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Hankel Transform of the First Form (q, r)-Dowling Numbers

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Abstract. In this paper, the Hankel transform of the generalized q-exponential polynomial of the first form (q, r)-Whitney numbers of the second kind is established using the method of Cigler. Consequently, the Hankel transform of the first form (q, r)-Dowling numbers is obtained as special case.

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1. Introduction

The r-Dowling numbers $D_{m,r}(n)$ are defined in [6] as the sum of r-Whitney numbers of the second $W_{m,r}(n, k)$ [9, 11]. More precisely,

$$D_{m,r}(n) := \sum_{k=0}^{n} W_{m,r}(n, k),$$

where n is a nonnegative integer and the parameters m and r may be real or complex numbers. These numbers are certain generalization of ordinary Bell numbers B_n [3], r-Bell numbers $B_r(n)$ [10], and noncentral Bell numbers $B_{n,a}$ [7]. That is, when $m = 1$, the r-Dowling numbers reduce to r-Bell numbers and noncentral Bell numbers. Furthermore, when $m = 1, r = 0$, these yield the ordinary Bell numbers.

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J. Layman [8] defined the Hankel transform of an integers sequence (a_n) as a sequence of the following determinants d_n of Hankel matrix of order n

$$d_n = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_{n+1} \\ a_2 & a_3 & a_4 & \dots & a_{n+2} \\ & & \dots\dots\dots\dots & & \\ a_n & a_{n+1} & a_{n+2} & \dots & a_{2n} \end{vmatrix}. \tag{1}$$

Aigner [1] derived the Hankel transform of the ordinary Bell numbers to be

$$det(B_{i+j})_{0\leq i,j\leq n} = \prod_{k=0}^{n} k! \tag{2}$$

which is exactly the Hankel transform obtained by Mezo [10] for r-Bell numbers using Layman's Theorem [8] on the invariance of Hankel transform.

Using the method of Aiger [1] and Layman's Theorem [8], the sequence of (r, β)-Bell numbers in [4, 12], denoted by $\{G_{n,r,\beta}\}$, has been shown to possess the following Hankel transform (see [13])

$$H(G_{n,r,\beta}) = \prod_{j=0}^{n} \beta^j j!.$$

It is worth mentioning that the (r, β)-Bell numbers are equivalent to the r-Dowling numbers $D_{m,r}(n)$, which are defined in [6] as

$$D_{m,r}(n) = \sum_{k=0}^{n} W_{m,r}(n, k)$$

where $W_{m,r}(n, k)$ denotes the r-Whitney numbers of the second kind introduced by Mezo in [9]. In [13], the authors have also tried to derive the Hankel transform of the sequence of q-analogue of (r, β)-Bell numbers. In this attempt, they used the q-analogue defined in [14]. But they failed to derive it.

Just recently, another definition of q-analogue of r-Whitney numbers of the second $W_{m,r}[n, k]_q$ was introduced in [16, 17] by means of the following triangular recurrence relation

$$W_{m,r}[n, k]_q = q^{m(k-1)+r} W_{m,r}[n-1, k-1]_q + [mk + r]_q W_{m,r}[n-1, k]_q, \tag{3}$$

where n and k are nonnegative integers, the parameters m and r may be real of complex numbers and

$$W_{m,r}[n, k]_q = \begin{cases} 1, & n = k \;\; and \;\; n \geq 0 \\ 0, & n < k \;\; or \;\; n, k < 0. \end{cases}$$

From this definition, two more forms of the q-analogue were defined in [16, 17] as

$$W^*_{m,r}[n, k]_q := q^{-kr-m\binom{k}{2}} W_{m,r}[n, k]_q \tag{4}$$

$$\widetilde{W}_{m,r}[n,k]_q := q^{kr}W^*_{m,r}[n,k]_q = q^{-m\binom{k}{2}}W_{m,r}[n,k]_q, \tag{5}$$

where $W^*_{m,r}[n,k]_q$ and $\widetilde{W}_{m,r}[n,k]_q$ denote the second and third forms of the q-analogue, respectively. Corresponding to these, three forms of q-analogues for r-Dowling numbers (or (q,r)-Dowling numbers) were defined as follows:

$$D_{m,r}[n]_q := \sum_{k=0}^{n} W_{m,r}[n,k]_q \tag{6}$$

$$D^*_{m,r}[n]_q := \sum_{k=0}^{n} W^*_{m,r}[n,k]_q \tag{7}$$

$$\widetilde{D}_{m,r}[n]_q := \sum_{k=0}^{n} \widetilde{W}_{m,r}[n,k]_q. \tag{8}$$

