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Minimal and Upper Cost Effective Domination Number in Graphs

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Abstract. Given a connected graph G, we say that $S \subseteq V(G)$ is a cost effective dominating set in G if, each vertex in S is adjacent to at least as many vertices outside S as inside S and that every vertex outside S is adjacent to at least one vertex in S. The minimum cardinality of a cost effective dominating set is the cost effective domination number of G. The maximum cardinality of a cost effective dominating set is the upper cost effective domination number of G, and is denoted by $\gamma_{ce}^{+}(G)$. A cost effective dominating set is said to be minimal if it does not contain a proper subset which is itself a cost effective dominating in G. The maximum cardinality of a minimal cost effective dominating set in a graph G is the minimal cost effective domination number of G, and is denoted by $\gamma_{mce}(G)$. In this paper we provide bounds on upper cost effective domination number and minimal cost effective domination number of a connected graph G and characterized those graphs whose upper and minimal cost effective domination numbers are either $1, 2$ or $n-1$. We also establish a Nordhaus-Gaddum type result for the introduced parameters and solve some realization problems.

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1. Introduction

Throughout this paper, we consider simple, finite and undirected graphs $G = (V(G), E(G))$. All basic terminologies used here are taken from [4]. For a subset $S \subseteq V(G)$, the symbol $|S|$ refers to the cardinality of S. In particular, $|V(G)|$ is the *order* of G.

Let G be a connected graph. For $v \in V(G)$, the *closed neighborhood* of v is the set $N_G[v]$ consisting of v and all vertices adjacent to v. The *open neighborhood* of v is the set

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$N_G(v) = N_G[v] \setminus \{v\}$. For $S \subseteq V(G)$, $N_G[S] = \cup_{v \in S} N_G[v]$ and $N_G(S) = \cup_{v \in S} N_G(v)$. S is said to be a *dominating set* in G if $N_G[S] = V(G)$. The minimum cardinality $\gamma(G)$ of a dominating set is called the *domination number* of G. A dominating set S in G is said to be a *minimal dominating set* if it has no proper subset which is itself a dominating set in G. The maximum cardinality of a minimal domination set in G is denoted by $\gamma_m(G)$.

A subset $S \subseteq V(G)$ is an *independent set* in G if $uv \notin E(G)$ for distinct pairs of vertices u and v in S. An *independent dominating set* in G is an independent set in G which is dominating in G. The minimum cardinality $\gamma_i(G)$ of an independent dominating set in G is called *independence domination number.*

A subset $S \subseteq V(G)$ is said to be a *cost effective set* in G if for every $v \in S$, $|N_G(v) \cap S| \leq |N_G(v) \setminus S|$. A subset $S \subseteq V(G)$ is said to be a *very cost effective set* in G if for every $v \in S$, $|N_G(v) \cap S| < |N_G(v) \setminus S|$. A subset $S \subseteq V(G)$ is said to be a (*very*)*cost effective dominating set* in G if S is both a (very) cost effective set and a dominating set in G. The minimum cardinality of a cost effective dominating set of a graph G is called the *cost effective domination number* of G, and is denoted by $\gamma_{ce}(G)$.

Motivated by [8], the following concepts are introduced by the authors in [7] . The maximum cardinality, denoted by $\gamma^+_{ce}(G)$, of a cost effective dominating set in G is called an *upper cost effective domination number* of G. A cost effective dominating set of cardinality $\gamma^+_{ce}(G)$ is called an *upper cost effective dominating set.* A cost effective dominating set $S \subseteq V(G)$ is said to be *minimal cost effective set* if S does not contain a proper subset which is itself a cost effective dominating set. The symbol $\gamma_{mce}(G)$ is used to denote the maximum cardinality of a minimal cost effective dominating set of G.

For convenience, we use the terms γ_{ce}-set, γ^+_{ce}-set and γ_{mce}-set to refer to the cost effective dominating sets with cardinality $\gamma_{ce}(G)$, $\gamma^+_{ce}(G)$ and $\gamma_{mce}(G)$, respectively.

The following results are due to T.W. Haynes et.al. and F.V. Fomin et.al.

Theorem 1. [2] For a connected graph G of order $n \geq 2$, $\gamma_{ce}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 2. [2] Let G be a connected graph

(*i*) If $\Delta(G) \leq 4$, then $\gamma(G) = \gamma_{ce}(G)$.

(*ii*) If $\gamma(G) \leq 3$, then $\gamma(G) = \gamma_{ce}(G)$.

Lemma 1. [2] Every independent dominating set in an isolate-free graph G is a very cost effective dominating set in G.

Theorem 3. [5] *Every maximal independent set is a minimal dominating set.*

2. Results

Proposition 1. *Let $C_1, C_2, \ldots, C_n$ be the components of a graph G, and let $S \subseteq V(G)$. Then S is a cost effective dominating set in G if and only if $S_k = S \cap V(C_k)$ is a cost effective dominating set in C_k for $k = 1, 2, \ldots, n$.*

Proof. It is clear that S is a dominating set in G if and only if S_k is a dominating set in C_k for each $k = 1, 2, \ldots, n$. Moreover, since for $v \in S_k$, we have

$$N_G(v) \cap S = N_{C_k} \cap S_k \text{ and } N_G(v) \setminus S = N_{C_k}(v) \setminus S_k,$$

S is a cost effective set in G if and only if S_k is a cost effective set in C_k for each $k = 1, 2, \ldots, n$. □

