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Inverse Limit of an Inverse System of BE-algebras

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Abstract. This paper covers the notion of the inverse limit of an inverse system of BE-algebras and investigates some of its properties. Moreover, this study deals with the completion of a BE-algebra.

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1. Introduction

The notion of BCK-algebras was initiated by Y. Imai and K. Iséki [1] in 1966 as a generalization of the concept of set theoretic difference and propositional calculi. In [3], K. H. Kim and Y. H. Yon introduced the dual BCK-algebra and study its relation to MV-algebra. As a generalization of dual BCK-algebra, H. S. Kim and Y. H. Kim [2] introduced the BE-algebra. Today, BE-algebras have been studied by many authors and many branches of mathematics have been applied to BE-algebras, such as probability theory, topology, fuzzy set theory and so on. In this paper we will apply the concept of inverse limit in the sense of category theory to some collections of BE-algebras. This notion in category theory has been studied in different kinds of categories. This study introduces the inverse limit of an inverse system of BE-algebras and investigates some of its properties. Through this concept, we present the idea of the completion of any BE-algebra.

An algebra $(X; *, 1_X)$ is called a *BE-algebra* if the following hold: for all $x, y, z \in X$, (BE1) $x * x = 1_X$; (BE2) $x * 1_X = 1_X$; (BE3) $1_X * x = x$; and (BE4) $x * (y * z) = y * (x * z)$. A relation "$\leq$" on X, called *BE-ordering*, is defined by $x \leq y$ if and only if $x * y = 1_X$. Throughout this paper, we denote a BE-algebra $(X, *, 1_X)$ simply by X if no confusion arises. A non-empty subset S of X is said to be a *subalgebra* of X if $x * y \in S$ for all

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$x, y \in S$. A non-empty subset F of X is said to be a *filter* of X if: (F1) $1_X \in F$; and (F2) $x * y \in F$ and $x \in F$ imply $y \in F$. A filter F of X is said to be *normal* if it satisfies the following condition: for all $x, y, z \in X$, $x * y \in F$ implies $(z * x) * (z * y) \in F$ and $(y*z)*(x*z) \in F$. A BE-algebra X is said to be *self distributive* if $x*(y*z) = (x*y)*(x*z)$ for all $x, y, z \in X$. It is called *commutative* if satisfies $(x*y)*y = (y*x)*x$ for all $x, y \in X$. It is said to be a *transitive BE-algebra* if it satisfies the condition: $y*z \leq (x*y)*(x*z)$ for all $x, y, z \in X$. If X is a transitive BE-algebra, then the relation "$\leq$" is transitive. Let X and Y be BE-algebras. A mapping $f : X \to Y$ is a *homomorphism* if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$, see [4].

Theorem 1. [4] *Let X and Y be BE-algebras. If $f : X \to Y$ is a homomorphism, then $f(1_X) = 1_Y$.*

Theorem 2. [4] *Every commutative BE-algebra X is transitive.*

Proposition 1. [6] *If X is a transitive BE-algebra, then every filter of X is normal.*

Theorem 3. [6] *Let X be a commutative BE-algebra. There is a bijection between congruence relations and filters of X.*

Definition 1. [5] *Let I be a set and $\leq$ be a binary operation on I. We call $I = (I, \leq)$ a directed partially ordered set or directed poset if it satisfies the following conditions: (i) $i \leq i$, for $i \in I$; (ii) $i \leq j$ and $j \leq k$ imply $i \leq k$, for $i, j, k \in I$; (iii) $i \leq j$ and $j \leq i$ imply $i = j$, for $i, j \in I$; and (iv) if $i, j \in I$, there exists some $k \in I$ such that $i, j \leq k$.*

Definition 2. *An inverse or projective system of BE-algebras over a directed poset I, consists of a collection $\{X_i \mid i \in I\}$ of BE-algebras indexed by I, and a collection of homomorphisms $\varphi_{ij} : X_i \to X_j$, defined whenever $i \geq j$, such that $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ whenever $i, j, k \in I$ and $i \geq j \geq k$. In addition, we assume that φ_{ii} is the identity mapping id_{X_i} on X_i.*

We shall denote such a system by $\{X_i, \varphi_{ij}, I\}$, or by $\{X_i, \varphi_{ij}\}$ if the index set I is clearly understood.