However, among the three forms of (q,r)-Dowling numbers, only the first form has not been given a Hankel transform. The third form was thoroughly studied in [17] and its Hankel transform was successfully derived, which is given by

$$H(\widetilde{D}_{m,r}[n]_q) = q^{m\binom{n+1}{3}-rn(n+1)}[0]_{q^m}![1]_{q^m}!\dots[n]_{q^m}![m]_q^{\binom{n+1}{2}}, \tag{9}$$

using the Hankel transform of q-exponential polynomials in [5], the Layman's Theorem in [8] and the Spivey-Steil Theorem in [19]. This method cannot be used to derive the Hankel transform of the first and second forms of q-analogues for r-Dowling numbers. But the method used by Cigler in [2] can be used to derive the Hankel transform for the second form of the (q,r)-Dowling numbers. The said Hankel transform was derived in [15], which is given by

$$H(D^*_{m,r}[n]_q) = [m]_q^{\binom{n}{2}}q^{\binom{n}{3}+r\binom{n}{2}}\prod_{k=0}^{n-1}[k]_{q^m}!$$

Corcino et al. [18] have made a preliminary investigation for the first form (q,r)-Dowling numbers $D_{m,r}[n]_q$ by establishing an explicit formula expressed in terms of the first form (q,r)-Whitney numbers of the second kind and (q,r)-Whitney-Lah numbers. In this present paper, the Hankel transform for the sequence $(D_{m,r}[n]_q)_{n=0}^{\infty}$ will be established using Cigler's method [2]. However, a more general form of $D_{m,r}[n]_q$, denoted by $\Phi_n[x,r,m]_q$, is considered, which is defined in polynomial form as follows:

$$\Phi_n\left[x,r,m\right]_q = \sum_{k=0}^{n} W_{m,r}[n,k]_q x^k \tag{10}$$

such that, when $x=1$, $\Phi_n[1,r,m]_q = D_{m,r}[n]_q$.

2. Generalized q-Exponential Polynomials

We may call $\Phi_n\left[x,r,m\right]_q$ to be the generalized q-exponential polynomial of q-analogue of r-Whitney numbers of the second kind. Note that we can rewrite (10) as

$$\Phi_{n-1}\left[x,r,m\right]_q = \sum_{k=0}^{n-1} W_{m,r}[n-1,k]_q x^k$$
$$\Phi_{n-1}\left[qx,r,m\right]_q = \sum_{k=0}^{n-1} W_{m,r}[n-1,k]_q q^{rk+m\binom{k}{2}+k} x^k. \tag{11}$$

The following theorem contains a recursive relation for $\Phi_n\left[x,r,m\right]_q$.

Theorem 2.1. *The generalized q-exponential polynomials $\Phi_n\left[x,r,m\right]_q$ of q-analogue of r-Whitney numbers of the second kind satisfy the following relation*

$$\Phi_n[x,r,m]_q = \left[q^r x + (q^m-1)q^r x^2 D_{q^m} + [r]_q + q^r[m]_q x D_{q^m}\right]\Phi_{n-1}[x,r,m]_q. \tag{12}$$

Proof. Using (3), equation (10) can be written as

$$\begin{aligned}
\Phi_n\left[x,r,m\right]_q &= \sum_{k=0}^{n} W_{m,r}[n,k]_q x^k \\
&= \sum_{k=0}^{n} q^{mk-m+r} W_{m,r}[n-1,k-1]_q x^k + \sum_{k=0}^{n}[mk+r]_q W_{m,r}[n-1,k]_q x^k \\
&= \sum_{k=0}^{n-1} q^{m(k+1)-m+r} W_{m,r}[n-1,k]_q x^{k+1} + \left([r]_q + q^r[m]_q x D_{q^m}\right)\Phi_{n-1}[x,r,m]_q \\
&= x\sum_{k=0}^{n-1} q^{mk+r} W_{m,r}[n-1,k]_q x^k + \left([r]_q + q^r[m]_q x D_{q^m}\right)\Phi_{n-1}[x,r,m]_q \\
&= xq^r\sum_{k=0}^{n-1} q^{mk} W_{m,r}[n-1,k]_q x^k + \left([r]_q + q^r[m]_q x D_{q^m}\right)\Phi_{n-1}[x,r,m]_q,
\end{aligned}$$

