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# Results on $C_2$-paracompactness

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**Abstract.** A *$C$-paracompact* is a topological space $X$ associated with a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ satisfying that $f \upharpoonright_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Furthermore, $X$ is called *$C_2$-paracompact if $Y$ is $T_2$ paracompact.* In this article, we discuss the above concepts and answer the problem of Arhangel'skiĭ. Moreover, we prove that the sigma product $\Sigma(0)$ can not be condensed onto a $T_2$ paracompact space.

**2020 Mathematics Subject Classifications**: 54C10, 54D20

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## 1. Introduction and preliminaries

In the present work, we give some new results about $C$-paracompactness and $C_2$-paracompactness [8] and answer a problem of Arhangel'skiĭ. Also, we prove that the sigma product $\Sigma(0)$ can not be condensed onto a $T_2$ paracompact space. Throughout this paper, $\langle x, y\rangle$ denotes an ordered pair, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{P}$ denotes the irrational numbers, and $\mathbb{R}$ denotes the set of real numbers. $T_2$ denotes the Hausdorff property. A $T_4$ space is a $T_1$ normal space and a Tychonoff space ($T_{3\frac{1}{2}}$) is a $T_1$ completely regular space. We do not assume Hausdorffness in the definition of compactness, countable compactness, local compactness, and paracompactness. So, a space is paracompact if any open cover has a locally finite open refinement. The regularity of Lindelöfness's definition is not assumed. The interior and the closure of a subset $A$ of a space $X$, are denoted by int$A$ and $\overline{A}$, respectively. An ordinal $\gamma$ consists of all ordinal $\alpha$ that satisfying $\alpha < \gamma$. The first infinite ordinal is $\omega_0$, the first uncountable ordinal is $\omega_1$, and the successor cardinal of $\omega_1$ is $\omega_2$.

We begin by recalling the following definition, see [8].

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**Definition 1.** (Arhangel'skiĭ, 2016.)

*A topological space $X$ is called $C$-paracompact* if there exist a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. A topological space $X$ is called *$C_2$-paracompact* if there exist a Hausdorff paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.

In [8, Theorem 2.2], the following theorem was proved.

**Theorem 1.** *If $X$ is Fréchet and $C$-paracompact ($C_2$-paracompact), then any function witnesses its $C$-paracompactness ($C_2$-paracompactness) is continuous.*

## 2. Results and Examples

Since the paracompactness is not multiplicative, it seems that both $C$-paracompactness and $C_2$-paracompactness are not multiplicative, but we still could not find a counterexample. We introduce here a case where $C$-paracompactness and $C_2$-paracompactness are multiplicative.

**Theorem 2.** *If $X$ is $C$-paracompact ($C_2$-paracompact) and $Z$ is a compact $T_2$ space, then $X \times Z$ is $C$-paracompact ($C_2$-paracompact).*

*Proof.* Let $Y$ be a paracompact ($T_2$ paracompact) space and $f : X \longrightarrow Y$ be a bijective function such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Consider the product space $Y \times Z$ which is paracompact ($T_2$ paracompact), because the product of any paracompact space with a compact space is paracompact, see [4, 5.1.36]. Define $g : X \times Z \longrightarrow Y \times Z$ by $g(\langle x, i\rangle) = \langle f(x), i\rangle$. Then $g$ is a bijective function and $g = f \times id_Z$, where $id_Z$ is the identity function on $Z$. Let $C$ be any compact subspace of $X \times Z$. Then $C \subseteq p_1(C) \times p_2(C)$, where $p_1$ and $p_2$ are the usual projection functions. $p_1(C)$ is a compact subspace of $X$ and $p_2(C)$ is a compact subspace of $Z$, thus $p_1(C) \times p_2(C)$ is a compact subspace of $X \times Z$. Now, $f \restriction_{p_1(C)} : p_1(C) \longrightarrow f(p_1(C))$ is a homeomorphism and $id_Z \restriction_{p_2(C)} : p_2(C) \longrightarrow p_2(C)$ is a homeomorphism. Thus $(f \times id_Z) \restriction_{(p_1(C) \times p_2(C))} : p_1(C) \times p_2(C) \longrightarrow f(p_1(C)) \times p_2(C)$ is a homeomorphism. We conclude that $g \restriction_C : C \longrightarrow g(C)$ is a homeomorphism because

$$g \restriction_C = ((f \times id_Z) \restriction_{(p_1(C) \times p_2(C))}) \restriction_C .$$