Definition 3. *Let Y be a BE-algebra, $\{X_i, \varphi_{ij}, I\}$ an inverse system of BE-algebras over a directed poset I, and let $\psi_i : Y \to X_i$ be a homomorphism for each $i \in I$. These mappings ψ_i are said to be compatible if $\varphi_{ij}\psi_i = \psi_j$ whenever $j \leq i$.*

Definition 4. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras over a directed poset I. A subalgebra X of $\prod_{i \in I} X_i$ together with compatible homomorphisms $\varphi_i : X \to X_i$ where $i \in I$ is an inverse limit or a projective limit of the inverse system $\{X_i, \varphi_{ij}, I\}$ if the following universal property is satisfied: whenever Y is a BE-algebra and $\{\psi_i : Y \to X_i$ $(i \in I)\}$ is a set of compatible homomorphisms, then there is a unique homomorphism $\psi : Y \to X$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$. We say that ψ is "induced" or "determined" by the compatible homomorphisms ψ_i. The maps $\varphi_i : X \to X_i$ are called projections.*

We shall denote the inverse limit of the inverse system $\{X_i, \varphi_{ij}, I\}$ by $\varprojlim_{i \in I} X_i$, $\varprojlim X_i$, $(\varprojlim X_i, \varphi_i)$ or (X, φ_i).

2. Some Properties of Inverse Limit of BE-algebras

Theorem 4. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras over a directed poset I. Then (X, φ_i) is an inverse limit of the inverse system $\{X_i, \varphi_{ij}, I\}$, where $X = \left\{(x_i) \in \prod_{i\in I} X_i \mid \text{ for all } i, j \in I \text{ such that } i \geq j, \varphi_{ij}(x_i) = x_j\right\}$ and $\varphi_i : X \to X_i$ is the restriction of the natural projection $\rho_i : \prod_{i\in I} X_i \to X_i$, that is, $\varphi_i = \rho_{i|_X}$.*

Proof. Let $(x_i), (y_i) \in X$. Then $\varphi_{ij}(x_i) = x_j$ and $\varphi_{ij}(y_i) = y_j$ whenever $i \geq j$. Thus, $\varphi_{ij}(x_i * y_i) = \varphi_{ij}(x_i) * \varphi_{ij}(y_i) = x_j * y_j$ whenever $i \geq j$. Since $(x_i) * (y_i) = (x_i * y_i)$, $(x_i) * (y_i) \in X$. Hence, X is a subalgebra of $\prod_{i\in I} X_i$. Let $\varphi_i : X \to X_i$ be the restriction of the natural projection $\rho_i : \prod_{i\in I} X_i \to X_i$. We will show that the BE-algebra X together with φ_i is the inverse limit of $\{X_i, \varphi_{ij}, I\}$.

Let $(x_i) \in X$. Then $\varphi_{ij}\varphi_i((x_i)) = \varphi_{ij}(x_i) = x_j = \varphi_j((x_i))$ whenever $i \geq j$. Thus, $\varphi_{ij}\varphi_i = \varphi_j$ whenever $i \geq j$. This implies that φ_i's are compatible. Let Y be a BE-algebra and $\{\psi_i : Y \to X_i\}$ be a set of compatible homomorphisms. Consider the mapping $\psi : Y \to X$ defined by $\psi(y) = (\psi_i(y))$ for each $y \in Y$. Since ψ_i's are compatible, $\varphi_{ij}\psi_i = \psi_j$ whenever $i \geq j$. Thus, $\varphi_{ij}(\psi_i(y)) = \psi_j(y)$ whenever $i \geq j$ and for $y \in Y$. Hence, $(\psi_i(y)) \in X$. Let $y \in Y$. Then $\varphi_i\psi(y) = \varphi_i((\psi_i(y)) = \psi_i(y)$ for all $i \in I$. Thus, $\varphi_i\psi = \psi_i$ for all $i \in I$. Now, we will show that ψ is unique. Suppose that $\phi : Y \to X$ is another homomorphism such that $\varphi_i\phi = \psi_i$ for all $i \in I$. Suppose further that there exists $y \in Y$ such that $\psi(y) \neq \phi(y)$. By the definition of X, there exists $i \in I$ such that $\varphi_i(\psi(y)) = \psi_i(y) \neq \varphi_i(\phi(y))$. This is a contradiction. Hence, $\psi(y) = \phi(y)$ for all $y \in Y$. Therefore, ψ is unique. Consequently, X is an inverse limit of $\{X_i, \varphi_{ij}, I\}$. □