where D_q denotes the q-derivative operator defined by

$$D_q f(x) = \frac{f(x)-f(qx)}{(1-q)x}. \tag{13}$$

Hence, using (11), we have

$$\Phi_n[x,r,m]_q = xq^r\Phi_{n-1}[q^m x,r,m]_q + \left([r]_q + q^r[m]_q x D_{q^m}\right)\Phi_{n-1}[x,r,m]_q. \tag{14}$$

Note that (13) can be expressed as

$$f(qx) = (q-1)xD_q f(x) + f(x)$$

$$f(q^m x) = (q^m - 1)xD_{q^m} f(x) + f(x)$$
$$f(q^m x) = ((q^m - 1)xD_{q^m} + 1)f(x).$$

This implies that

$$\Phi_{n-1}[q^m x, r, m]_q = (1 + (q^m - 1)xD_{q^m})\, \Phi_{n-1}[x, r, m]_q.$$

Thus, equation (14) can further be written as

$$\begin{aligned}\Phi_n[x, r, m]_q &= xq^r\,(1 + (q^m - 1)xD_{q^m})\, \Phi_{n-1}[x, r, m]_q \\ &\quad + ([r]_q + q^r[m]_q xD_{q^m})\, \Phi_{n-1}[x, r, m]_q \\ &= \big[q^r x + (q^m - 1)q^r x^2 D_{q^m} \\ &\quad + [r]_q + q^r[m]_q xD_{q^m}\big]\, \Phi_{n-1}[x, r, m]_q,\end{aligned}$$

which is exactly the desired relation.

Remark 2.2. Let $\hat{D}_{qx} = \big[q^r x + (q^m - 1)q^r x^2 D_{q^m} + [r]_q + q^r[m]_q xD_{q^m}\big]$. Then, (12) can be written as

$$\Phi_n[x, r, m]_q = \hat{D}_{qx}\Phi_{n-1}[x, r, m]_q. \tag{15}$$

By repeated application of (15),

$$\begin{aligned}\Phi_n[x, r, m]_q &= \hat{D}_{qx}\Phi_{n-1}[x, r, m]_q \\ &= \hat{D}_{qx}\left(\hat{D}_{qx}\Phi_{n-2}[x, r, m]_q\right) \\ &= \hat{D}^2_{qx}\Phi_{n-2}[x, r, m]_q \\ &\vdots \\ &= \hat{D}^n_{qx}\Phi_0[x, r, m]_q \\ &= \hat{D}^n_{qx}.\end{aligned}$$

3. Hankel Transform of $D_{m,r}[n]_q$

Let $\langle\langle x\rangle\rangle_{r,m,k} = \prod_{j=0}^{k-1} \frac{(x-[r+jm]_q)}{q^{r+jm}} = q^{-rk-m\binom{k}{2}}\langle x\rangle_{r,m,k}$.
The horizontal generating function of $W^*_{m,r}[n, k]_q$ is given by:

$$\sum_{k=0}^{n} W_{m,r}[n, k]_q\langle x\rangle_{r,m,k} = x^n.$$

where $x = [t]_k$. Define a linear functional $G_{r,q}$ by

$$G_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,n}\right) = a^n$$

and a linear operator $V_{r,q}$ by

$$V_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,n}\right) = x^n.$$

Then

$$\begin{aligned}
V_{r,q}\left(x^n\right) &= \sum_{k=0}^{n} W^*_{m,r}[n,k]_q V_{r,q}\left(\langle x\rangle_{r,m,k}\right)\\
&= \sum_{k=0}^{n} W^*_{m,r}[n,k]_q q^{rk+m\binom{k}{2}} V_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,k}\right)\\
&= \sum_{k=0}^{n} W^*_{m,r}[n,k]_q q^{rk+m\binom{k}{2}} x^k\\
&= \Phi_n[x,r,m]_q
\end{aligned}$$