Corollary 1. *Let G be a graph of order n. Then $\gamma_{ce}(G) = n$ if and only if $G = \overline{K}_n$.*

Proof. Suppose that $\gamma_{ce}(G) = n$. Then $S = V(G)$ is the only cost effective set in G. Let $u, v \in V(G)$ with $u \neq v$. Since $N_G(v) \setminus S = \varnothing$,

$$|N_G(v) \cap S| \leq |N_G(v) \setminus S| = 0.$$

Consequently, $|N_G(v) \cap S| = 0$, that is, $N_G(v) \cap S = \varnothing$. Thus, $u \notin N_G(v)$ so that $uv \notin E(G)$. Since u and v are arbitrary, $G = \overline{K}_n$. The converse follows immediately from Proposition 1. □

Lemma 2. *Let G be a nontrivial connected graph, and $S \subseteq V(G)$. If S is a cost effective set in G, then every subset of S is also a cost effective set in G.*

Proof. If $S = \varnothing$, then the conclusion is trivial. Suppose that $S \subseteq V(G)$ is a nonempty cost effective set in G. Let $u \in S$ and put $S^* = S \setminus \{u\}$. Let $v \in S^*$. If $u \in N_G(v)$, then

$$N_G(v) \cap S^* = (N_G(v) \cap S) \setminus \{u\} \text{ and } N_G(v) \setminus S = (N_G(v) \setminus S^*) \setminus \{u\}$$

so that

$$\begin{aligned} |N_G(v) \cap S^*| &< |N_G(v) \cap S| \\ &\leq |N_G(v) \setminus S| \\ &< |N_G(v) \setminus S^*|. \end{aligned}$$

If $u \notin N_G(v)$, then

$$N_G(v) \cap S^* = N_G(v) \cap S \text{ and } N_G(v) \setminus S^* = N_G(v) \setminus S$$

so that

$$|N_G(v) \cap S^*| \leq |N_G(v) \setminus S^*|.$$

This shows that S^* is a cost effective set. If $A \subsetneq S$, then A can be obtained from S by removing one vertex at a time. As shown above, removal of a vertex from a cost effective set results to a cost effective set. Thus, A is a cost effective set in G. □

Corollary 2. *Let G be a nontrivial connected graph. Then every minimal cost effective dominating set in G is a minimal dominating set. Consequently, $\gamma_{mce}(G) \leq \gamma_m(G)$.*

Remark 1. *A minimal dominating set need not be a minimal cost effective dominating set.*

Remark 2. *Any independent dominating set in a connected nontrivial graph is a minimal cost effective dominating set in G.*

Corollary 3. *Let G be a nontrivial connected graph . Then*

$$i(G) \leq \gamma_{mce}(G) \leq \gamma_m(G).$$

Proof. This follows from Remark 2 and Corollary 2. □

Remark 3. *For any nontrivial connected graph G,*

$$\gamma_{ce}(G) \leq \gamma_{mce}(G) \leq \gamma^+_{ce}(G).$$

Theorem 4. *For any connected graph G of order $n \geq 2$, if G is not complete, then $\gamma_{mce}(G) \geq 2$, and consequently, $\gamma^+_{ce}(G) \geq 2$.*

Proof. Let S be a maximal independent set of G. Since $G \neq K_n$, $|S| > 2$. By Theorem 3, S is a dominating set in G. Thus, S is a minimal cost effective dominating set in G by Corollary 2. Consequently, $\gamma_{mce}(G) \geq 2$. □

Theorem 5. *For any complete graph K_n of order $n \geq 2$,*

$$\gamma^+_{ce}(K_n) = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. Let $S \subseteq V(K_n)$ with $|S| = \left\lfloor \frac{n+1}{2} \right\rfloor$. For each $u \in S$,

$$|N_{K_n}(u) \cap S| = |S| - 1 = \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \quad \text{and} \quad |N_{K_n}(u) \setminus S| = n - |S|.$$

Now, $2\left\lfloor \frac{n+1}{2} \right\rfloor \leq n+1$ so that

$$|N_{K_n}(u) \cap S| = \left\lfloor \frac{n+1}{2} \right\rfloor - 1 \leq n - \left\lfloor \frac{n+1}{2} \right\rfloor = |N_{K_n}(u) \setminus S|.$$

Thus, S is a cost effective dominating set in K_n. Hence,

$$\gamma^+_{ce}(K_n) \geq |S| = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Let S be a γ^+_{ce}-set of K_n. For each $u \in S$, $|N_{K_n}(u) \cap S| = |S| - 1$ and

$$|N_{K_n}(u) \setminus S| = |V(K_n) \setminus S| = |V(K_n)| - |S| = n - |S|.$$

Since S is cost effective,

$$|S| - 1 \leq n - |S|,$$

or equivalently, $2|S| \leq n+1$. Thus, $|S| \leq \frac{1}{2}(n+1)$. Since S is a γ^+_{ce}-set,

$$\gamma^+_{ce}(K_n) = |S| = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

□

Corollary 4. *Let G be a nontrivial connected graph of order $n \geq 2$. Then*

(i) $\gamma_{ce}(G) = 1$ *if and only if* $G = K_1 + H$ *for some subgraph H of G;*

(ii) $\gamma_{mce}(G) = 1$ *if and only if* $G = K_n$*;*

(iii) $\gamma^+_{ce}(G) = 1$ *if and only if* $G = K_2$.

