EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 2, 2021, 380-395
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

Characterizations and Identities for Isosceles Triangular Numbers

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Abstract. Triangular numbers have been of interest and continuously studied due to their beautiful representations, nice properties, and various links with other figurate numbers. For positive integers n and l, the nth l-isosceles triangular number is a generalization of triangular numbers defined to be the arithmetic sum of the form

$$T(n, l) = 1 + (1 + l) + (1 + 2l) + \cdots + (1 + (n - 1)l).$$

In this paper, we focus on characterizations and identities for isosceles triangular numbers as well as their links with other figurate numbers. Recursive formulas for constructions of isosceles triangular numbers are given together with necessary and sufficient conditions for a positive integer to be a sum of isosceles triangular numbers. Various identities for isosceles triangular numbers are established. Results on triangular numbers can be viewed as a special case.

2020 Mathematics Subject Classifications: 11B99, 11E25

Key Words and Phrases: Integer Sequences, Triangular numbers, Isosceles triangular numbers, Polygonal Numbers, Identities

1. Introduction

A triangular number is a number that can be represented as an equilateral triangular arrangement of points equally spaced. Precisely, the nth triangular number is the number of points composing an equilateral triangle with n points on a side which equals the sum of the n natural numbers of the form

$$T(n) = 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}. \quad (1)$$

The nth triangular number can be represented as points in an equilateral triangle as in Figure 1.

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DOI: https://doi.org/10.29020/nybg.ejpam.v14i2.3952

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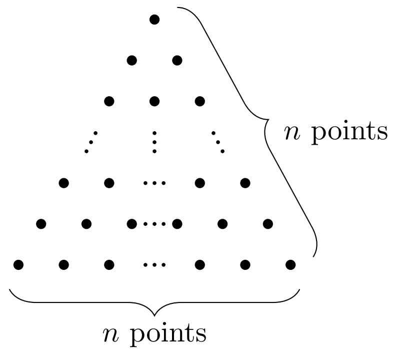


Figure 1: Triangular Number $T(n)$

Triangular numbers have been introduced and studied since the 6th century BC. Due to their nice properties and wide links with other mathematical objects, such numbers have been extensively studied (see [1–3] and references therein). Subsequently, characterizations of triangular numbers and interesting relationships among triangular numbers and other figurate numbers have been established (see [2, 4, 10]). Various subsequences of triangular numbers with nice properties have been presented in [2, 8, 11]. Identities of triangular numbers have been of interest and studied in [1–3, 7, 9, 12].

In [5], an isosceles triangular number has been introduced as a generalization of triangular numbers and it is defined to be a number that can be represented as an isosceles triangular arrangement of points. Precisely, the nth l-isosceles triangular number, denoted by $T(n,l)$, is defined to be the arithmetic sum of the form

$$T(n,l) = 1 + (1+l) + (1+2l) + \cdots + (1+(n-1)l) = n + \frac{n(n-1)l}{2} = n + lT(n-1).$$

Alternatively, an l-isosceles triangular number can be viewed as a special case of generalized trapezoidal numbers in [6]. Clearly, the isosceles triangular number $T(n,l)$ becomes the nth triangular number $T(n)$ whenever $l=1$. In the case where $l=2$, it can be easily seen that $T(n,2)=n^2$ is a square number. For convenience, the notions of the l-isosceles triangular number $T(0,l)$ and the triangular number $T(0)$ are used in some contexts and they are set to be zero. In [5], parity and some properties of isosceles triangular numbers have been studied. The nth l-isosceles triangular number $T(n,l)$ can be represented as an isosceles triangular arrangement of points in Figure 2.

Illustrative examples of isosceles triangular numbers $T(4,2)=16$ and $T(4,3)=22$ are given in Example 1.

Example 1. *The positive integers* 16 *and* 22 *are* 2*-isosceles and* 3*-isosceles triangular numbers, respectively. They can be represented as isosceles triangular shapes as follows.*

A polygonal number is a number represented as points or pebbles arranged in the shape of a regular polygon (see [2]). For positive integers n and m, the nth m-gonal number is defined to be