Consider the polynomial

$$g_{n,q}(x,a,r,m) = \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q \langle\langle x\rangle\rangle_{r,m,n-k}.$$

Then

$$\begin{aligned}
V_{r,q}\left(g_{n,q}(x,a,r,m)\right) &= \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q V_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,n-k}\right)\\
&= \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix}_q x^{n-k}\\
&= p_{n,q}(x,a).
\end{aligned}$$

This implies that $V^{-1}_{r,q} p_{n,q}(x,a) = g_{n,q}(x,a,r,m)$. Now,

$$V_{r,q} x g_{n,q}(x,a,r,m) = V_{r,q} x V^{-1}_{r,q} p_{n,q}(x,a).$$

Applying the operator to $p_{n,q}(x,a)$, we get

$$\begin{aligned}
V_{r,q} x g_{n,q}(x,a,r,m) &= V_{r,q} x V^{-1}_{r,q} p_{n,q}(x,a)\\
&= q^r x p_{n,q}(x,a) + (q^m - 1) q^r x^2 D_{q^m} p_{n,q}(x,a) + [r]_q p_{n,q}(x,a)\\
&\quad + q^r [m]_q x D_{q^m} p_{n,q}(x,a).
\end{aligned}$$

Note that

$$\begin{aligned}
x p_{n,q}(x,a) &= \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n\\ k\end{bmatrix} x^{n+1-k}\\
&= \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \left(\begin{bmatrix} n+1\\ k\end{bmatrix} - q^{n+1-k}\begin{bmatrix} n\\ k-1\end{bmatrix}\right) x^{n+1-k}
\end{aligned}$$

$$= \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^{n+1-k} - \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} q^{n+1-k} \begin{bmatrix} n \\ k-1 \end{bmatrix} x^{n+1-k}$$
$$+ (-a)^{n+1} q^{\binom{n+1}{2}} - (-a)^{n+1} q^{\binom{n+1}{2}}.$$

$$\begin{aligned} xp_{n,q}(x,a) &= \sum_{k=0}^{n+1}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix} x^{n+1-k} - \sum_{k=-1}^{n-1}(-a)^{k+1} q^{\binom{k+1}{2}} q^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \\ &\quad - (-a)^{n+1} q^{\binom{n+1}{2}} \\ &= p_{n+1,q}(x,a) - \sum_{k=0}^{n-1}(-a)^{k+1} q^{\binom{k+1}{2}+n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \\ &\quad - (-a)^{n+1} q^{\binom{n+1}{2}} \\ &= p_{n+1,q}(x,a) - \sum_{k=0}^{n}(-a)^{k+1} q^{\binom{k+1}{2}+n-k} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \\ &= p_{n+1,q}(x,a) + aq^n \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} \\ &= p_{n+1,q}(x,a) + aq^n p_{n,q}(x,a). \end{aligned}$$

So, $q^r xp_{n,q}(x,a) = q^r p_{n+1,q}(x,a) + aq^{n+r} p_{n,q}(x,a)$. With $D_q p_{n,q}(x,a) = [n]_q p_{n-1,q}(x,a)$, we have

$$D_{q^m} p_{n,q}(x,a) = [n]_{q^m} p_{n-1,q}(x,a).$$

Hence,

$$(q^m-1)q^r x^2 D_{q^m} p_{n,q}(x,a) = (q^m-1)q^r x^2 [n]_{q^m} p_{n-1,q}(x,a) = (q^{mn}-1)q^r x^2 p_{n-1,q}(x,a)$$