Proof. Statement (i) is clear, while statement (ii) follows immediately from Theorem 4. Suppose that $\gamma^+_{ce}(G) = 1$. Then G is complete, by Theorem 4. Now, $\gamma^+_{ce}(K_n) = \lfloor \frac{n+1}{2} \rfloor$ by Theorem 5. Thus, $\gamma^+_{ce}(K_n) = 1$ if and only if $n = 2$. □

Theorem 6. *For $m, n \geq 1$,*

(i) $\gamma_{ce}(K_{m,n}) = \min\{m, n, 2\}$

(ii) $\gamma_{mce}(K_{m,n}) = \gamma^+_{ce}(K_{m,n}) = \max\{m, n\}$.

Proof. The result is obvious if $n = 1$ or $m = 1$. Suppose that $m, n \geq 2$. Put $G = K_{m,n}$. We claim that $S \subseteq V(G)$ is a cost effective dominating set in G if and only if either S is a partite set of $K_{m,n}$ or S intersects each partite set and $2 \leq |S| \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. Since partite sets U, V of $K_{m,n}$ are independent dominating sets of G, they are minimal cost effective dominating sets in G by Proposition 2. Assume $|U| = m$ and $|V| = n$. Let S be a cost effective dominating set in G which is not a partite set. Since S is a dominating set, S intersects both partite sets. Let $v \in S$. If $v \in U$, then

$$|S \cap V| = |N_G(v) \cap S| \leq |N_G(v) \setminus S| = |V \setminus S|.$$

Consequently, $|S \cap V| \leq \lfloor \frac{n}{2} \rfloor$. Similarly, $|S \cap U| \leq \lfloor \frac{m}{2} \rfloor$. Thus, $2 \leq |S| \leq \lfloor \frac{n}{2} \rfloor + \lfloor \frac{m}{2} \rfloor$. The converse is obvious. Therefore, $\gamma_{ce}(G) = \min\{m, n, 2\} = 2$ and $\gamma_{mce}(G) = \max\{m, n\} = \gamma^+_{ce}(G)$. □

Theorem 7. (i) *For $n \geq 2$,*

$$\gamma^+_{ce}(P_n) = \left\lfloor \frac{2n}{3} \right\rfloor \text{ and } \gamma_{mce}(P_n) = \left\lceil \frac{n}{2} \right\rceil.$$

(ii) For $n \geq 3$,

$$\gamma^+_{ce}(C_n) = \left\lfloor \frac{2n}{3} \right\rfloor \text{ and } \gamma_{mce}(C_n) = \left\lceil \frac{n-1}{2} \right\rceil$$

Proof. For (i), let $P_n = [v_1, v_2, \ldots, v_n]$ be a path of order $n \geq 2$. The result is clear if $n = 2$. Suppose that $n \geq 3$, and let $n = 3k + j$, with $k \geq 1$ and $0 \leq j \leq 2$. Note that for $j = 0, 1$, the set $\{v_1, v_3, v_4, \ldots, v_{3k-3}, v_{3k-2}, v_{3k}\}$ is a cost effective dominating set in P_n. On the other hand, if $j = 2$, then set $\{v_1, v_3, v_4, \ldots, v_{3k-3}, v_{3k-2}, v_{3k}, v_{3k+1}\}$ is a cost effective dominating set in P_n. In any case, $\gamma^+_{ce}(P_n) \geq \lfloor \frac{2n}{3} \rfloor$. Now, let $S \subseteq V(P_n)$ be a γ^+_{ce}-set in P_n. Being a cost effective dominating set, the vertices v_i, v_{i+1}, v_{i+2} cannot be all in S for all $i = 1, 2, \ldots, n-2$. In particular, if $j = 0$, then $|S| \leq 2k$, and this is attained with $v_{3k} \in S$. Apparently, in view of this, $|S| \leq 2k$ for $j = 1$ and $|S| \leq 2k + 1$ for $j = 2$. Indeed, $\gamma^+_{ce}(P_n) = \lfloor \frac{2n}{3} \rfloor$.

Similarly, to prove the second part of (i), we write $n = 2k + j$, with $0 \leq j \leq 1$. Since the sets $\{v_1, v_3, v_5, \ldots, v_{2k-1}\}$ and $\{v_1, v_3, v_5, \ldots, v_{2k-1}, v_{2k+1}\}$ are minimal cost effective dominating sets in P_n for $j = 0$ and $j = 1$, respectively, we have $\gamma_{mce}(P_n) \geq \lceil \frac{n}{2} \rceil$. Conversely, let $S \subseteq V(P_n)$ be a minimal cost effective dominating set in P_n. Then for all $i = 1, 2, \ldots, n-3$, $|S \cap \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}| \leq 2$. Thus, $|S| \leq \lceil \frac{2n}{4} \rceil = \lceil \frac{n}{2} \rceil$.