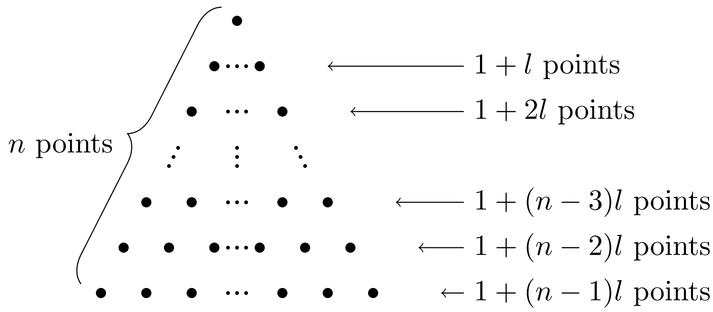
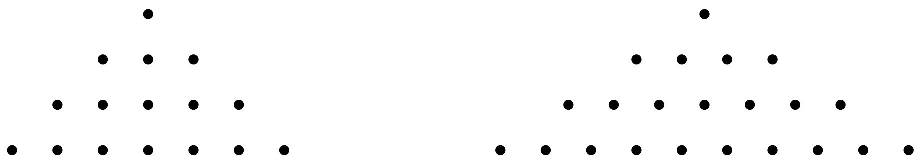


Figure 2: Isosceles triangular number $T(n, l)$



(a) $T(4, 2) = 16$ (b) $T(4, 3) = 22$

Figure 3: Isosceles triangular numbers $T(4, 2) = 16$ and $T(4, 3) = 22$

$$\begin{aligned} P(n, m) &= 1 + (1 + (m-2)) + (1 + 2(m-2)) + \cdots + (1 + (m-2)(n-1)) \\ &= n + \frac{n(n-1)(m-2)}{2}. \end{aligned}$$

It is not difficult to see that $T(n, l) = P(n, l + 2)$, i.e., an isosceles triangular number is actually a shifted version of a polygonal number. However, in this manuscript, the notion of isosceles triangular numbers is used to present the generality of triangular numbers and nice appearance of some identities.

As mentioned above, various characterizations and identities for triangular numbers have been established. For isosceles triangular numbers, only a few works have been done on their characterizations and identities. It is therefore of interest to investigate such problems. The first goal of this paper is to generalize the facts (see [3]) that "a positive integer N is a triangular number if and only if $9N + 1$ is a triangular number" and "N is a triangular number if and only if $8N + 1$ is square" to isosceles triangular numbers. Secondly, we aim to generalize some identities for triangular numbers in [4] and [7] to isosceles triangular numbers. To the best of our knowledge, the characterizations and identities for isosceles triangular numbers presented in this paper have not been established either in terms of isosceles triangular numbers or polygonal numbers.

The paper is organized as follows. Characterizations of isosceles triangular numbers are presented in Section 2 as well as their links with other figurate numbers. In Section 3, some identities for isosceles triangular numbers are established as generalizations of triangular numbers. Summary and remarks are provided in Section 4.

2. Characterizations of Isosceles Triangular Numbers

In this section, some characterizations of isosceles triangular numbers are given as well as links with other figurate numbers. Some classical results on the characterizations of triangular numbers can be viewed as a special case.

2.1. Characterizations and Recursive Constructions of Isosceles Triangular Numbers

In this subsection, we focus on a generalization of the well-known fact "a positive integer N is a triangular number if and only if $9N+1$ is a triangular number" in [3].

In the following theorem, a characterization of l-isosceles triangular numbers is presented. A recursive construction of l-isosceles triangular numbers can be deduced directly from the theorem.

Theorem 1. *Let N and l be positive integers. Let r and s be positive integers of the form $r=(2l+1)^2$ and $s=\frac{l^3-3l^2+4}{2}$. Then N is an l-isosceles triangular number if and only if $rN+s$ is an l-isosceles triangular number.*

Proof. Assume that N is an l-isosceles triangular number. Then $N=n+\frac{n(n-1)l}{2}$ for some positive integer n. It follows that

$$\begin{aligned}
rN+s &= (2l+1)^2\left(n+\frac{n(n-1)l}{2}\right)+\frac{l^3-3l^2+4}{2}\\
&= \frac{n^2l(2l+1)^2+(2-l)n(2l+1)^2+l^3-3l^2+4}{2}\\
&= \frac{4l^3n^2+4l^2n^2+ln^2+8l^2n+8ln+2n-4l^3n-4l^2n-ln+l^3-3l^2+4}{2}\\
&= \frac{(4l^3+4l^2+l)n^2-(4l^3-4l^2-7l-2)n+(l^3-3l^2+4)}{2}\\
&= \frac{l\left((4l^2+4l+1)n^2-(4l^2-6l-4)n+(l^2-4l+4)\right)-(l-2)\left((2l+1)n-(l-2)\right)}{2}\\
&= \frac{l\left((2l+1)n-(l-2)\right)^2-(l-2)\left((2l+1)n-(l-2)\right)}{2}\\
&= \frac{2\left((2l+1)n-(l-2)\right)+l\left((2l+1)n-(l-2)\right)^2-l\left((2l+1)n-(l-2)\right)}{2}\\
&= \left((2l+1)n-(l-2)\right)+\frac{\left((2l+1)n-(l-2)\right)\left(\left((2l+1)n-(l-2)\right)-1\right)l}{2}\\
&= T((2l+1)n-(l-2),l).
\end{aligned}$$

Hence, $rN+s$ is an l-isosceles triangular number.