**Corollary 1.** *If $X$ is $C_2$-paracompact ($C$-paracompact), then so is $X \times I$, where $I$ is the closed unit interval $[0, 1]$ considered with its usual Euclidean metric topology.*

We still do not know an answer of the converse of the above theorem which is the following statement: *If $X \times I$ is $C_2$-paracompact, is then $X$ $C_2$-paracompact ?* Observe that if $X$ is $C_2$-paracompact and $Y$ is $T_4$, then the natural projection $p : X \times Y \longrightarrow Y$

may not be closed. For example, $\omega_1$ is $C_2$-paracompact being $T_2$ locally compact [8] and $\omega_1 + 1$ is $T_2$ compact, hence $T_4$, but $p : \omega_1 \times (\omega_1 + 1) \longrightarrow \omega_1 + 1$ is not closed, see [4, 3.10.16].

Referring to Theorem 1, we introduce here another case when a product of two $C_2$-paracompact spaces will be $C_2$-paracompact.

**Theorem 3.** *If $X$ and $Z$ are $C_2$-paracompact spaces such that $X$ is Fréchet and countably compact, then $X \times Z$ is $C_2$-paracompact.*

*Proof.* Let $Y$ and $Y'$ be $T_2$ paracompact spaces, $f : X \longrightarrow Y$ and $f' : Z \longrightarrow Y'$ be bijective function such that the restriction of each of them on any compact subspace is a homeomorphism. Define $g : X \times Z \longrightarrow Y \times Y'$ by $g(\langle x, z\rangle) = \langle f(x), f'(z)\rangle$, i.e., $g = f \times f'$. Then $g$ is bijective. Now, $X$ is Fréchet gives that $f$ is continuous, see Theorem 1. Since $X$ is countably compact and $f$ continuous surjective, then $Y$ is countably compact. Hence $Y$ is compact because any $T_2$ countably compact paracompact is compact [4, 5.1.20]. Since a product of a $T_2$ paracompact space with a $T_2$ compact space is $T_2$ paracompact [4, 5.1.36], then $Y \times Y'$ is $T_2$ paracompact. Now, similar argument as in the proof of Theorem 2 shows that the restriction of $g$ on any compact subspace $C$ of $X \times Z$ will be a homeomorphism.

Recall that a topological space $( X , \mathcal{T} )$ is called *lower compact* if there exists a coarser topology $\mathcal{T}'$ on $X$ such that $( X , \mathcal{T}' )$ is $T_2$-compact, [8]. It was proved in [8, 2.21] that "*if $X$ is $C_2$-paracompact countably compact Fréchet, then $X$ is lower compact*". It turns out that lower compactness is enough in Theorem 3.

**Theorem 4.** *If $X$ is lower compact and $Z$ is $C_2$-paracompact, then $X \times Z$ is $C_2$-paracompact.*

*Proof.* Let $\mathcal{T}$ denote the topology on $X$. Let $\mathcal{T}'$ be a $T_2$ compact topology on $X$ coarser than $\mathcal{T}$. Pick a $T_2$ paracompact space $Y'$ and a bijective function $f' : Z \longrightarrow Y'$ such that the restriction of $f'$ on any compact subspace is a homeomorphism. Define $g : X \times Z \longrightarrow X \times Y'$ by $g(\langle x, z\rangle) = \langle x, f'(z)\rangle$, i.e., $g = id_X \times f'$, where $X$ in the codomain is considered with the topology $\mathcal{T}'$. Observe that the restriction of the identity function $id_X : ( X , \mathcal{T} ) \longrightarrow ( X , \mathcal{T}' )$ on any compact subspace $A$ in $( X , \mathcal{T} )$ is a homeomorphism, see [4, 3.1.13]. Also, the codomain of $g$, $X \times Y'$ is $T_2$ paracompact being a product of a $T_2$ compact space $( X , \mathcal{T}' )$, with a $T_2$ paracompact space $Y'$, [4, 5.1.36]. Now, similar argument as in the proof of Theorem 2 shows that the restriction of $g$ on any compact subspace $C$ of $X \times Z$ will be a homeomorphism.

Here is an example showing that the Fréchet property is essential in Theorem 1.