For (ii), let $C_n = [v_1, v_2, \ldots, v_n]$ be a cycle of order $n \geq 3$. The result is clear if $n = 3$. Suppose that $n \geq 4$, and let $n = 3k + j$, with $k \geq 1$ and $0 \leq j \leq 2$. Note that for $j = 0, 1$, the set $\{v_1, v_3, v_4, \ldots, v_{3k-3}, v_{3k-2}, v_{3k}\}$ is a cost effective dominating set in C_n. On the other hand, if $j = 2$, then set $\{v_1, v_3, v_4, \ldots, v_{3k-3}, v_{3k-2}, v_{3k}, v_{3k+1}\}$ is a cost effective dominating set in C_n. In any case, $\gamma^+_{ce}(C_n) \geq \lfloor \frac{2n}{3} \rfloor$. Now, let $S \subseteq V(C_n)$ be a γ^+_{ce}-set in C_n. This means that the vertices v_i, v_{i+1}, v_{i+2} cannot be all in S for all $i = 1, 2, \ldots, n-2$. In particular, if $j = 0$, then $|S| \leq 2k$, and this is attained with $v_{3k} \in S$. Apparently, in view of this, $|S| \leq 2k$ for $j = 1$ and $|S| \leq 2k + 1$ for $j = 2$. Indeed, $\gamma^+_{ce}(C_n) = \lfloor \frac{2n}{3} \rfloor$.

To prove the second part of (ii), we write $n = 2k + j$, with $0 \leq j \leq 1$. Since the sets $\{v_1, v_3, v_5, \ldots, v_{2k-1}\}$ and $\{v_1, v_3, v_5, \ldots, v_{2k-2}, v_{2k}\}$ are minimal cost effective dominating sets in C_n for $j = 0$ and $j = 1$, respectively, we have $\gamma_{mce}(C_n) \geq \lceil \frac{n-1}{2} \rceil$. Conversely, let $S \subseteq V(C_n)$ be a minimal cost effective dominating set in C_n. Note that for any $n \geq 3$, $v_n \notin S$. Also, for all $i = 1, 2, \ldots, n-3$, $|S \cap \{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}| \leq 2$. Thus, $|S| \leq \lceil \frac{n-1}{2} \rceil$. Therefore, $\gamma_{mce}(C_n) = \lceil \frac{n-1}{2} \rceil$. □

Proposition 2. *For any double star graph* $S_{r,s}$ *where* $r, s \geq 1$,

(i) $\gamma_{ce}(S_{r,s}) = 2$

(ii) $\gamma^+_{ce}(S_{r,s}) = r + s = \gamma_{mce}(S_{r,s})$.

Proof. Let u and v be the two central vertices of $G = S_{r,s}$, and let U and V be the sets of all leaves adjacent to u and v, respectively, with $|U| = r$ and $|V| = s$. Let S be a cost effective dominating set in G. Note that if $u \in S$, then $U \cap S = \varnothing$ and if $u \notin S$, then $U \subseteq S$. Similar statements apply if $v \in S$ and $V \cap S = \varnothing$. Thus, S is one of the following: (i) $S = \{u, v\}$, (ii) $S = \{u\} \cup V$, (iii) $S = \{v\} \cup U$, (iv) $S = U \cup V$.

Therefore, $\gamma_{ce}(G) = 2$ and $\gamma^+_{ce}(G) = |U| + |V| = r + s$. Note further that if $u, v \notin S$, then $S \setminus \{x\}$ is not a dominating set in G, hence not a cost effective dominating set in G for any $x \in U \cup V$. This means that $U \cup V$ is a minimal cost effective dominating set in G. Therefore, $\gamma_{mce}(G) = |U| + |V| = r + s$. □

Corollary 5. *If G is any of the following graphs: P_n $(n \geq 2)$, C_n $(n \geq 3)$, $K_{m,n}$ $(m, n \geq 2)$, K_n $(n \geq 1)$ and $S_{r,s}$, then $\gamma_{mce}(G) = \gamma_m(G)$.*

Theorem 8. *Let G be a connected graph of order $n \geq 2$. Then,*

(i.) $\gamma_{mce}(G) = n - 1$ *if and only if* $G = K_{1,n-1}$

(ii.) $\gamma^+_{ce}(G) = n - 1$ *if and only if* $G = K_1 + \bigcup_{j=1}^{t} K_{r_j}$, *where* $1 \leq r_j \leq 2$ *and* $\sum r_j = n - 1$.

Proof. Let $S \subseteq V(G)$ be a cost effective set with $|S| = n - 1$, and let $v \in V(G) \setminus S$. Since G is connected and S is cost effective, it follows that $uv \in E(G)$ and $|N_G(u) \cap S| \leq 1$ for all $u \in S$. If there exists $w \in S$ such that $|N_G(w) \cap S| = 1$ then $S' = S \setminus \{w\}$ is a cost effective set dominating set in G. If S is a minimal cost effective set, then $\langle S \rangle = \overline{K_{n-1}}$. Thus, $G = K_{1,n-1}$. Otherwise, $G = K_1 + \bigcup_{j=1}^{t} K_{r_j}$, where $1 \leq r_j \leq 2$ and $\sum r_j = n - 1$. The converse is clear. □

Theorem 9. *For a connected graph G of order $n \geq 5$, $\gamma^+_{ce}(G) \geq 3$.*

Proof. Suppose that G is a complete graph of order $n \geq 5$. If $S \subseteq V(G)$ is a γ^+_{ce}-set in G, then by Theorem 5, $\gamma^+_{ce}(G) = |S| = \left\lfloor \frac{1}{2}(n + 1) \right\rfloor \geq \left\lfloor \frac{1}{2}(5 + 1) \right\rfloor = 3$. Suppose that G is not complete. Then there exist $u, v \in V(G)$ such that $uv \notin E(G)$. Then $\{u, v\}$ is a cost effective set in G. Suppose that $N_G[\{u, v\}] \neq V(G)$. Let $v_1 \in V(G) \setminus N_G[\{u, v\}]$ and put $S_1 = \{u, v, v_1\}$. If $N_G[S_1] = V(G)$, then S_1 is a cost effective dominating set in G. Suppose that $N_G[S_1] \neq V(G)$. Let $v_2 \in V(G) \setminus N_G[S_1]$ and put $S_2 = \{u, v, v_1, v_2\}$. If $N_G[S_2] = V(G)$, then S_2 is a cost effective dominating set in G. Continuing in this manner, there exists a positive integer $k \geq 1$ such that $v_k \notin N_G[S_{k-1}]$ and $N_G[S_k] = V(G)$ wth $S_0 = \{u, v\}$. Thus, S_k is a cost effective dominating set in G. Consequently, $\gamma^+_{ce}(G) \geq 3$.