Conversely, assume that $rN+s$ is an l-isosceles triangular number. Then

$$rN+s=n+\frac{n(n-1)l}{2}$$

for some positive number n. Equivalently, we have

$$\begin{aligned} N &= \frac{2n + ln^2 - ln - 2s}{2r} \\ &= \frac{2n + ln^2 - ln - (l^3 - 3l^2 + 4)}{2(2l+1)^2} \\ &= \frac{l\left(n^2 + 2n(l-2) + (l-2)^2\right) - (l-2)(2l+1)(n+l-2)}{2(2l+1)^2} \\ &= \frac{l(n+l-2)^2 - (l-2)(2l+1)(n+l-2)}{2(2l+1)^2} \\ &= \frac{\frac{l(n+l-2)^2}{(2l+1)^2} - \frac{(l-2)(n+l-2)}{2l+1}}{2} \\ &= \frac{\frac{2(n+l-2)}{2l+1} + \frac{l(n+l-2)^2}{(2l+1)^2} - \frac{l(n+l-2)}{2l+1}}{2} \\ &= \frac{n+l-2}{2l+1} + \frac{\left(\frac{n+l-2}{2l+1}\right)\left(\frac{n+l-2}{2l+1} - 1\right)l}{2}. \end{aligned}$$

Since $N = \frac{2n + ln^2 - ln - (l^3 - 3l^2 + 4)}{2(2l+1)^2} = \frac{(ln - l^2 + l + 2)(n+l-2)}{2(2l+1)^2}$ is a positive integer, we have $(2l+1)^2 | (ln - l^2 + l + 2)(n+l-2)$. If $(2l+1)|(n+l-2)$, then $\frac{(n+l-2)}{(2l+1)} \in \mathbb{N}$. Suppose that $(2l+1) \nmid (n+l-2)$. Since $ln - l^2 + l + 2 = l(n+l-2) - (2l+1)(l-2)$ and $\gcd(l, 2l+1) = 1$, we have $(2l+1) \nmid l(n+l-2)$ which implies that $(2l+1) \nmid (ln - l^2 + l + 2)$. Hence, $(2l+1)^2 \nmid (ln - l^2 + l + 2)(n+l-2)$ which is a contradiction. Then $(2l+1)|(n+l-2)$ which implies that $\frac{(n+l-2)}{(2l+1)}$ is a positive integer. Therefore,

$$N = \frac{n+l-2}{2l+1} + \frac{\left(\frac{n+l-2}{2l+1}\right)\left(\frac{n+l-2}{2l+1} - 1\right)l}{2} = T\left(\frac{n+l-2}{2l+1}, l\right)$$

is an l-isosceles triangular number.

By setting $l = 1$, we have the classical fact "N is a triangular number if and only if $9N + 1$ is a triangular number". For $l \in \{2, 3, 4\}$, we have the following results.

- A positive integer N is 2-isosceles triangular if and only if $25N$ is a 2-isosceles triangular number. Equivalently, N is a square number.
- A positive integer N is 3-isosceles triangular if and only if $49N + 2$ is a 3-isosceles triangular number.

- A positive integer N is 4-isosceles triangular if and only if $81N + 10$ is a 4-isosceles triangular number.

In general, each positive integer l, a recursive construction of l-isosceles triangular numbers can be deduced directly from Theorem 1.

2.2. Isosceles Triangular Numbers and Square Numbers

In this subsection, we focus on a generalization of the classical fact "N is a triangular number if and only if $8N + 1$ is square" (see [3]).

Necessary and sufficient conditions for an l-isosceles triangular number to be square are given in the following theorem.

Theorem 2. *Let N and l be positive integers. Then N is an l-isosceles triangular number if and only if $8Nl + (l-2)^2$ is square and $\sqrt{8Nl + (l-2)^2} \equiv (l+2) \pmod{2l}$.*

Proof. Assume that N is an l-isosceles triangular number. Then $N = n + \dfrac{n(n-1)l}{2}$ for some positive integer n. It follows that

$$\begin{aligned} 8Nl + (l-2)^2 &= 8nl + 4n(n-1)l^2 + (l-2)^2 \\ &= (2nl)^2 - 2(2nl)(l-2) + (l-2)^2 \\ &= (2nl - l + 2)^2 \end{aligned}$$

is square and $\sqrt{8Nl + (l-2)^2} \equiv 2nl - l + 2 \equiv l + 2 \pmod{2l}$.