Let (X, φ_i) be an inverse limit of the inverse system of BE-algebras $\{X_i, \varphi_{ij}, I\}$. By definition, the maps $\varphi_i : X \to X_i$ are compatible. Thus, the universal property of the inverse limit shows that there exists a unique homomorphism $\varphi : X \to X$ such that $\varphi_i\varphi = \varphi_i$ for all $i \in I$. Since $\varphi_i id_X = \varphi_i$ for all $i \in I$ and φ is unique, $\varphi = id_X$. This observation is stated in the following remark.

Remark 1. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras over a directed poset I and let (X, φ_i) be an inverse limit of $\{X_i, \varphi_{ij}, I\}$. Then the homomorphism $id_X : X \to X$ satisfies $\varphi_i id_X = \varphi_i$ for all $i \in I$ and is the only homomorphism with this property.*

Theorem 5. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras over a directed poset I. Then the inverse limit is unique up to isomorphism, that is, if (X, φ_i) and (Y, ψ_i) are two limits of the inverse system $\{X_i, \varphi_{ij}, I\}$, then there is a unique isomorphism $\varphi : X \to Y$ such that $\psi_i\varphi = \varphi_i$ for each $i \in I$.*

Proof. Suppose that (X, φ_i) and (Y, ψ_i) are two inverse limits of the inverse system $\{X_i, \varphi_{ij}, I\}$. Since the maps $\psi_i : Y \to X_i$ are compatible, the universal property of the inverse limit (X, φ_i) shows that there exists a unique homomorphism $\psi : Y \to X$ such that $\varphi_i\psi = \psi_i$ for all $i \in I$. Similarly, there exists a unique homomorphism $\varphi : Y \to X$ such that $\psi_i\varphi = \varphi_i$ for all $i \in I$. It follows that $\psi_i = \varphi_i\psi = \psi_i\varphi\psi$ for all $i \in I$. Thus, by Remark 1, $\varphi\psi = id_Y$. Similarly, $\psi\varphi = id_X$. Therefore, φ is an isomorphism. □

Note that if S is a subalgebra of a transitive BE-algebra X, then for all $x, y, z \in S$, $y * z \leq (x * y) * (x * z)$. Thus, S is also transitive. Also, in [4], the direct product of transitive BE-algebras is transitive and every commutative and self-distributive BE-algebra is transitive. Thus, we have the following result.

Proposition 2. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras. If each X_i is transitive (resp. commutative, self-distributive) for all $i \in I$, then $\varprojlim X_i$ is transitive (resp. commutative, self-distributive).*

Proposition 3. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras and let $X = \varprojlim X_i$ be its corresponding inverse limit. Suppose that $i_0 \in I$ such that $i_0 \geq i_1, \ldots, i_t$ and $\varphi_{i_0 i_k}(A_{i_0}) \subseteq A_{i_k}$ where $A_{i_k} \subseteq X_{i_k}$ for all $k = 0, 1, \ldots, t$. Then*

$$X \cap \left[\left(\prod_{i \neq i_0} X_i\right) \times A_{i_0}\right] = X \cap \left[\left(\prod_{i \neq i_0, \ldots, i_t} X_i\right) \times A_{i_0} \times \cdots \times A_{i_t}\right].$$

Proof. Let $(x_i) \in X \cap \left[\left(\prod_{i \neq i_0} X_i\right) \times A_{i_0}\right]$. Then $(x_i) \in X$ and $(x_i) \in \left(\prod_{i \neq i_0} X_i\right) \times A_{i_0}$. Thus, $\varphi_{ij}(x_i) = x_j$ for all $i \geq j$ and $x_{i_0} \in A_{i_0}$. Hence, $\varphi_{i_0 j}(x_{i_0}) = x_j$ for all $j \leq i_0$. Since $i_0 \geq i_1, \ldots, i_t$, $\varphi_{i_0 i_k}(x_{i_0}) = x_{i_k}$ for all $k = 1, \ldots, t$. Since $\varphi_{i_0 i_k}(A_{i_0}) \subseteq A_{i_k}$ for all $k = 1, \ldots, t$, we have $x_{i_k} \in A_{i_k}$ for all $k = 1, \ldots, t$. This implies that $(x_i) \in \left(\prod_{i \neq i_0, \ldots, i_t} X_i\right) \times A_{i_0} \times \cdots \times A_{i_t}$. Since $(x_i) \in X$, we get $(x_i) \in X \cap \left[\left(\prod_{i \neq i_0, \ldots, i_t} X_i\right) \times A_{i_0} \times \cdots \times A_{i_t}\right]$. The other inclusion is easy to show. $\square$