**Example 1.** *Consider $\omega_2$, the successor cardinal number of the cardinal number $\omega_1$. Let $[\omega_2]^{\leq \omega_1} = \{ E \subset \omega_2 : |E| \leq \omega_1 \}$. Let $i \notin \omega_2$ and put $X = \{i\} \cup \omega_2$. For each $\alpha \in \omega_2$, let $\{\alpha\}$ be open and an open neighborhood of $i$ is of the form $U = \{i\} \cup (\omega_2 \setminus E)$ where*

*$E \in [\omega_2]^{\leq\omega_1}$. Then $X$ is not Fréchet because $i \in \overline{\omega_2}$ and the only convergent sequence is the eventually constant sequence. (So, $X$ is not even of countable tightness.)*

*Observe that $X$ is $T_2$ paracompact. BUT we will not treat $X$ in this way. A subspace $A$ of $X$ is compact if and only if $A$ is finite. So, by [8, Theorem 2.7], $X$ is $C_2$-paracompact and $X = Y$ with the discrete topology and the identity function witness the $C_2$-paracompactness of $X$ and clearly the identity function can not be continuous because $X$ is not discrete.* ■

Observe that $X$ in Example 1 is not of countable tightness as $i \in \overline{\omega_2}$ but there is no countable subset $A$ of $\omega_2$ satisfies $i \in \overline{A}$. Here is an example showing that the countable tightness property is not enough in Theorem 1.

**Example 2.** *For each $i \in \mathbb{N}$, let $X_i = \{a_i\} \cup \{a_{i,j} : j \in \mathbb{N}\}$ be such that $X_n \cap X_m = \emptyset$ for each $n, m \in \mathbb{N}$ with $n \neq m$. Let $a \notin \cup_{i\in\mathbb{N}} X_i$ and put $X = \{a\} \cup (\cup_{i\in\mathbb{N}} X_i)$. Generate a topology on $X$ by the following neighborhood system: For each $i, j \in \mathbb{N}$, let $\mathcal{B}(a_{i,j}) = \{\{a_{i,j}\}\}$. For each $i \in \mathbb{N}$, let $\mathcal{B}(a_i) = \{a_i\} \cup \{a_{i,j} : j \geq k$, where $k \in \mathbb{N}\}$. For members of $\mathcal{B}(a)$ we take all sets obtained from $X$ by removing a finite numbers of $X_i$'s and a finite number of points of $a_{i,j}$ in all the remaining $X_i$'s. So, if $U \in \mathcal{B}(a)$, then $U$ is of the form $U = \{a\} \cup (\cup_{i\in(\mathbb{N}\setminus E)} X_i')$ where $E$ is a finite subset of $\mathbb{N}$ and $X_i' = X_i \setminus E_i$ where $E_i$ is a finite subset of $\{a_{i,j} : j \in \mathbb{N}\}$. It is well-known that $X$ is zero-dimensional normal space which is not Fréchet [4, 1.6.19]. Let $Z = X \setminus \{a_i : i \in \mathbb{N}\}$. Then $Z$ as a subspace of $X$ is not sequential [4, 1.6.20]. But since $Z$ is countable, then it is of countable tightness. Now, a subspace $C$ of $Z$ is compact if and only if $C$ is finite. Since $Z$ is also $T_1$, then by [8, Theorem 2.7], $Y = Z$ with the discrete topology and the identity function witness the $C_2$-paracompactness of $Z$ and since $Z$ is not discrete, then the identity function is not continuous.* ■

Let $X$ be any set containing more than one element. Fix an element $p \in X$. The topology $\mathcal{T} = \{\emptyset\} \cup \{W \subseteq X : p \in W\}$ is called *the particular point topology on $X$*, see [9].

**Theorem 5.** *Let $(X, \mathcal{T})$ be a Fréchet $\sigma$-compact non-compact space such that $\mathcal{T}$ is coarser than a particular point topology $\mathcal{T}_p$ on $X$, where $p \in X$, then $(X, \mathcal{T})$ can not be $C$-paracompact.*

*Proof.* Suppose that $(X, \mathcal{T})$ is $C$-paracompact. Pick a paracompact space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Since $X$ is Fréchet, then $f$ is continuous, see Theorem 1. So, for any non-empty open subset $W$ of $Y$ we have that $f^{-1}(W)$ is open in $X$, hence $p \in f^{-1}(W)$ which gives that $f(p) \in W$.