Conversely, assume that $8Nl+(l-2)^2$ is square and $\sqrt{8Nl + (l-2)^2} \equiv (l+2) \pmod{2l}$. Then there exists a positive integer m such that $\sqrt{8Nl + (l-2)^2} = 2lm+(l+2)$. It follows that

$$\begin{aligned} 8Nl + (l-2)^2 &= (2lm + (l+2))^2 \\ &= 4l^2m^2 + 4lm(l+2) + (l+2)^2, \end{aligned}$$

and hence,

$$\begin{aligned} N &= \frac{4l^2m^2 + 4lm(l+2) + (l+2)^2 - (l-2)^2}{8l} \\ &= \frac{8ml + 8l + 4(m+1)ml^2}{8l} \\ &= (m+1) + \frac{(m+1)ml}{2} \\ &= T(m+1, l). \end{aligned}$$

Therefore, N is an l-isosceles triangular number as desired.

For $l = 1$, $8Nl + (l-2)^2 = 8N + 1$ is always odd. Hence, if $8Nl + (l-2)^2$ is square, then $\sqrt{8Nl + (l-2)^2} \equiv \sqrt{8N+1} \equiv 1 \equiv (l+2) \pmod{2l}$. Using Theorem 2, the following well-know result can be derived immediately.

Corollary 1 ([3]). *Let N be a positive integer. Then N is a triangular number if and only if $8N+1$ is square.*

In general, we have the following results on the sum of k l-isosceles triangular numbers.

Theorem 3. *Let l, k, and N be positive integers. Then N is a sum of k l-isosceles triangular numbers if and only if there exist positive integers $u_1, u_2, \ldots, u_k$ such that*

(i) $8lN + k(l-2)^2 = u_1^2 + u_2^2 + \cdots + u_k^2$, *and*

(ii) $u_i \equiv l+2 \pmod{2l}$ *for all* $i = 1, 2, \ldots, k$.

Proof. Assume that N is a sum of k l-isosceles triangular numbers. Then

$$N = \sum_{i=1}^{k} T(m_i, l)$$

for some positive integers m_i. It follows that

$$N = \sum_{i=1}^{k} \left(m_i + \frac{m_i(m_i-1)l}{2} \right) = \sum_{i=1}^{k} \frac{2m_i + m_i^2 l - m_i l}{2}$$

and

$$\begin{aligned} 8lN + k(l-2)^2 &= k(l-2)^2 + 4l \sum_{i=1}^{k} \left(2m_i + m_i^2 l - m_i l\right) \\ &= \sum_{i=1}^{k} \left(4m_i^2 l^2 - 4m_i l(l-2) + (l-2)^2\right) \\ &= \sum_{i=1}^{k} (2m_i l - (l-2))^2. \end{aligned}$$

For each $i \in \{1, 2, \ldots, k\}$, let $u_i = 2m_i l - (l-2)$. It follows that

$$8lN + k(l-2)^2 = u_1^2 + u_2^2 + \cdots + u_k^2$$

and $u_i \equiv 2lm_i - (l-2) \equiv l+2 \pmod{2l}$ for all $i \in \{1, 2, \ldots, k\}$.

Conversely, assume that there exist positive integers $u_1, u_2, \ldots, u_k$ such that

$$8lN + k(l-2)^2 = u_1^2 + u_2^2 + \cdots + u_k^2$$

and $u_i \equiv l+2 \pmod{2l}$ for all $i = 1, 2, \ldots, k$. For each $i \in \{1, 2, \ldots, k\}$, let $m_i = \frac{u_i + (l-2)}{2l}$. Since $u_i \equiv l+2 \pmod{2l}$, it follows that m_i is a positive integer. It is not difficult to verify that $N = \sum_{i=1}^{k} T(m_i, l)$.

For $k = 1$, Theorem 3 becomes Theorem 2. By setting $l = 1$ in Theorem 3, we have the following well-known result.

Corollary 2 ([3]). *Let k and N be positive integers. Then N is a sum of k triangular numbers if and only if $8N + k$ is a sum of k odd squares.*

From Theorem 3, a positive integer N is a sum of two l-isosceles triangular numbers if and only if there exist positive integers u and v such that $8lN + 2(l-2)^2 = u^2 + v^2$ and $u \equiv v \equiv l + 2 \pmod{2l}$. An alternative characterization for this case is given in the next theorem.