and

$$q^r [m]_q x D_{q^m} p_{n,q}(x,a) = q^r [m]_q x [n]_{q^m} p_{n-1,q}(x,a)$$

Thus,

$$\begin{aligned} &V_{r,q} x g_{n,q}(x,a,r,m) = V_{r,q} x V_{r,q}^{-1} p_{n,q}(x,a) \\ &\quad = q^r p_{n+1,q}(x,a) + aq^{n+r} p_{n,q}(x,a) + [r]_q p_{n,q}(x,a) \\ &\qquad + (q^{mn}-1)q^r x^2 p_{n-1,q}(x,a) + q^r [m]_q x [n]_{q^m} p_{n-1,q}(x,a) \\ &\quad = q^r (p_{n+1,q}(x,a) + q^n a p_{n,q}(x,a)) + q^r p_{n+1,q}(x,a) + aq^{n+r} p_{n,q}(x,a) + [r]_q p_{n,q}(x,a) \\ &\qquad + (q^{mn}-1)q^r [p_{n+1,q}(x,a) + (q^n a + q^{n-1} a) p_{n,q}(x,a) + q^{2n-2} a^2 p_{n-1,a}(x,a)] \\ &\qquad + q^r [m]_q [n]_{q^m} (p_{n,q}(x,a) + q^{n-1} a p_{n-1,q}(x,a) \\ &\quad = q^r (2 + (q^{mn}-1)) p_{n+1,q}(x,a) \\ &\qquad + (q^{n+r} a + aq^{n+r} + [r]_q + (q^{mn}-1) q^r (q^n a + q^{n-1} a) + q^r [m]_q [n]_{q^m}) p_{n,q}(x,a) \end{aligned}$$

$$+((q^{mn}-1)q^rq^{2n-2}a^2+q^r[m]_q[n]_{q^m}q^{n-1}a)p_{n-1,q}(x,a)$$

Applying the operator $V_{r,q}^{-1}: p_{n,q}(x,a)\mapsto g_{n,q}(x,a,r,m)$, then

$$\begin{aligned}xg_{n,q}(x,a,r,m)&=q^r(2+(q^{mn}-1))g_{n+1,q}(x,a,r,m)\\&+(2q^{n+r}a+[r]_q+(q^{mn}-1)q^r(q^na+q^{n-1}a)+q^r[m]_q[n]_{q^m})g_{n,q}(x,a,r,m)\\&+((q^{mn}-1)q^rq^{2n-2}a^2+q^r[m]_q[n]_{q^m}q^{n-1}a)g_{n-1,q}(x,a,r,m)\end{aligned}$$

We set

$$h_{n,q}(x,a,r,m)=q^{m\binom{n}{2}+rn}g_{n,q}(x,a,r,m).$$

That is,

$$g_{n,q}(x,a,r,m)=q^{-m\binom{n}{2}-rn}h_{n,q}(x,a,r,m).$$

Then,

$$\begin{aligned}xq^{-m\binom{n}{2}-rn}h_{n,q}(x,a,r,m)&=q^r(2+(q^{mn}-1))q^{-m\binom{n+1}{2}-r(n+1)}h_{n+1,q}(x,a,r,m)\\&+(2q^{n+r}a+[r]_q+(q^{mn}-1)q^r(q^na+q^{n-1}a)+q^r[m]_q[n]_{q^m})q^{-m\binom{n}{2}-rn}h_{n,q}(x,a,r,m)\\&+((q^{mn}-1)q^rq^{2n-2}a^2+q^r[m]_q[n]_{q^m}q^{n-1}a)q^{-m\binom{n-1}{2}-r(n-1)}h_{n-1,q}(x,a,r,m)\end{aligned}$$

$$\begin{aligned}xh_{n,q}(x,a,r,m)&=q^r(2+(q^{mn}-1))q^{-mn-r}h_{n+1,q}(x,a,r,m)\\&+(2q^{n+r}a+[r]_q+(q^{mn}-1)q^r(q^na+q^{n-1}a)+q^r[m]_q[n]_{q^m})h_{n,q}(x,a,r,m)\\&+((q^{mn}-1)q^rq^{2n-2}a^2+q^r[m]_q[n]_{q^m}q^{n-1}a)q^{m(n-1)+r}h_{n-1,q}(x,a,r,m)\end{aligned}$$

$$\begin{aligned}xh_{n,q}(x,a,r,m)&=(2+(q^{mn}-1))q^{-mn}h_{n+1,q}(x,a,r,m)\\&+(2q^{n+r}a+[r]_q+(q^{mn}-1)q^r(q^na+q^{n-1}a)+q^r[m]_q[n]_{q^m})h_{n,q}(x,a,r,m)\\&+((q^{mn}-1)q^rq^{2n-2}a^2+q^r[m]_q[n]_{q^m}q^{n-1}a)q^{m(n-1)+r}h_{n-1,q}(x,a,r,m)\end{aligned}\tag{16}$$