Now, write $X = \bigcup_{n\in\mathbb{N}} X_n$ where $X_n$ is compact for each $n \in \mathbb{N}$ and $X_n \subset X_{n+1}$ for each $n \in \mathbb{N}$, i.e., the $X_n$'s are increasing. Since $X$ is not compact, there exists an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ such that for any finite subset $F$ of $\Lambda$ there exists an element $x \in X$ such that $x \notin \bigcup_{\alpha\in F} U_\alpha$. Now, $\mathcal{U}$ is an open cover for $X_1$ and $X_1$ is compact. Let $F_1$ be a finite subset of $\Lambda$ such that $X_1 \subseteq \bigcup_{\alpha\in F_1} U_\alpha = V_1$. Pick $a_2 \in X \setminus V_1$ and let $i_2 \in \mathbb{N}$ be the minimal so that $a_2 \in X_{i_2}$, i.e., if $j < i_2$, then $a_2 \notin X_j$. Now, $\mathcal{U}$ is an open cover for

$X_{i_2}$ and $X_{i_2}$ is compact. Let $F_2$ be a finite subset of $\Lambda$ such that $X_{i_2} \subseteq \bigcup_{\alpha \in F_2} U_\alpha = V_2$. If $m \in \mathbb{N}$ so that $a_m \in X$, $i_m \in \mathbb{N}$, $F_m$ finite subset of $\Lambda$, and $V_m$ are chosen, then Pick $a_{m+1} \in X \setminus V_m$ and let $i_{m+1} \in \mathbb{N}$ be the minimal so that $a_{m+1} \in X_{i_{m+1}}$. Continue, $\mathcal{U}$ is an open cover for $X_{i_{m+1}}$ and $X_{i_{m+1}}$ is compact. Let $F_{m+1}$ be a finite subset of $\Lambda$ such that $X_{i_{m+1}} \subseteq \bigcup_{\alpha \in F_{m+1}} U_\alpha = V_{m+1}$. So, we have constructed two countably infinite families of $X_{i_m}$'s, $V_m$'s such that $X_{i_m} \subset X_{i_{m+1}}$ and $X_{i_{m+1}} \setminus V_m \neq \emptyset$ for each $m \in \mathbb{N}$ as $a_{m+1} \in X_{i_{m+1}} \setminus V_m$.

Now, for each $m \in \mathbb{N}$, $f \restriction_{X_{i_m}} : X_{i_m} \longrightarrow f(X_{i_m})$ is a homeomorphism, where $i_1 = 1$. We have $V_m \cap X_{i_{m+1}}$ is open in $X_{i_{m+1}}$ for each $m \in \mathbb{N}$, thus $f(V_m \cap X_{i_{m+1}})$ is open in $f(X_{i_{m+1}})$ for each $m \in \mathbb{N}$. Hence, for each $m \in \mathbb{N}$ there exists an open subset $W_m$ of $Y$ such that $W_m \cap f(X_{i_{m+1}}) = f(V_m \cap X_{i_{m+1}})$. Observe that $f(a_{m+1}) \notin W_m$ for each $m \in \mathbb{N}$. Since the family $\{ V_m : m \in \mathbb{N} \}$ is an open cover for $X$, then we have that the family $\{ W_m : m \in \mathbb{N} \}$ is an open cover for $Y$ consisting of distinct proper subsets of $Y$. Since each non-empty open subset of $Y$ must contain the element $f(p)$, then the open cover $\{ W_m : m \in \mathbb{N} \}$ of $Y$ has no locally finite open refinement, which is a contradiction. Therefore, $( X , \mathcal{T} )$ is not $C$-paracompact.

Non-compactness assumption is essential in Theorem 5, for example, consider on $\mathbb{R}$ the topology $\mathcal{T} = \{\emptyset, \mathbb{R}, \{p\}\}$, where $p \in \mathbb{R}$.