Theorem 4. *Let l and N be positive integers. Then N is a sum of two l-isosceles triangular numbers if and only if there exist positive integers u and v such that*

(i) $4lN + (l-2)^2 = u^2 + v^2$, *and*

(ii) $u + v \equiv l + 2 \pmod{2l}$ *and* $u - v \equiv l + 2 \pmod{2l}$.

Proof. Assume that N is a sum of two l-isosceles triangular numbers. Then $N = T(m,l) + T(n,l)$ for some positive integers m and n, i.e.,

$$N = m + \frac{m(m-1)l}{2} + n + \frac{n(n-1)l}{2} = \frac{2m + m(m-1)l + 2n + n(n-1)l}{2}.$$

It follows that

$$\begin{aligned}
4lN + (l-2)^2 &= 2l\,(2m + m(m-1)l + 2n + n(n-1)l) + (l-2)^2 \\
&= 4ml + 2m^2l^2 - 2ml^2 + 4nl + 2n^2l^2 - 2nl^2 + (l-2)^2 \\
&= (lm)^2 + (ln)^2 + (l-2)^2 + 2l^2mn - 2(l-2)lm \\
&\quad - 2(l-2)ln + (lm)^2 - 2mnl^2 + (ln)^2 \\
&= (l(m+n) - (l-2))^2 + (l(m-n))^2.
\end{aligned}$$

Let $u = l(m+n) - (l-2)$ and $v = l(m-n)$. We therefore have $4lN + (l-2)^2 = u^2 + v^2$, $u + v \equiv 2lm - l + 2 \equiv l + 2 \pmod{2l}$, and $u - v \equiv 2ln - l + 2 \equiv l + 2 \pmod{2l}$.

Conversely, assume that there exist positive integers u and v such that

$$4lN + (l-2)^2 = u^2 + v^2,$$

$u + v \equiv l + 2 \pmod{2l}$, and $u - v \equiv l + 2 \pmod{2l}$. Without loss of generality, assume that $u \geq v$. Let $m = \frac{u + v + (l-2)}{2l}$ and $n = \frac{u - v + (l-2)}{2l}$. Since $u + v \equiv l + 2 \pmod{2l}$ and $u - v \equiv l + 2 \pmod{2l}$, it follows that m and n are positive integers. It is not difficult to verify that $N = T(m,l) + T(n,l)$.

3. Identities

In this section, we focus on generalizations of classical identities for triangular numbers (see, for example, [1–3, 7, 9, 12]). Each identity for l-isosceles triangular numbers is given and followed by the corresponding identity for triangular numbers.

The identity concerning the $(m+n)$th l-isosceles triangular numbers is given in the next theorem.

Theorem 5. *$T(m+n,l) = T(m,l) + T(n,l) + lmn$ for all positive integers l, m, and n.*

Proof. Let l, m, and n be positive integers. Then

$$\begin{aligned} T(m+n,l) &= m+n+\frac{m+n(m+n-1)l}{2} \\ &= m+n+\frac{(m^2-m+n^2-n+2mn)l}{2} \\ &= \frac{2m+2n+m^2l-ml+n^2l-nl+2lmn}{2} \\ &= \frac{2m+m(m-1)l+2n+n(n-1)l+2lmn}{2} \\ &= m+\frac{m(m-1)l}{2}+n+\frac{n(n-1)l}{2}+lmn \\ &= T(m,l)+T(n,l)+lmn. \end{aligned}$$

As desired, we have $T(m+n,l) = T(m,l) + T(n,l) + lmn$.

By setting $l = 1$, we have the classical identity $T(m+n) = T(m) + T(n) + mn$ in [12, Equation (4a)].

In the following theorem, the identity concerning the mnth l-isosceles triangular numbers is presented.

Theorem 6. *$T(mn,l) = T(m,l)T(n,l)+(2-l)lT(m-1)T(n-1)$ for all positive integers m, n, and l.*

Proof. Let m, n, and l be positive integers. Using a direct calculation, we have

$$\begin{aligned} T(mn,l) &= mn+\frac{mn(mn-1)l}{2} \\ &= \frac{4mn+2m^2n^2l-2mnl}{4} \\ &= \frac{(2m+m^2l-ml)(2n+n^2l-nl)}{4}+\frac{(2l-l^2)(m^2-m)(n^2-n)}{4} \\ &= \left(m+\frac{m(m-1)l}{2}\right)\left(n+\frac{n(n-1)l}{2}\right)+(2-l)l\left(\frac{(m-1)m}{2}\right)\left(\frac{(n-1)n}{2}\right) \\ &= T(m,l)T(n,l)+(2-l)lT(m-1)T(n-1). \end{aligned}$$

This completes the proof.

