^{0}\right)\right)\right)\\ u_1^1(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_0^1(x,t)}{\partial x}\right)\right)\right)\\ u_{n+1}^1(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_n^1(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\\ \begin{cases}v_0^1(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^{0},v^{0}\right)\right)\right)\\ v_1^1(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0^1(x,t)}{\partial x}\right)\right)\right)\\ v_{n+1}^1(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_n^1(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\end{cases} \tag{70}$$

Let's suppose that one can find u^0 and v^0 as $N_1\left(u^0(x,t)v^0(x,t)\right)=0$ and $N_2\left(u^0(x,t)v^0(x,t)\right)=0$, we obtain the following Laplace-SBA algorithm:

$$\begin{cases}\begin{cases}u_0^1(x,t)=e^{-t}x^2\\ u_1^1(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_0^1(x,t)}{\partial x}\right)\right)\right)\\ u_{n+1}^1(x,t)=\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_n^1(x,t)}{\partial x}\right)\right)\right) \quad ; \quad \forall\ n\geq 2\end{cases}\\ \begin{cases}v_0^1(x,t)=e^{-t}x^2\\ v_1^1(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0^1(x,t)}{\partial x}\right)\right)\right)\\ v_{n+1}^1(x,t)=\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_n^1(x,t)}{\partial x}\right)\right)\right) \quad ; \quad \forall\ n\geq 2\end{cases}\end{cases} \tag{71}$$

From (71), we obtain:

$$\begin{cases}\begin{cases}u_0^1(x,t)=e^{-t}x^2\\ u_1^1(x,t)=4e^{-t}-4+4-4e^{-t}=0\\ u_{n+1}^1(x,t)=0 \quad ; \quad \forall\ n\geq 2\end{cases}\\ \begin{cases}v_0^1(x,t)=e^{-t}x^2\\ v_1^1(x,t)=4e^{-t}-4+4-4e^{-t}=0\\ v_{n+1}^1(x,t)=0 \quad ; \quad \forall\ n\geq 2\end{cases}\end{cases} \tag{72}$$

Therefore, we get:

$$\begin{cases}u^1(x,t)=u_0^1(x,t)=x^2e^{-t}\\ v^1(x,t)=v_0^1(x,t)=x^2e^{-t}\end{cases} \tag{73}$$

For $k=2$, we have the following Laplace-SBA algorithm:

$$\begin{cases}\begin{cases}u_0^2(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^1,v^1\right)\right)\right)\\ u_1^2(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_0^2(x,t)}{\partial x}\right)\right)\right)\\ u_{n+1}^2(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_n^2(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\\ \begin{cases}v_0^2(x,t)=e^{-t}x^2+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(N_1\left(u^1,v^1\right)\right)\right)\\ v_1^2(x,t)=4e^{-t}-4+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0^2(x,t)}{\partial x}\right)\right)\right)\\ v_{n+1}^2(x,t)=+\mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_n^2(x,t)}{\partial x}\right)\right)\right)\quad;\quad n\geq 0\end{cases}\end{cases}\tag{74}$$

We remark that:

$$\begin{cases}N_1\left(u^1,v^1\right)=2u^1(x,t)\dfrac{\partial u^1(x,t)}{\partial x}-\dfrac{\partial}{\partial x}\left(u^1(x,t)v^1(x,t)\right)\\ \qquad=2\left(x^2e^{-t}\right)\dfrac{\partial}{\partial x}\left(x^2e^{-t}\right)-\dfrac{\partial}{\partial x}\left(x^4e^{-2t}\right)\\ \qquad=4x^3e^{-2t}-4x^3e^{-2t}=0\\ N_2\left(u^1,v^1\right)=2v^1(x,t)\dfrac{\partial v^1(x,t)}{\partial x}-\dfrac{\partial}{\partial x}\left(u^1(x,t)v^1(x,t)\right)\\ \qquad=2\left(x^2e^{-t}\right)\dfrac{\partial}{\partial x}\left(x^2e^{-t}\right)-\dfrac{\partial}{\partial x}\left(x^4e^{-2t}\right)\\ \qquad=4x^3e^{-2t}-4x^3e^{-2t}=0\end{cases}\tag{75}$$

and (74) becomes:

$$\begin{cases} \begin{cases} u_0^2(x,t) = e^{-t}x^2 \\ u_1^2(x,t) = 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_0^2(x,t)}{\partial x}\right)\right)\right) \\ u_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u_n^2(x,t)}{\partial x}\right)\right)\right) \quad ; \quad \forall \; n \geq 2 \end{cases} \\ \begin{cases} v_0^2(x,t) = e^{-t}x^2 \\ v_1^1(x,t) = 4e^{-t} - 4 + \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_0^2(x,t)}{\partial x}\right)\right)\right) \\ v_{n+1}^2(x,t) = \mathcal{L}_t^{-1}\left(\frac{1}{s}\mathcal{L}_t\left(\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial v_n^2(x,t)}{\partial x}\right)\right)\right) \quad ; \quad \forall \; n \geq 2 \end{cases} \end{cases} \tag{76}$$

We remark that (76) is the same algorithm that (71). Thus obtain:

$$\begin{cases} u^2(x,t) = x^2e^{-t} \\ v^2(x,t) = x^2e^{-t} \end{cases} \tag{77}$$

Using the same the procedure for $k \geq 3$, we get:

$$\begin{cases} u^1(x,t) = u^2(x,t) = \cdots = u^k(x,t) = x^2e^{-t} \\ v^1(x,t) = v^2(x,t) = \cdots = v^k(x,t) = x^2e^{-t} \end{cases} \tag{78}$$

Thus, the solution of example 3 is:

$$\begin{cases} u(x,t) = \lim\limits_{k \longrightarrow +\infty} u^k(x,t) = x^2e^{-t} \\ v(x,t) = \lim\limits_{k \longrightarrow +\infty} v^k(x,t) = x^2e^{-t} \end{cases} \tag{79}$$

4. Conclusion

The results of this paper show that the use of this method allowed to obtain the exact solutions of the coupled Burger's equations. Consequently, Laplace-SBA method is a powerful mathematical tool for solving any system of Burger's equations with initial conditions. However, further research is needed to verify the relevance and effectiveness of this method in the resolution of various nonlinear differential equations.