Corollary 1. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras and let $X = \varprojlim X_i$ be its corresponding inverse limit. Then*

$$X \cap \left[\left(\prod_{i \neq i_0} X_i\right) \times \{1_{X_{i_0}}\}\right] = X \cap \left[\left(\prod_{i \neq i_0, \ldots, i_t} X_i\right) \times \{1_{X_{i_0}}\} \times \cdots \times \{1_{X_{i_t}}\}\right]$$

where $i_0 \geq i_1, \ldots, i_t$.

Proof. Since $\varphi_{i_0 i_k}$ is a homomorphism for all $i_0 \geq i_1, \ldots, i_t$ and for all $k = 1, \ldots, t$, by Theorem 1, $\varphi_{i_0 i_k}(1_{X_{i_0}}) = 1_{X_{i_k}}$ for all $k = 1, \ldots, t$. Thus, $\varphi_{i_0 i_k}(\{1_{X_{i_0}}\}) = \{1_{X_{i_k}}\}$ for all $i_0 \geq i_1, \ldots, i_t$ and for all $k = 1, \ldots, t$. By Proposition 3, the result holds. $\square$

Theorem 6. *Let $\{X_i, \varphi_{ij}, I\}$ be an inverse system of BE-algebras and let $(X = \varprojlim X_i, \varphi_i)$ be its corresponding inverse limit where $\varphi_i : X \to X_i$ is the restriction of the natural projection $\rho_i : \prod_{i \in I} X_i \to X_i$. Then for all $j \in I$, $X \cap \left[\left(\prod_{i \neq j} X_i\right) \times \{1_{X_j}\}\right] = Ker\varphi_j$.*

Proof. For any $j \in I$, we have

$$\begin{aligned} Ker\varphi_j &= \{(x_i) \in X \mid \varphi_j((x_i)) = \rho_j((x_i)) = x_j = 1_{X_j}\} \\ &= \{(x_i) \in X \mid x_j = 1_{X_j}\} \\ &= X \cap \left[\left(\prod_{i \neq j} X_i \right) \times \{1_{X_j}\} \right]. \end{aligned}$$

□

3. Completion of a BE-algebra

Proposition 4. *Let X be a BE-algebra and I be the set of congruences of X. Then $(I, \leq)$ is a directed poset where the binary operation $\leq$ on I is defined by $\theta \leq \phi$ whenever $\phi \subseteq \theta$ for all $\theta, \phi \in I$.*

Theorem 7. *Let X be a BE-algebra and I be the set of congruences of X. Then $\{X/\phi, \varphi_{\phi\theta}, I\}$ is an inverse system where $\theta \leq \phi$ whenever $\phi \subseteq \theta$ for all $\theta, \phi \in I$ and $\varphi_{\phi\theta} : X/\phi \to X/\theta$ is the epimorphism defined by $\varphi_{\phi\theta}([x]_\phi) = [x]_\theta$.*

Proof. By Proposition 4, $(I, \leq)$ is a directed poset. We will show that $\{X/\phi, \varphi_{\phi\theta}, I\}$ is an inverse system, that is, $\varphi_{\theta\lambda}\varphi_{\phi\theta} = \varphi_{\phi\lambda}$ whenever $\phi \geq \theta \geq \lambda$. Now, $\varphi_{\theta\lambda}\varphi_{\phi\theta}([x]_\phi) = \varphi_{\theta\lambda}(\varphi_{\phi\theta}([x]_\phi)) = \varphi_{\theta\lambda}([x]_\theta) = [x]_\lambda = \varphi_{\phi\lambda}([x]_\phi)$. Therefore, $\varphi_{\theta\lambda}\varphi_{\phi\theta} = \varphi_{\phi\lambda}$ whenever $\phi \geq \theta \geq \lambda$. Note that $\varphi_{\theta\theta} : X/\theta \to X/\theta$ is defined by $\varphi_{\theta\theta}([x]_\theta) = [x]_\theta$. Thus, $\varphi_{\theta\theta} = id_{X/\theta}$. Therefore, $\{X/\phi, \varphi_{\phi\theta}, I\}$ is an inverse system. □