The classical identity $T(mn) = T(m)T(n) + T(m-1)T(n-1)$ for triangular numbers in [12, Equation (17a)] follows easily when $l = 1$.

One of the classical facts about triangular numbers is that a sum of two consecutive triangular numbers is square (see [12, Equation (1)]). Precisely, $T(n)+T(n+1) = (n+1)^2$ is square for all positive integers n. In the following theorem, we prove that $l = 1$ is the necessary and sufficient condition for the sum $T(n,l) + T(n+1,l)$ to be square for all positive integers n.

Theorem 7. *Let l be a positive integer. Then $T(n,l)+T(n+1,l)$ is square for all positive integers n if and only if $l = 1$.*

Proof. Assume that $l \geq 2$. First, we note that $T(n,l) + T(n+1,l) = n^2l + 2n + 1$ for all positive integers n. It is not difficult to see that $T(1,l) + T(1+1,l) = l+3$ is square if and only if $2^2(l+3)$ is square. For $l \geq 2$, if $T(1,l) + T(1+1,l) = l+3$ is square, then $l \geq 6$ and $2^2(l+3) \geq 36$. Hence, $T(2,l) + T(2+1,l) = 4l+5 = 2^2(l+3) - 7$ cannot be square. Therefore, $T(n,l) + T(n+1,l)$ is non-square for $n = 1$ or $n = 2$.

The converse follows directly from [12, Equation (1)].

Next, we focus on some identities for isosceles triangular numbers induced by a recurrence relation.

Theorem 8. *$nT(n+1,l) - (l-1)n = (n+2)T(n,l)$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then

$$\begin{aligned}
nT(n+1,l) - (l-1)n &= n\left((n+1) + \frac{(n+1)nl}{2}\right) - nl + n \\
&= \frac{2n^2 + 2n + n^3l + n^2l - 2nl + 2n}{2} \\
&= \frac{2n^2 + n^3l - n^2l + 4n + 2n^2l - 2nl}{2} \\
&= (n+2)\left(\frac{2n + n^2l - nl}{2}\right) \\
&= (n+2)\left(n + \frac{n(n-1)l}{2}\right) \\
&= (n+2)T(n,l).
\end{aligned}$$

Hence, $nT(n+1,l) - (l-1)n = (n+2)T(n,l)$ as desired.

By setting $l = 1$, the identity $nT(n+1) = (n+2)T(n)$ for triangular numbers in [7, Equation (1.7)] follows.

Theorem 9. *$T(2n+1,l) - T(2n,l) = T(n+1,l) - T(n-1,l) + (l-1)$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then

$$\begin{aligned} T(2n+1,l) - T(2n,l) &= 2n+1+\frac{(2n+1)(2n)l}{2} - 2n - \frac{(2n)(2n-1)l}{2} \\ &= \frac{4n+2+4n^2l+2nl-4n-4n^2l+2nl}{2} \\ &= \frac{2n+2+n^2l+nl-2n+2-n^2l+3nl-2l+2l-2}{2} \\ &= n+1+\frac{(n+1)(n)l}{2} - (n-1) - \frac{(n-1)(n-2)l}{2} + l - 1 \\ &= T(n+1,l) - T(n-1,l) + (l-1). \end{aligned}$$

This completes the proof.

From the theorem above, the relation $T(2n+1) - T(2n) = T(n+1) - T(n-1)$ in [9, Equation (7.14)] can be obtained directly when $l = 1$.

Theorem 10. *$T(2n,l) = 3T(n,l) + lT(n-1) + (l-1)n$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then

$$\begin{aligned} 3T(n,l) + lT(n-1) + (l-1)n &= 3\left(n + \frac{n(n-1)l}{2}\right) + l\left(\frac{(n-1)n}{2}\right) + (l-1)n \\ &= \frac{6n+3n^2l-3nl+n^2l-nl+2nl-2n}{2} \\ &= \frac{4n+(4n^2-2n)l}{2} \\ &= 2n + \frac{2n(2n-1)l}{2} \\ &= T(2n,l). \end{aligned}$$

As desired, we have $T(2n,l) = 3T(n,l) + lT(n-1) + (l-1)n$.