Suppose that $N_G[\{u, v\}] = V(G)$. Note that, since $|V(G)| \geq 5$, G has at least three more vertices v_1, v_2 and v_3 other than u and v. Further, since $\{u, v\}$ is a dominating set in G, at least two vertices in $V(G) \setminus \{u, v\}$ are adjacent to either u or v or both. Consider the following cases.

Case 1 : Suppose that $v_1 v_2 \notin E(G)$.

Subcase 1 : Consider having v_1, v_2 being adjacent only to either u or v. Assume that v_1 and v_2 are adjacent to u. Since $S^* = \{v_1, v_2, v\}$ is a cost effective set in G, one can construct a cost effective dominating set S beginning with S^* as done above. Consequently, $|S| \geq 3$.

Subcase 2 : Consider v_1 being adjacent to both u and v and v_2 being adjacent only to either u or v. Note that since $\{u, v\}$ is a dominating set in G, $v_3 \in N_G(\{u, v\})$. If $v_3 \notin N_G(v_1) \cap N_G(v_2)$, then the set $S^* = \{v_1, v_2, v_3\}$ is a cost effective set in G, and one can construct a cost effective dominating set S beginning with S^*. On the other hand, if $v_3 \in N_G(v_1) \cap N_G(v_2)$, then the set $S = \{u, v, v_3\}$ is a cost effective dominating set in G. Thus, $|S| \geq 3$.

Subcase 3 : Suppose that v_1, v_2 are both adjacent to u and v. If $v_3 \in N_G(v_1) \cap N_G(v_2)$, then the set $S^* = \{u, v, v_3\}$ is a cost effective dominating set in G. If $v_3 \notin N_G(v_1) \cap N_G(v_2)$, then one can start with the cost effective set $S^* = \{v_1, v_{,2}, v_3\}$ to construct a cost effective dominating set S in G.

Case 2 : Suppose that $v_1v_2 \in E(G)$.

Subcase 1 : Suppose that v_1, v_2 are adjacent only to either u or v. Assume that v_1, v_2 are adjacent to u. Since the set $S^* = \{v_1, v_2, v\}$ is a cost effective set in G, one can construct a cost effective dominating set S in G starting with S^*. Clearly, $|S| \geq 3$.

Subcase 2 : Suppose that v_1 is adjacent to both u and v and v_2 is adjacent only to either u or v. Assume that $v_2 \in N_G(u)$. Then the set $S = \{u, v, v_2\}$ is a cost effective dominating set in G.

Subcase 3 : Suppose that v_1, v_2 are both adjacent to u and v. If $v_3 \in N_G(v_1) \cap N_G(v_2)$, then the set $S = \{u, v, v_2\}$ is a cost effective dominating set in G. If $v_3 \notin N_G(v_1) \cap N_G(v_2)$, then the set $S^* = \{v_1, v_2, v_3\}$ is a cost effective set in G, and one can construct a cost effective dominating set S in G starting with S^*.

In both cases, $\gamma_{ce}^+(G) \geq 3$. □

Corollary 6. *If G is a connected graph of order $n \geq 3$, then $\gamma_{ce}^+(G) = 2$ if and only if G is one of the following graphs: P_3, P_4, C_3, C_4, K_4 or $K_2 + \overline{K}_2$.*

Proof. Suppose that $\gamma_{ce}^+(G) = 2$ and let $S = \{u, v\}$ be a γ_{ce}^+-set in G. Let $D = V(G) \setminus S$. If $|D| = 1$, then either $G = C_3$ or $G = P_3$. If $|D| = 2$, then G is any of the following: P_4, C_4, $K_2 + \overline{K}_2$, or K_4. Suppose that $|D| \geq 3$. Then $|V(G)| \geq 5$. By Theorem 9, $\gamma_{ce}^+(G) > 2$, and the desired conclusion follows. For the converse, it is easy to verify that if G is one of the following graphs: P_3, P_4, C_3, C_4, $K_2 + \overline{K}_2$, or K_4, then $\gamma_{ce}^+(G) = 2$. □

Theorem 10. *Let G be a connected noncomplete graph.*

(*i*) *If $\gamma_m(G) = 2$, then $\gamma_{mce}(G) = 2$.*

(*ii*) *If $\gamma_{mce}(G) = 2$, then for each pair of nonadjacent vertices u and v of G, $\{u, v\}$ is a dominating set in G*

Proof. For (*i*), observe that by Theorem 4 and Corollary 2,

$$2 \leq \gamma_{mce}(G) \leq 2.$$

Thus, $\gamma_{mce}(G) = 2$. To prove (*ii*), suppose that $\gamma_{mce}(G) = 2$ and let $u, v \in V(G)$ with $uv \notin E(G)$. Then $\{u, v\}$ is a cost effective set in G, and as previously done, one can construct a minimal cost effective dominating set S in G beginning with the vertices u and v. Since $\gamma_{mce}(G) = 2$, $|S| = 2$ and $S = \{u, v\}$. Thus, $S = \{u, v\}$ is a dominating set in G. □