It is clear that

$$\begin{aligned}G_{r,q}(h_{n,q}(x,a,r,m))&=G_{r,q}\left(q^{m\binom{n}{2}+rn}g_{n,q}(x,a,r,m)\right)\\&=q^{m\binom{n}{2}+rn}\sum_{k=0}^{n}(-a)^kq^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_qG_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,n-k}\right)\\&=q^{m\binom{n}{2}+rn}\sum_{k=0}^{n}(-a)^kq^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_qa^{n-k}\\&=q^{m\binom{n}{2}+rn}p_{n,q}(a,a)\\&=0,\end{aligned}$$

$$G_{r,q}\left(\langle\langle x\rangle\rangle_{r,m,0}\right)=a^0=1$$

which implies

$$G_{r,q}(1) = 1.$$

and

$$\begin{aligned} g_{0,q}(x,a,r,m) &= \sum_{k=0}^{0} (-a)^k q^{\binom{k}{2}} \begin{bmatrix} 0 \\ k \end{bmatrix}_q \langle\langle x \rangle\rangle_{r,m,0-k} \\ &= (-a)^0 q^{\binom{0}{2}} \begin{bmatrix} 0 \\ 0 \end{bmatrix}_q \langle\langle x \rangle\rangle_{r,m,0} \\ &= 1. \end{aligned}$$

It follows that

$$h_{0,q}(x,a,r,m) = q^{m\binom{0}{2}+0} g_{0,q}(x,a,r,m) = 1$$

and

$$G_{r,q}\left(h_{0,q}(x,a,r,m)\right) = G_{r,q}(1) = 1.$$

Clearly, $G_{r,q}\left([x]_q h_{n,q}(x,a,r,m)\right) = 0$ and from (16),

$$xh_{n,q}(x,a,r,m) = g(n)h_{n+1,q}(x,a,r,m) + f(n)h_{n,q}(x,a,r,m) + c(n)h_{n-1,q}(x,a,r,m)$$

where

$$\begin{aligned} g(n) &= (2 + (q^{mn} - 1))q^{-mn} \\ f(n) &= (2q^{n+r}a + [r]_q + (q^{mn} - 1)q^r(q^n a + q^{n-1}a) + q^r[m]_q[n]_{q^m}) \\ c(n) &= ((q^{mn} - 1)q^r q^{2n-2}a^2 + q^r[m]_q[n]_{q^m} q^{n-1}a)q^{m(n-1)+r}. \end{aligned}$$

Then,

$$\begin{aligned} x^2 h_{n,q}(x,a,r,m) &= xxh_{n,q}(x,a,r,m) \\ &= x\left[g(n)h_{n+1,q}(x,a,r,m) + f(n)h_{n,q}(x,a,r,m) + c(n)h_{n-1,q}(x,a,r,m)\right] \\ &= g(n)xh_{n+1,q}(x,a,r,m) + f(n)xh_{n,q}(x,a,r,m) \\ &\quad + c(n)xh_{n-1,q}(x,a,r,m) \\ &= g(n)g(n+1)h_{n+2,q}(x,a,r,m) + g(n)f(n+1)h_{n+1,q}(x,a,r,m) \\ &\quad + g(n)c(n+1)h_{n,q}(x,a,r,m) + g(n)f(n)h_{n+1,q}(x,a,r,m) \\ &\quad + f^2(n)h_{n,q}(x,a,r,m) + f(n)c(n)h_{n-1,q}(x,a,r,m) \\ &\quad + c(n)g(n-1)h_{n,q}(x,a,r,m) + c(n)f(n-1)h_{n-1,q}(x,a,r,m) \\ &\quad + c(n)c(n-1)h_{n-2,q}(x,a,r,m) \\ &= g(n)g(n+1)h_{n+2,q}(x,a,r,m) \\ &\quad + \left[g(n)f(n+1) + g(n)f(n)\right] h_{n+1,q}(x,a,r,m) \\ &\quad + \left[g(n)c(n+1) + f^2(n) + c(n)g(n-1)\right] h_{n,q}(x,a,r,m) \\ &\quad + \left[f(n)c(n) + c(n)f(n-1)\right] h_{n-1,q}(x,a,r,m) \end{aligned}$$