Lemma 1. *$lT(n-1) + (l-1)n = T(n-1) + (l-1)T(n)$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then we have

$$\begin{aligned} T(n-1) + (l-1)T(n) &= \frac{(n-1)n}{2} + (l-1)\left(\frac{n(n+1)}{2}\right) \\ &= \frac{n^2-n+n^2l+nl-n^2-n}{2} \\ &= \frac{n^2l-nl+2nl-2n}{2} \\ &= l\left(\frac{(n-1)n}{2}\right) + (l-1)n \end{aligned}$$

$$= lT(n-1) + (l-1)n$$

as desired.

From Theorem 10 and Lemma 1, we have the following identity.

Corollary 3. *$T(2n,l) = 3T(n,l) + T(n-1) + (l-1)T(n)$ for all positive integers n and l.*

By setting $l = 1$, we have $T(2n) = 3T(n) + T(n-1)$ for all positive integers n as in [7, Equation (1.12)].

Theorem 11. *$T(2n+1,l) = 3T(n,l) + T(n+1,l) + 2(l-1)n$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then

$$\begin{aligned}
T(2n+1,l) &= 2n+1+\frac{(2n+1)(2n)l}{2} \\
&= \frac{4n+2+4n^2l+2nl}{2} \\
&= \frac{6n+3n^2l-3nl+2n+2+n^2l+nl+4nl-4n}{2} \\
&= 3\left(n+\frac{n(n-1)l}{2}\right) + \left(n+1+\frac{(n+1)(n)l}{2}\right) + 2ln - 2n \\
&= 3T(n,l) + T(n+1,l) + 2(l-1)n.
\end{aligned}$$

Hence, the proof is completed.

Theorem 12. *$lT(2n+1,l) = 3lT(n,l) + lT(n+1,l) + 4nT(l-1)$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then we have

$$\begin{aligned}
lT(2n+1,l) &= l\left(2n+1+\frac{(2n+1)(2n)l}{2}\right) \\
&= \frac{4nl+2l+4n^2l^2+2nl^2}{2} \\
&= \frac{6nl+3n^2l^2-3nl^2+2nl+2l+n^2l^2+nl^2+4nl^2-4nl}{2} \\
&= 3\left(n+\frac{n(n-1)l}{2}\right) + l\left(n+1+\frac{(n+1)(n)l}{2}\right) + 4n\left(\frac{(l-1)l}{2}\right) \\
&= 3lT(n,l) + lT(n+1,l) + 4nT(l-1)
\end{aligned}$$

as required.

From the two theorems above, the well-known relation $T(2n+1) = 3T(n) + T(n+1)$ derived from [9, Equations (7.14) and (7.22)] can be obtained by setting $l = 1$.

Theorem 13. *$n^2T(k-1,l)+kT(n,l) = T(nk,l)-n^2(k-1)(l-1)$ for all positive integers n, k, and l.*

Proof. Let n, k, and l be positive integers. Then

$$\begin{aligned} n^2T(k-1,l)+kT(n,l) &= n^2\left(k-1+\frac{(k-1)(k-2)l}{2}\right)+k\left(n+\frac{n(n-1)l}{2}\right) \\ &= \frac{2n^2k-2n^2+n^2k^2l-3n^2kl+2n^2l+2nk+n^2kl-nkl}{2} \\ &= \frac{2nk+n^2k^2l-nkl-2n^2kl+2n^2k+2n^2l-2n^2}{2} \\ &= \frac{2nk+n^2k^2l-nkl}{2}-n^2kl+n^2k+n^2l-n^2 \\ &= nk+\frac{nk(nk-1)l}{2}-n^2(k-1)(l-1) \\ &= T(nk,l)-n^2(k-1)(l-1). \end{aligned}$$

Therefore, we have $n^2T(k-1,l)+kT(n,l) = T(nk,l)-n^2(k-1)(l-1)$ as desired.

By setting $l = 1$, the identity $n^2T(k-1)+kT(n) = T(nk)$ for triangular numbers follows.

Theorem 14. *$n^2T(k-1,l)+kT(n-1,l) = T(nk-1,l)-(n^2-1)(k-1)(l-1)$ for all positive integers n, k, and l.*

Proof. Let n, k, and l be positive integers. Then

$$\begin{aligned} &n^2T(k-1,l)+kT(n-1,l) \\ &\quad = n^2\left(k-1+\frac{(k-1)(k-2)l}{2}\right)+k\left(n-1+\frac{(n-1)(n-2)l}{2}\right) \\ &\quad = \frac{2n^2k-2n^2+n^2k^2l-3n^2kl+2n^2l+2nk-2k+n^2kl-3nkl+2kl}{2} \\ &\quad = \frac{2nk-2+n^2k^2l-3nkl+2l-2n^2kl+2n^2l+2kl-2l+2n^k-2n^2-2k+2}{2} \\ &\quad = \frac{2nk-2+n^2k^2l-3nkl+2l}{2}-(n^2kl-n^2l-kl+l-n^2k+n^2+k-1) \\ &\quad = (nk-1)+\frac{(nk-1)(nk-2)l}{2}-(n^2-1)(k-1)(l-1) \\ &\quad = T(nk-1,l)-(n^2-1)(k-1)(l-1) \end{aligned}$$

as desired.