Theorem 8. *Let X be a BE-algebra and I be the set of congruences on X. Consider the inverse system $\{X/\phi, \varphi_{\phi\theta}, I\}$. Then*

$$\hat{X} = \left\{ ([x]_\phi) \in \prod_{\phi \in I} X/\phi \mid \text{ for all } \phi, \theta \in I \text{ such that } \phi \geq \theta, \varphi_{\phi\theta}([x]_\phi) = [x]_\theta \right\}$$

together with the projections $\varphi_\phi : \hat{X} \to X/\phi$ for all $\phi \in I$ is an inverse limit of $\{X/\phi, \varphi_{\phi\theta}, I\}$.

Proof. The result follows from Theorem 4.

We call $\hat{X}$ of Theorem 8 as the *completion* of the BE-algebra X.

Lemma 1. *Let X be a BE-algebra and $\{X/\phi, \varphi_{\phi\theta}, I\}$ be the defining inverse system of the completion $\hat{X}$ of X. Then the canonical epimorphisms $\varphi_\theta : X \to X/\theta$ are compatible where $\theta \in I$.*

Proof. Let $x \in X$. Then $\varphi_{\phi\theta}(\varphi_\phi(x)) = \varphi_{\phi\theta}([x]_\phi) = [x]_\theta = \varphi_\theta(x)$ whenever $\phi \geq \theta$. Therefore, the canonical epimorphisms φ_θ's are compatible. □

Theorem 9. *Let X be a BE-algebra and $\{X/\phi, \varphi_{\phi\theta}, I\}$ be the defining inverse system of the completion $\hat{X}$ of X. Then the canonical epimorphisms $\varphi_\theta : X \to X/\theta$ induce a homomorphism $\gamma : X \to \hat{X}$ defined by $\gamma(x) = (\varphi_\theta(x)) = ([x]_\theta)$.*

Proof. By Lemma 1, $\{\varphi_\theta : X \to X/\theta\}$ is a set of compatible homomorphism. Since X is a BE-algebra and $\hat{X} = \varprojlim X/\theta$, by Definition 4 and from the proof of Theorem 4, there exists a homomorphism $\gamma : X \to \hat{X}$ with $\gamma(x) = (\varphi_\theta(x)) = ([x]_\theta)$ for each $x \in X$. □

Theorem 10. *Let X be a BE-algebra and $\{X/\phi, \varphi_{\phi\theta}, I\}$ be the defining inverse system of the completion $\hat{X}$ of X. Then $\gamma^{-1}(([x]_\theta)) = \bigcap_{\theta\in I}[x]_\theta$ for all $([x]_\theta) \in \hat{X}$.*

Proof. Let $y \in \gamma^{-1}(([x]_\theta))$. Then $\gamma(y) = ([x]_\theta)$. By the definition of γ, $\gamma(y) = ([y]_\theta)$. Thus, $([y]_\theta) = ([x]_\theta)$. Hence, $[y]_\theta = [x]_\theta$ for all $\theta \in I$. This implies that $y \in \bigcap_{\theta\in I}[x]_\theta$. It follows that $\gamma^{-1}(([x]_\theta)) \subseteq \bigcap_{\theta\in I}[x]_\theta$.

Let $y \in \bigcap_{\theta\in I}[x]_\theta$. Then $y \in [x]_\theta$ for all $\theta \in I$. Thus, $[y]_\theta = [x]_\theta$ for all $\theta \in I$. Hence, $([y]_\theta) = ([x]_\theta)$. Since $\gamma(y) = ([y]_\theta)$, it follows that $\gamma(y) = ([x]_\theta)$. Hence, $y \in \gamma^{-1}(([x]_\theta))$. Therefore, $\bigcap_{\theta\in I}[x]_\theta \subseteq \gamma^{-1}(([x]_\theta))$. □

Suppose X is commutative BE-algebra. By Theorem 3, there is a one to one correspondence between the set of congruences of X and the set of filters of X. For every congruence θ, there is a corresponding $\equiv^F$, where F is a filter, such that $\theta = \equiv^F$. With this, we can just write every congruence θ as a filter F in X.