References

[1] K. Abbaoui, Y. Cherruault,Convergence of Adomian method applied to differential equations, Mathematical and computer Modellings 20(5), (1994) 103-109. DOI: https://doi.org/10.1016/0895-1221(94)00144-8

[2] M.A. Abdoua, A. A. Solimanb, "Variational iteration method for solving Burger's and coupled Burger's equations" Journal of computational and Applied Mathematic 181 (2005) 245-251. DOI: https://doi.org/10.1016/j.cam.2004.11.032

[3] M.Al-Mazmumy, The Modified Adomian Decomposition Method for Solving Nonlinear Coupled Burger's Equations, Nonlinear Analysis and Differential Equations, 3(2015) 111-122.DOI: https://dx.doi.org/10.12988/nade.2015.41226

[4] J .Biazar, M. Eslami, H. Ghazvini. Homotopy pertubation method for systems of partial differential equations.Int.J. Nonlinear Sci. Num. Simul-8(3) (2003) 411-416. DOI: 10.1515/IJNSNS.2007.8.3.413

[5] M.Dehghan, A Hamidi, M.Shakou Rifar, The solution of coupled Burger's equations using Adomian-pade technique . Appl. Math. Comput.189(2007) 1034-1047. DOI:10.1016/j.amc.2006.11.179

[6] Hassan Eltayeb, Said Mesloub, Adem Kilieman, Anote on a singular coupled Burgers equation ad double Laplace transform method .Journal of Nonlinear Sciences and Applications 11(2018) 635-643. DOI: http://dx.doi.org/10.22436/jnsa.011.05.05

[7] J.H.He, Variational iteration method a kind of nolinear analytical technique: some examples. Int J.Nonlinear Mech .34 (1999) 699-08

[8] A.A Hemeda Homotopy Perturbation Method for Systems of Nonlinear Coupled Equations. Appplied Mathematical Sciences 6(96) (2012) 4787-4800.

[9] Hongqing Zhu, Huazhong Shu, Meiyu Ding, Numerical solutions of two-dimensional Burger's equations by discrete Adomian decomposition method Computers and Mathematics with Applications 60(2010) 840-848. DOI: https://doi.org/10.1016/j.camwa.2010.05.031

[10] M.Mirzazadeh, Z. Ayati. New homotopy perturbation method for system of Burgers equations. Alexandria Engineering Journal (55) (2016) 1619-1624.DOI: https://doi.org/10.1016/j.aej.2016.02.003

[11] WM. Moslem, R Sabry, Zakharov-Kuz, Netsov-Burger's equation for dust ion acoustic waves. Chaos solitons Fractals 36(2008) 628-634. Doi:10.1016/j.chaos.2006.06.097

[12] G. D. Nkaya, F. Bassono, R. Yaro, J. Bonazebi Yindoula, G.Bissanga, "Application of the SBA Method for Solving the partial Differential Equation ", Journal of Mathematics Research.10(6) 2018. DOI:10.5539/jmr.v10n6p98.

[13] Y. Pare, Y. Minoungou, A. Wassiha Nébié, "Resolution of nonlinear convection-diffusion-reaction equations of Cauchy's kind by the Laplace SBA method", European Journal of Pure and Applied Mathematical. 12(3), (2019) 771-789. DOI:10.29020/nybg.ejpam.v12i3.3430

[14] M.M. Rashidi, E. Erfani, New analytical method for solving Burger's and nonlinear heat transfer equations and comparison with HAM. Comput.Phys Commun.180(2009) 1539-1544. DOI:10.1016/j.cpc.2009.04.009

[15] Joseph Bonazebi Yindoula, Youssouf Pare, Gabriel Bissanga, Francis Bassono. Application of the Adomian Decomposition Method (ADM) and the SOME BLAISE ABBO (SBA) method to solving Diffusion-Reaction equation, Advances in Theoretical and Applied Mathematics; 9(2014), $N°2$, 97-104.

[16] B. ABBO, Nouvel algorithme numérique de résolution des Equations Différentielles Ordinaires (EDO) et des Equations aux dérivées Partielles (EDP) non linéaires. Thèse de doctorat unique, Université de de Ouagadougou, janvier 2007, UFR/SEA; Département de Mathématiques et Informatique (Burkina Faso).

[17] Stevy Mikamona Mayembo, Joseph Bonazebi-Yindoula, Youssouf Pare, Gabriel Bissanga. Application of the SBA method to solve the nonlinear biological population models. European journal of pure and applied mathematics. Vol. 12, No. 3,2019, 1-10. ISSN 1307-5543 -www.ejpam.com

[18] Pare Youssouf, Résolution de quelques équations fonctionnelles par la méthode SBA (Somé Blaise-Abbo), Diss, Université de Ouagadougou, Mai 2010.