The following example answers three kinds of invariants. We used two well-known spaces, the Alexandroff duplicate space and the closed extension space. Recall that for any $T_1$ space $X$, let $X' = X \times \{1\}$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we denote the element $\langle x, 1 \rangle$ in $X'$ by $x'$ and for a subset $B \subseteq X$ let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in $X$ with $x \in U \}$. Let $\mathcal{T}$ denote the unique topology on $A(X)$ which has $\{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ as its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of* $X$ [3]. In [8], it was shown that *" if $X$ is $C_2$-paracompact, then so is its Alexandroff duplicate $A(X)$.".*

**Example 3.** *Consider the Alexandroff duplicate space $A(\mathbb{R})$ of $\mathbb{R}$ with its usual metric topology. It is $C_2$-paracompact, see [8, Theorem 28]. Now, let $i = \sqrt{-1} \notin \mathbb{R}$ and put $X = \mathbb{R} \cup \{i\}$. Let $\mathcal{T}$ be the closed extension topology on $X$ generated from $\mathbb{R}$ with its usual metric topology and $i$. So, $\mathcal{T} = \{\emptyset\} \cup \{ W \cup \{i\} : W \subseteq \mathbb{R}; W$ is open in the usual metric topology $\}$.*
*$( X , \mathcal{T} )$ is not $C$-paracompact because it is Fréchet, being first countable, non-compact, and coarser than the particular point topology on $X$ where the particular point is $i$, see Theorem 5. Define $g : A(\mathbb{R}) \longrightarrow X$ by*

$$g(x) = \begin{cases} i & ; \text{if } x \in \mathbb{R}' \\ x & ; \text{if } x \in \mathbb{R} \end{cases}$$

*$g$ is an open surjection function. Thus $C$-paracompactness and $C_2$-paracompactness are neither invariant, open invariant, nor quotient invariant.* ■

Now we show that $C$-paracompactness and $C_2$-paracompactness are both not hereditary. Recall that a space $X$ is called *$C$-normal* if there exist a normal space $Y$ and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$ [1]. It is clear that any $C_2$-paracompact space is $C$-normal [8].

**Example 4.** *Consider $2^{\omega_1}$, where $2 = \{0, 1\}$ with the discrete topology. Consider the subspace of $2^{\omega_1}$ consisting of all points with at most countably many non-zero coordinates, i.e., the sigma product $\Sigma(0)$. Put $X = 2^{\omega_1} \times \Sigma(0)$. Raushan Buzyakova proved that $X$ can not be mapped onto a normal space $Y$ by a bijective continuous function [2].* In [7], M. Saeed proved that $X$ is not $C$-normal, hence $X$ is not $C_2$-paracompact. Since $X$ is a Tychonoff non-compact space, any compactification of $X$ is $C_2$-paracompact while $X$ is not. ■

We still do not know if $C$-paracompactness ($C_2$-paracompactness) is hereditary with respect to closed subspaces or not.

Now, here is our first main result. Arhangel'skiĭ stated the following problem, see [8]: "*Is there a $T_4$ space which is not $C_2$-paracompact?*". We will answer this problem in positive.

**Example 5.** *Consider the sigma product $\Sigma(0)$ as a subspace of $2^{\omega_1}$, where $2 = \{0, 1\}$ with the discrete topology, see Example 4. We have that $\Sigma(0)$ is $T_4$ [5, Theorem 7.4], countably compact [6, Theorem 6.10], Fréchet [4, 3.10.D], hence it is a $k$-space [4, 3.10.D]. Also $\Sigma(0)$ is not paracompact because it is contained a copy of $\omega_1$ as a closed subspace [5, Theorem 7.2]. Suppose that $\Sigma(0)$ is $C_2$-paracompact. By Theorem 2, $X = 2^{\omega_1} \times \Sigma(0)$ is $C_2$-paracompact. This contradicts M. Saeed's result [7] and Buzyakova's result [2] because any $T_2$ paracompact space is normal.* ■

Here is our second main result. Recall that a function $f : X \longrightarrow Y$ is called *condensation* if it is bijective and continuous. The sigma product $\Sigma(0)$ is a $k$-space [4, 3.10.D]. Considering the theorem "*a function $f$ of a $k$-space $X$ to a topological space $Y$ is continuous if and only if for every compact space $C \subseteq X$ the restriction $f \restriction_C : C \longrightarrow Y$ is continuous*", [4, 3.3.21], we conclude the following:

**Corollary 2.** *The sigma product $\Sigma(0)$ can not be condensed onto any $T_2$ paracompact space.*

**Open Problem:** Is $C_2$-paracompactness multiplicative ?

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