For $l = 1$, we have the identity $n^2T(k-1)+kT(n-1) = T(nk-1)$ for all positive integers k and n.

In the following theorems, some identities concerning squares of l-isosceles triangular numbers are presented.

Theorem 15. *$T(n+1,l)^2 - T(n,l)^2 = (n+1)^3 + (l-1)n\big((l+1)n^2+3n+1\big)$ for all positive integers l and n.*

Proof. Let l and n be positive integers. Then

$$\begin{aligned}
T(n+1,l)^2 - T(n,l)^2 &= \left((n+1)+\frac{(n+1)nl}{2}\right)^2 - \left(n+\frac{n(n-1)l}{2}\right)^2\\
&= \frac{4n^2+8n+4n^3l+8n^2l+4n+4+4nl+n^4l^2+2n^3l^2+n^2l^2}{4}\\
&\quad - \frac{4n^2+4n^3l-4n^2l+n^4l^2-2n^3l^2+n^2l^2}{4}\\
&= \frac{8n+12n^2l+4+4nl+4n^3l^2}{4}\\
&= 2n+3n^2l+1+nl+n^3l^2\\
&= (n^3+3n^2+3n+1)+(nl-n)(n^2l+n^2+3n+1)\\
&= (n+1)^3+(l-1)n((l+1)n^2+3n+1).
\end{aligned}$$

Hence, $T(n+1,l)^2 - T(n,l)^2 = (n+1)^3 + (l-1)n\big((l+1)n^2+3n+1\big)$ as desired.

The identity $T(n+1)^2 - T(n)^2 = (n+1)^3$ for triangular numbers in [7, Equation (1.5)] is a special case of the above theorem where $l = 1$.

Theorem 16. *$T(n,l)^2 + T(n-1,l)^2 = lT(n^2,l) - (2n^2-2n+1)(2T(l,n-1)-(l+1))$ for all positive integers n and l.*

Proof. Let n and l be positive integers. Then

$$\begin{aligned}
&T(n,l)^2 + T(n-1,l)^2\\
&\quad= \left(n+\frac{n(n-1)l}{2}\right)^2 + \left(n-1+\frac{(n-1)(n-2)l}{2}\right)^2\\
&\quad= \left(\frac{2n+n^2l-nl}{2}\right)^2 + \left(\frac{2n-2+n^2l-3nl+2l}{2}\right)^2\\
&\quad= \frac{8n^2+8n^3l-20n^2l+2n^4l^2-8n^3l^2+14n^2l^2+20nl-8n+4-8l-12nl^2+4l^2}{4}\\
&\quad= \frac{4n^2+4n^3l-10n^2l+n^4l^2-4n^3l^2+7n^2l^2+10nl-4n+2-4l-6nl^2+2l^2}{2}\\
&\quad= \left(\frac{2n^2l+n^4l^2-n^2l^2}{2}\right) - (2n^2-2n+1)\left(\frac{4l+2nl^2-2l^2-2nl+2l-2l-2}{2}\right)\\
&\quad= l\left(n^2+\frac{n^2(n^2-1)l}{2}\right) - (2n^2-2n+1)\left(2(l+\frac{l(l-1)(n-1)}{2})-l-1\right)\\
&\quad= lT(n^2,l) - (2n^2-2n+1)(2T(l,n-1)-(l+1)).
\end{aligned}$$

The proof is therefore completed.

The identity $T(n)^2 + T(n-1)^2 = T(n^2)$ in [4, Equation (19a)] is a special case of the theorem where $l = 1$.

4. Conclusion and Remarks

Generalizations of triangular numbers have been studied in terms of isosceles triangular numbers. Some characterizations of such numbers are given as well as links with square numbers. Some identities concerning isosceles triangular numbers have been established as generalizations of classical identities for triangular numbers.

It would be interesting to give generalizations of other identities for triangular numbers in terms of isosceles triangular numbers. The study of relationships among isosceles triangular numbers and various types of figurate is an interesting problem as well.

Acknowledgements

The authors would like to thank the referees and the editor for their useful and insightful comments. S. Jitman was supported by the Thailand Research Fund under Research Grant RSA6280042.

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