$$+ c(n)c(n-1)h_{n-2,q}(x,a,r,m).$$

Applying the linear functional $G_{r,q}$ to $[x]_q^2 h_{n,q}(x,a,r,m)$ gives,

$$G_{r,q}\left(x^2 h_{n,q}(x,a,r,m)\right) = 0$$
$$\vdots$$
$$G_{r,q}\left(x^k h_{n,q}(x,a,r,m)\right) = 0$$

for $k < n$. For $k = n$,

$$\begin{aligned}x^n h_{n,q}(x,a,r,m) =& g(n)x^{n-1}h_{n+1,q}(x,a,r,m) + f(n)x^{n-1}h_{n,q}(x,a,r,m)\\ &+ c(n)x^{n-1}h_{n-1,q}(x,a,r,m).\end{aligned}$$

Then,

$$\begin{aligned}G_{r,q}\left(x^n h_{n,q}(x,a,r,m)\right) =& g(n)G_{r,q}\left(x^{n-1}h_{n+1,q}(x,a,r,m)\right) + f(n)G_{r,q}\left(x^{n-1}h_{n,q}(x,a,r,m)\right)\\ &+ c(n)G_{r,q}\left(x^{n-1}h_{n-1,q}(x,a,r,m)\right)\\ =& c(n)G_{r,q}\left(x^{n-1}h_{n-1,q}(x,a,r,m)\right)\\ =& c(n)c(n-1)G_{r,q}\left(x^{n-2}h_{n-2,q}(x,a,r,m)\right)\\ =& c(n)c(n-1)c(n-2)G_{r,q}\left(x^{n-3}h_{n-3,q}(x,a,r,m)\right)\\ &\vdots\\ =& c(n)c(n-1)c(n-2)\dots c(1)G_{r,q}\left(x^0 h_{0,q}(x,a,r,m)\right)\\ =& \left[\prod_{i=1}^{n} c(i)\right](1)\\ =& \prod_{i=1}^{n} c(i)\end{aligned}$$

Since $x^n h_{n,q}(x,a,r,m)$ is a sequence of orthogonal polynomials with respect to linear functional $G_{r,q}$,

$$d_{n,q} = G_{r,q}\left(x^n h_{n,q}(x,a,r,m)\right) = \prod_{i=1}^{n} c(i)$$

where

$$c(i) = ((q^{mi}-1)q^r q^{2i-2}a^2 + q^r[m]_q[i]_{q^m}q^{i-1}a)q^{m(i-1)+r}$$

Then

$$\begin{aligned}d_{n,q}(n,0) &= G_{r,q}\left[[x]_q^n h_{n,q}(x,a,r,m)\right]\\ &= \prod_{i=0}^{n-1} d_{i,q}\end{aligned}$$

$$= \prod_{i=0}^{n-1} \left\{ \prod_{j=1}^{i} \left[((q^{mj}-1)q^r q^{2j-2}a^2 + q^r [m]_q [j]_{q^m} q^{j-1} a) q^{m(j-1)+r} \right] \right\}$$

$$= \prod_{i=0}^{n-1} \left\{ \prod_{j=1}^{i} \left[a q^{m(j-1)+2r} ((q^{mj}-1)q^{2j-2}a + [m]_q [j]_{q^m} q^{j-1}) \right] \right\}$$

$$= \prod_{i=0}^{n-1} q^{2ri} q^{m(1+2+3+\cdots+(i-1))} a^i \prod_{j=1}^{i} \left[((q^{mj}-1)q^{2j-2}a + [m]_q [j]_{q^m} q^{j-1}) \right]$$

$$= \prod_{i=0}^{n-1} q^{2ri} q^{m\binom{i}{2}} a^i \prod_{j=1}^{i} \left[((q^{mj}-1)q^{2j-2}a + [m]_q [j]_{q^m} q^{j-1}) \right]$$

$$= q^{2r(0+1+2+3+\cdots+(n-1))+m\left[\binom{2}{2}+\binom{3}{2}+\cdots+\binom{n-1}{2}\right]} a^{0+1+2+\cdots+(n-1)}$$

$$\prod_{i=0}^{n-1} \prod_{j=1}^{i} \left[((q^{mj}-1)q^{2j-2}a + [m]_q [j]_{q^m} q^{j-1}) \right]$$

$$= q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} \prod_{j=1}^{i} \left[((q^{mj}-1)q^{2j-2}a + [m]_q [j]_{q^m} q^{j-1}) \right]$$

$$= q^{2r\binom{n}{2}+m\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} q^{\binom{i}{2}} \prod_{j=1}^{i} \left[[mj]_q \left(1 - q^j \left(\frac{1-q}{q} \right) a \right) \right]$$

$$= q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} \prod_{j=1}^{i} \left[[mj]_q \left(1 - q^{j-1}(1-q)a \right) \right].$$

This result is stated formally in the following theorem.