Remark 4. *The converse of each of the statements in Theorem 10 need not be true.*

Example 1. *Let G be a noncomplete connected graph of order $n \geq 3$. Then $\gamma_{mce}(G) = 2$ if G is any of the following:*

(*i*) $G = \overline{K_2} + K_n$, $n \geq 1$;

(*ii*) *G is obtained from K_n by adding a pendant edge where $n \geq 2$; or*

(*iii*) *G is the K_2-gluing of K_3 and K_n, $n \geq 3$.*

2.1. Nordhauss-Gaddum Type Results

Remark 5. *Following a similar proof, the statement in Corollary 1 remains true if $\gamma_{ce}(G)$ is changed to $\gamma_{ce}(G)^+$ or $\gamma_{mce}(G)$.*

Let Ξ be an infinite collection of all connected graph G such that $\overline{G}$ is also connected.

Theorem 11. *For all $G \in \Xi$ of order $n \geq 4$,*

(*i*) $4 \leq \gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) \leq 2n - 4$; *and*

(*ii*) $4 \leq \gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) \leq n^2 - 4n + 4$.

In particular,

$\gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) = 4$ *if and only if* $n = 4$, *and*

$\gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) = 4$ *if and only if* $n = 4$.

Proof. Let $G \in \Xi$ be of order $n \geq 4$. By Theorem 8 and Remark 5, $\gamma^+_{ce}(G) \neq n$ and $\gamma^+_{ce}(G) \neq n - 1$. Thus, Theorem 4 yield

$$4 \leq \gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) \leq (n-2) + (n-2) = 2n - 4$$

and

$$4 = (2)(2) \leq \gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) \leq (n-2)(n-2) = n^2 - 4n + 4.$$

Suppose that $\gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) = 4$. Then, necessarily, $\gamma^+_{ce}(G) = 2$ and $\gamma_{ce}(\overline{G}) = 2$. By Theorem 9, $n = 4$. Conversely, if $n = 4$, then $\gamma^+_{ce}(G)+\gamma^+_{ce}(\overline{G}) = 4$ Similarly, $\gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) = 4$ if and only if $n = 4$. □

Corollary 7. *If $G \in \Xi$ of order $n \geq 4$, then*

(*i*) $\gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) = 4$ *if and only if* $G = P_4$, *(ii)* $\gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) = 4$ *if and only if* $G = P_4$.

Proof. The result follows from Theorem 11 and Corollary 6. □

Theorem 11 and Theorem 9 imply that if $G \in \Xi$ is of order $n \geq 5$, then

$$6 \leq \gamma^+_{ce}(G) + \gamma^+_{ce}(\overline{G}) \leq 2n - 4 \quad (1)$$

and

$$9 \leq \gamma^+_{ce}(G)\gamma^+_{ce}(\overline{G}) \leq n^2 - 4n + 4. \quad (2)$$

Since $\gamma^+_{ce}(C_5) = \gamma^+_{ce}(\overline{C_5}) = 3$, bounds in Equations 1 and 2 are sharp.

Following the proof of Theorem 11, the following is true.

Theorem 12. *For all* $G \in \Xi$ *of order* $n \geq 4$,

(i) $4 \leq \gamma_{mce}(G) + \gamma_{mce}(\overline{G}) \leq 2n - 4$; *and*

(ii) $4 \leq \gamma_{mce}(G)\gamma_{mce}(\overline{G}) \leq n^2 - 4n + 4$.

Proof. Let $G \in \Xi$ be of order $n \geq 4$. Note that whenever G is K_n or $K_{1,n-1}$, $\overline{G}$ is disconnected. Thus, Theorem 8 and Corollary 4 imply that

$$\gamma_{mce}(G) + \gamma_{mce}(\overline{G}) \leq (n-2) + (n-2) = 2n - 4$$

and

$$\gamma_{mce}(G)\gamma_{mce}(\overline{G}) \leq (n-2)(n-2) = n^2 - 4n + 4.$$

The left inequalities follow from Theorem 4. □

Remark 6. *The bounds given in Theorem 12 are sharp.*

To see this, consider the graph $G = P_4$. Observe that $\gamma_{mce}(P_4) = 2 = \gamma_{mce}(\overline{P_4})$.

2.2. Realization Problem

Theorem 13. *For every positive integers* a, b, c *with* $1 \leq a \leq b \leq c$ *there exists a connected graph* G *such that* $\gamma_{ce}(G) = a$, $\gamma_{mce}(G) = b$ *and* $\gamma^+_{ce}(G) = c$.

Proof. Suppose that $a = b = c$. Write $V(K_{1,a-1}) = \{x, u_1, u_2, \ldots, u_{a-1}\}$ as in Figure 1. Obtain G from $K_{1,a-1}$ by adding pendant edges v_ju_j, $j = 1, 2, \ldots, a-1$ and then adding new edges $v_{a-1}w_{a-1}$ and $w_{a-1}x$ as shown in Figure 1.