Theorem 11. *Let X be a commutative BE-algebra and let $\{F_i \mid i \in I\}$ be the family of filters of X such that $F_i \subseteq F_j$ whenever $i \geq j$. Then $Ker\gamma = \bigcap_{i\in I} F_i$.*

Proof. Let X be a commutative BE-algebra. Then

$$\begin{aligned} Ker\gamma &= \{x \in X \mid \gamma(x) = ([1_X]_{F_i})\ \forall i \in I\} \\ &= \{x \in X \mid ([x]_{F_i}) = ([1_X]_{F_i})\ \forall i \in I\} \\ &= \{x \in X \mid x \in [1_X]_{F_i} = F_i\ \forall i \in I\} \\ &= \bigcap_{i\in I} F_i. \end{aligned}$$

□

Lemma 2. *Let X be a BE-algebra. If F and G are normal filters of X, then $F \cap G$ is also a normal filter of X.*

Theorem 12. *Let X be a BE-algebra and let $\mathcal{I}$ be a family of normal filters of X such that $F \cap G \in \mathcal{I}$ for every $F, G \in \mathcal{I}$. Then $\{X/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system of BE-algebras where $\varphi_{FG} : X/F \to X/G$ is the epimorphism defined by $\varphi_{FG}([x]_F) = ([x]_G)$ whenever $F \subseteq G$.*

Proof. We will denote by $F \geq G$ whenever $F \subseteq G$. We will show that $(\mathcal{I}, \leq)$ is a directed poset. Let $F, G, H \in \mathcal{I}$. Since $F \subseteq F$ for all $F \in \mathcal{I}$, $F \leq F$ for all $F \in \mathcal{I}$. Suppose that $F \leq G$ and $G \leq H$. Then $G \subseteq F$ and $H \subseteq G$. By set inclusion, $H \subseteq F$. Hence, $F \leq H$. Suppose that $F \leq G$ and $G \leq F$. Then $G \subseteq F$ and $F \subseteq G$. By set inclusion, $F = G$. Note that $F \cap G \subseteq F$, $F \cap G \subseteq G$ and $F \cap G \in \mathcal{I}$. Thus, $F \leq F \cap G$ and $G \leq F \cap G$. Hence, for every $F, G \in \mathcal{I}$, there exists $H = F \cap G \in \mathcal{I}$ such that $F, G \leq H$. Therefore, $(\mathcal{I}, \leq)$ is a directed poset. We will show that $\{X/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system, that is, $\varphi_{GH}\varphi_{FG} = \varphi_{FH}$ whenever $F \geq G \geq H$. Now, $\varphi_{GH}\varphi_{FG}([x]_F) = \varphi_{GH}(\varphi_{FG}([x]_F)) = \varphi_{GH}([x]_G) = [x]_H = \varphi_{FH}([x]_F)$. Therefore, $\varphi_{GH}\varphi_{FG} = \varphi_{FH}$ whenever $F \geq G \geq H$. Note that $\varphi_{FF} : X/F \to X/F$ is defined by $\varphi_{FF}([x]_F) = [x]_F$. Thus, $\varphi_{FF} = id_{X/F}$. Therefore, $\{X/F, \varphi_{FG}, \mathcal{I}\}$ is an inverse system. □

If $\mathcal{I}$ is the family of all normal filters of X such that X/F is finite for all $F \in \mathcal{I}$, then we call the inverse limit of the inverse system $\{X/F, \varphi_{FG}, \mathcal{I}\}$ the *normal completion* of X.