Theorem 3.1. *The Hankel transform of $\Phi_n[x,r,m]_q$ corresponding to the 0th Hankel determinant is given by*

$$H\left(\Phi_n[x,r,m]_q\right) = q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} \prod_{j=1}^{i} \left[[mj]_q \left(1 - q^{j-1}(1-q)a \right) \right]. \tag{17}$$

Note that when $m = 1$, (17) yields

$$H\left(\Phi_n[x,r,1]_q\right) = q^{2r\binom{n}{2}+2\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} [i]_q!((1-q)a;q)_i$$

where

$$(x;q)_i = \prod_{j=0}^{i-1} \left(1 - q^j x \right).$$

This is exactly the result obtained by Cigler [2].

As a direct consequence of Theorem 3.1, we have the following corollary, which contains the main result of this paper.

Corollary 3.2. *The Hankel transform of the sequence* $(D_{m,r}[n]_q)_{n=0}^{\infty}$ *is given by*

$$H\left(D_{m,r}[n]_q\right) = q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} \prod_{i=0}^{n-1}((1-q)a;q)_i \prod_{j=1}^{i}[mj]_q.$$

Theorem 3.3. *The Hankel transform of* $\Phi_n[x,r,m]_q$ *corresponding to the* 1*st Hankel determinant is given by*

$$d_{n,q}(n,1) = q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1}((1-q)a;q)_i \prod_{j=1}^{i}[mj]_q$$
$$\sum_{k=0}^{n}(-1)^n[x]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}}.$$

Proof. From Gram-Schmidt orthogonalization process, we obtain

$$d_{n,q}(n,1) = d_{n,q}(n,0)(-1)^n p_{n,q}(0)$$

where $p_{n,q}(0)$ is a sequence of orthogonal polynomials i.e.,

$$g_{n,q}(x,a,r,m) = \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle\langle x\rangle\rangle_{r,m,k} = p_{n,q}(x)$$

which implies

$$p_{n,q}(0) = \sum_{k=0}^{n}(-a)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \langle\langle 0\rangle\rangle_{r,m,k}.$$

Since,

$$\begin{aligned}
\langle\langle 0\rangle\rangle_{r,m,k} &= \prod_{j=0}^{k-1} \frac{([0]_q - [r+jm]_q)}{q^{r+jm}} \\
&= \prod_{j=0}^{k-1} \frac{-[r+jm]_q}{q^{r+jm}} \\
&= \left(\frac{-[r]_q}{q^r}\right)\left(\frac{-[r+j]_q}{q^{r+m}}\right)\left(\frac{-[r+(k-1)m]_q}{q^{r+(k-1)m}}\right) \\
&= (-1)^k \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}}.
\end{aligned}$$

Then,

$$\begin{aligned} p_{n,q}(0) &= \sum_{k=0}^{n} (-a)^k \, q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}} \\ &= \sum_{k=0}^{n} (-1)^k a^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^k \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}} \\ &= \sum_{k=0}^{n} a^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}} \end{aligned}$$

which implies

$$(-1)^n p_{n,q}(0) = \sum_{k=0}^{n} (-1)^n [x]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}}.$$

Hence,

$$\begin{aligned} d_{n,q}(n,1) &= d_{n,q}(n,0)(-1)^n p_{n,q}(0) \\ &= q^{2r\binom{n}{2}+(m+1)\binom{n}{3}} a^{\binom{n}{2}} \prod_{i=0}^{n-1} ((1-q)a;q)_i \prod_{j=1}^{i} [mj]_q \\ &\quad \sum_{k=0}^{n} (-1)^n [x]_q^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{j=0}^{k-1} \frac{[r+jm]_q}{q^{r+jm}} \end{aligned}$$

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