Write $D = \{u_{a-1}, v_{a-1}, w_{a-1}, x\}$. Observe that for any $s, t \in D$, the sets $\{v_1, v_2, v_3 \ldots, v_{a-2}, s, t\}$, $\{u_1, u_2, \ldots, u_{a-2}, p, r\}$ where $p, r \in D \setminus \{x\}$ and sets of the form $\{u_j, v_k : j \neq k, j, k = 1, 2, \ldots, a-2\} \cup \{s, t\}$ are the only cost effective dominating sets in G. Therefore, $\gamma_{ce}(G) = a$, $\gamma_{mce}(G) = b$ and $\gamma^+_{ce}(G) = c$.

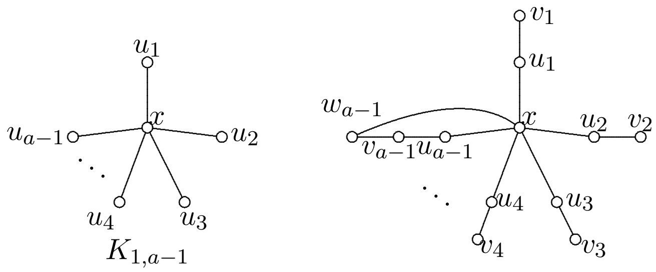

Figure 1: G obtained from $K_{1,a-1}$

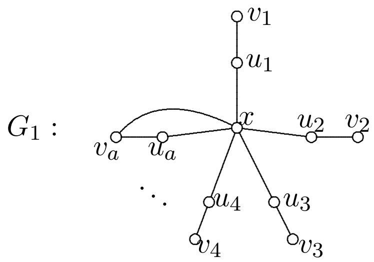

Figure 2: G_1 obtained from $K_{1,a}$

Suppose that $a = b$ and $c = b + 1$. Obtain the graph $G = G_1$ from $K_{1,a}$ by adding pendant edges u_jv_j, $j = 1, 2, \dots, a$ and an edge xv_a as shown in Figure 2. Observe that the sets $\{u_1, u_2, \dots, u_a\}$, $\{v_1, v_2, \dots, v_a\}$, $\{x, v_1, v_2, \dots, v_{a-1}\}$ and sets $\{v_j, u_k : j \neq k, j, k = 1, 2, \dots, a\}$ are the only minimal cost effective dominating sets in G. Thus, $\gamma_{mce}(G) = a = b$. Also, note that the set $\{x, v_1, v_2, \dots, v_a\}$ is a γ^+_{ce}-set in G. Therefore $\gamma^+_{ce}(G) = a + 1 = b + 1 = c$.

Suppose that $a = b$ and $c = b + k$ for $k \geq 2$. Let $V(K_{2k}) = \{w_1, w_2, \dots, w_{2k}\}$. Obtain $G = G_2$ from G_1 by joining complete graph K_{2k} to exactly one of the end vertices of G_1, say v_1, as shown in Figure 3.

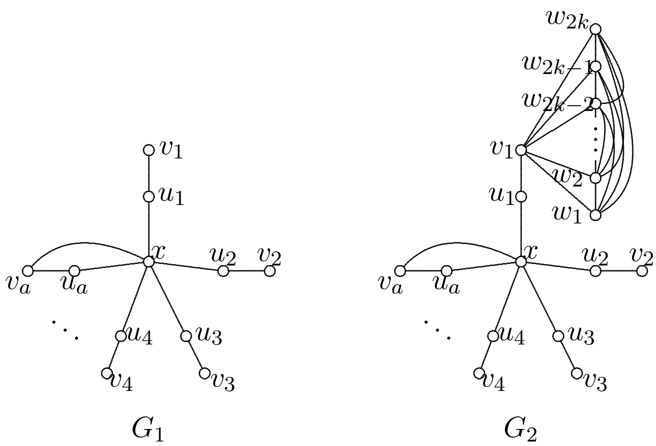

Figure 3: G_2 obtained from K_1

Observe that for each $r \in \{u_a, v_a\}$, the set $\{r, u_2, u_3, \dots, u_{a-1}, v_1\}$ is a γ_{ce}-set in

G. Thus, $\gamma_{ce}(G) = 1 + a - 1 = a$. Also, $\gamma_{mce}(G) = b$ which is determined by the set $\{x, v_2, v_3, \ldots, v_{a-1}, z\}$ where $z \in V(K_{2k}+v_1)$. Moreover, note that the set $\{x, v_2, v_3, \ldots, v_a, w_1, w_2, \ldots, w_{k-1}, w_k\}$ is a γ^+_{ce}-set in G. Therefore, $\gamma^+_{ce}(G) = 1+a-1+k = a+k = b+k = c$.

Suppose that $b = a + 1$ and $c = b$. Obtain $G = G_3$ is from $K_{1,a}$ by adding pendant edges u_jv_j, $j = 1, 2, \ldots, a$ as shown in Figure 4. Then $\gamma_{ce}(G) = a$ which is determined by

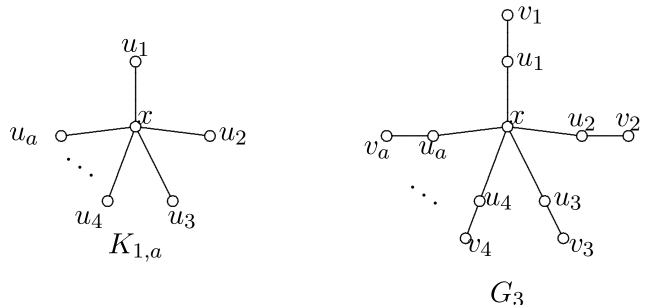

Figure 4: G_3 obtained from $K_{1,a}$

the set $\{u_1, u_2, \ldots, u_a\}$. Also, $\gamma_{mce}(G) = a + 1 = b$ and $\gamma^+_{ce}(G) = c$ which are determined by the set $\{v_1, v_2, \ldots, v_a, x\}$. Suppose that $c = b + k$, $k \geq 2$. Obtain G by joining the complete graph K_{2k} to exactly one of end vertices of G_3 say in v_1 , as shown in Figure 5.