Example 1. *Consider the commutative BE-algebra $X = \{1, a, b, c, \}$ with the operation $*$ defined by the Cayley table shown below.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

By Theorem 3, there is a bijection between the congruence relations and filters of X. Thus, the set I of congruence relations of X is completely determined by the set of all filters of X. The filters of X are $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$, $F_4 = \{1, c\}$, $F_5 = \{1, a, b\}$, $F_6 = \{1, a, c\}$, $F_7 = \{1, b, c\}$ and $F_8 = X$. Since X is commutative, by Theorem 2, X is transitive. By Proposition 1, all filters of X are normal. Hence, $\equiv^{F_i}$ is a congruence relation on X where $i = 1, \ldots, 8$. Thus, $I = \{\equiv^{F_i} \mid i = 1, \ldots, 8\}$. We will just denote $\equiv^{F_i}$ by F_i for all $i \in I$. Note that $\equiv^{F_i} \subseteq \equiv^{F_j}$ if $F_i \subseteq Fj$. Now,

$$X/F_1 = \{[1]_{F_1}, [a]_{F_1}, [b]_{F_1}, [c]_{F_1}\} \text{ where } [1]_{F_1} = \{1\}, [a]_{F_1} = \{a\}, [b]_{F_1} = \{b\} \text{ and } [c]_{F_1} = \{c\};$$

$$X/F_2 = \{[1]_{F_2}, [b]_{F_2}, [c]_{F_2}\} \text{ where } [1]_{F_2} = F_2, [b]_{F_2} = \{b\} \text{ and } [c]_{F_2} = \{c\};$$

$$X/F_3 = \{[1]_{F_3}, [a]_{F_3}, [c]_{F_3}\} \text{ where } [1]_{F_3} = F_3, [a]_{F_3} = \{a\} \text{ and } [c]_{F_3} = \{c\};$$

$$X/F_4 = \{[1]_{F_4}, [a]_{F_4}, [b]_{F_4}\} \text{ where } [1]_{F_4} = F_4, [a]_{F_4} = \{a\} \text{ and } [b]_{F_4} = \{b\};$$

$$X/F_5 = \{[1]_{F_5}, [c]_{F_5}\} \text{ where } [1]_{F_5} = F_5 \text{ and } [c]_{F_5} = \{c\};$$

$$X/F_6 = \{[1]_{F_6}, [b]_{F_6}\} \text{ where } [1]_{F_6} = F_6 \text{ and } [b]_{F_6} = \{b\};$$

$$X/F_7 = \{[1]_{F_7}, [a]_{F_7}\} \text{ where } [1]_{F_7} = F_7 \text{ and } [a]_{F_7} = \{a\}; \text{ and}$$

$X/F_8 = \{[1]_{F_8}\}$ *where* $[1]_{F_8} = X$.

Therefore, $\{X/F_i, \varphi_{F_iF_j}, I\}$ *is an inverse system of BE-algebras where* $\varphi_{F_iF_j} : X/F_i \to X/F_j$ *is the canonical epimorphism whenever* $F_i \geq F_j$*, that is,* $F_i \subseteq F_j$*. In particular,* $\{X/F_i, \varphi_{F_iF_j}, I\}$ *is a normal completion of* X*.*

Example 2. *Consider the inverse system* $\{X/F_i, \varphi_{F_iF_j}, I\}$ *in Example 1. Then* $\varprojlim X/F_i = \{([x]_{F_i}) \in \prod_{i=1}^{8} X/F_i \mid \forall\ F_i, F_j \in I \text{ such that } F_i \geq F_j, \varphi_{F_iF_j}([x]_{F_i}) = [x]_{F_j}\}$*. Now, the elements*

$$\begin{aligned}
\alpha_1 &= ([1]_{F_1}, [1]_{F_2}, [1]_{F_3}, [1]_{F_4}, [1]_{F_5}, [1]_{F_6}, [1]_{F_7}, [1]_{F_8}),\\
\alpha_2 &= ([a]_{F_1}, [1]_{F_2}, [a]_{F_3}, [a]_{F_4}, [1]_{F_5}, [1]_{F_6}, [a]_{F_7}, [1]_{F_8}),\\
\alpha_3 &= ([b]_{F_1}, [b]_{F_2}, [1]_{F_3}, [b]_{F_4}, [1]_{F_5}, [b]_{F_6}, [1]_{F_7}, [1]_{F_8}), \text{ and}\\
\alpha_4 &= ([c]_{F_1}, [c]_{F_2}, [c]_{F_3}, [1]_{F_4}, [c]_{F_5}, [1]_{F_6}, [1]_{F_7}, [1]_{F_8})
\end{aligned}$$

of $\prod_{i=1}^{8} X/F_i$ *are the only elements that satisfies the condition of* $\varprojlim X/F_i$*. Thus,* $\varprojlim X/F_i = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$*.*

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