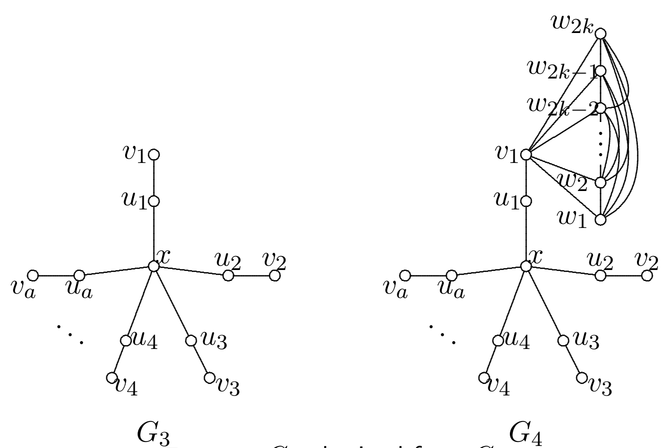

Figure 5: G_4 obtained from G_3

Now, $\gamma_{ce}(G) = a$ which is determined by the set $\{u_2, u_3, \ldots, u_a, v_1\}$. Note that $\gamma_{mce}(K_{2k}) = 1$. Thus, $\gamma_{mce}(G) = a + 1 = b$ which is determined by the set $\{x, v_2, v_3, \ldots, v_a, z\}$ $z \in V(K_{2k}+v_1)$. Also, observe that the set $\{w_1, w_2, \ldots, w_k\}$ is a γ^+_{ce}-set in K_{2k}. Thus, $\{x, v_1, v_2, \ldots, v_a, w_1, \ldots, w_k\}$ is a γ^+_{ce}-set in G. Therefore, $\gamma^+_{ce}(G) = 1 + a + k = b + k = c$.

Suppose that $b = a+k$ and $c = b$. Obtain $G = G_5$ from G_3 by adding k pendant edges v_ah_j, $j = 1, 2, \ldots, k$ as shown in Figure 6. Then $\gamma_{ce}(G) = a$ which is determined by the

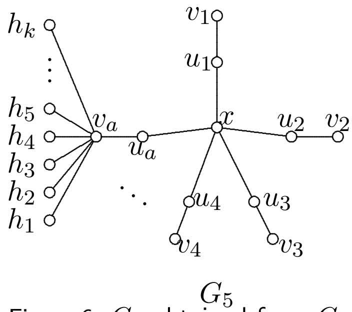

Figure 6: G_5 obtained from G_3

set $\{u_1, u_2, \ldots, u_{a-1}, u_a\}$. Also, observe that the set $\{v_1, v_2, \ldots, v_{a-1}, x, h_1, h_2, \ldots, h_k\}$ is a γ_{mce}-set in G and is the only cost effective dominating set in G which is of maximum order. Thus, $\gamma_{mce}(G) = a - 1 + 1 + k = a + k = b = c = \gamma^+_{ce}(G)$. Now, suppose that $c = b + t$, $t \geq 1$. Obtain $G = G_6$ from G_5 by joining complete graph K_{2t} to exactly one of the end vertices of G_4, say v_1, as shown in Figure 7.

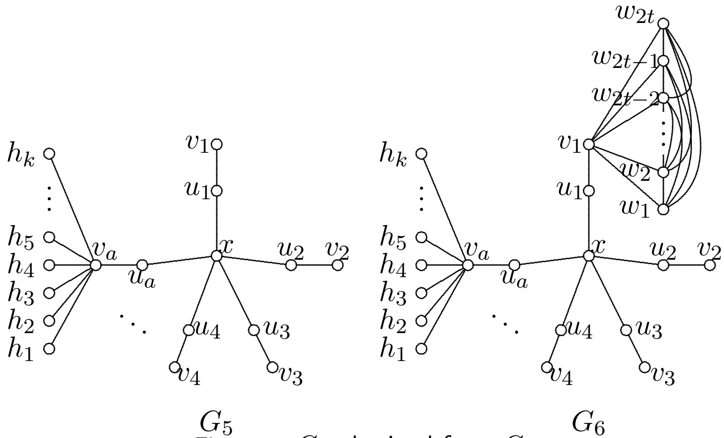

Figure 7: G_6 obtained from G_5

Then the set $\{u_2, u_3, \ldots, u_{a-1}, v_1, v_a\}$ is a γ_{ce}-set in G. Thus, $\gamma_{ce}(G) = a - 2 + 2 = a$. Also, since $\gamma_{mce}(K_{2t} + v_1) = 1$, $\gamma_{mce}(G) = a + k = b$ which is determined by the set $\{v_1, v_2, \ldots, v_{a-1}, x, h_1, \ldots, h_k\}$. Also, observe that since the set $\{v_1, v_2, \ldots, v_{a-1}, x, h_1, \ldots, h_k, w_1, \ldots, w_t\}$ is a γ^+_{ce}-set in G, $\gamma^+_{ce}(G) = a-1+1+k+t = a+k+t = b+k = c$. □

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