



A Topology on a Hyper BCI-algebra Generated by a Hyper-order

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Abstract. In this paper, we introduce an operator on a hyper BCI-algebra via application of a left hyper-order. The family consisting of the images of subsets under the operator turns out to be a base for some topology on the hyper BCI-algebra. We investigate some important properties of the induced topology on certain hyper BCI-algebras. In particular, we show that the generated topology on a non-trivial hyper subalgebra of an ordered hyper BCI-algebra coincides with the relative topology on this hyper subalgebra.

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1. Introduction

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [3] introduced the notion of a BCI-algebra which is a generalization of BCK-algebra. R. A. Alo and E. Y. Deeba [1] attempted to study the topological aspects of the BCK-structures. They studied and investigated various topologies on BCK-algebras analogous to that which had already been studied on lattices. In [4], Y. B. Jun et al. initiated the study of topological BCI-algebras (briefly, TBCI-algebras). In their study a BCI-algebra $(H, *, 0)$ is furnished with a topology in such a way that the associated operation $*$: $H \times H \rightarrow H$ of the BCI-algebra is continuous, where the Cartesian product $H \times H$ is furnished with the product topology.

During the 8th Congress of Scandinavian Mathematicians, F. Marty [6] introduced the theory of hyperstructure (sometimes called multialgebras). Following its introduction, various algebraic hyperstructures have been defined and many important results have

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appeared. Some recent studies on hyperstructures are on soft hypervector spaces and hyper-deductive systems done by Muhiuddin et al. in [8], [9], and [10]. As one may find, these hyperstructures have many applications in both pure and applied sciences.

In [5], Y.B. Jun et al. introduced and studied the concept of a hyper BCK-algebra. In [7], Muhiuddin et al. studied fuzzy soft hyper BCK-ideals in hyper BCK-algebras. In [13], Xin applied hyperstructures to BCI-algebras giving rise to the the concept of a hyper BCI-algebra. A study on a graph induced by a hyper BCI-algebra is done in [11].

Previous studies on the topological aspects of certain algebraic hyperstructures motivated us to study the topological structure of a hyper BCI-algebra when it carries a topology other the one considered in earlier studies. In this study, we purposely use the hyper-order associated with the hyperstructure to topologize it. Specifically, we topologize a given hyper BCI-algebra by considering a family of subsets which will form a base for some topology on the hyper BCI-algebra. These subsets are generated via left application of the hyper-order associated with the hyper BCI-algebra. Topological properties of the resulting space are investigated in various aspects. In particular, we show that the topology generated on a non-trivial hyper subalgebra of an ordered hyper BCI-algebra coincides with the relative (subspace) topology.

2. Preliminaries

A *hyperoperation* on a nonempty set H is a map from $H \times H$ into the nonempty subsets of H , $P^*(H) = P(H) \setminus \{\emptyset\}$. Let \otimes be a hyperoperation on H and $(x, y) \in H \times H$. Then its image under \otimes , denoted by $x \otimes y$, is called the *hyperproduct* of x and y . If A and B are nonempty subsets of H , then $A * B$ is given by $A \otimes B = \bigcup_{a \in A, b \in B} a \otimes b$. We shall use $x \otimes y$

instead of $x \otimes \{y\}$, $\{x\} \otimes y$, or $\{x\} \otimes \{y\}$. When $A \subseteq H$ and $x \in H$, we agree to write $A \otimes x$ instead of $A \otimes \{x\}$. Similarly, we write $x \otimes A$ for $\{x\} \otimes A$. In effect, $A \otimes x = \bigcup_{a \in A} a \otimes x$ and

$$x \otimes A = \bigcup_{a \in A} x \otimes a.$$

A *hyper BCI-algebra* $(H, \otimes, 0)$ (see [5]) is a nonempty set H endowed with a hyperoperation “ \otimes ” and a constant 0 such that: for all $x, y, z \in H$,

$$(B_1) ((x \otimes z) \otimes (y \otimes z)) \ll x \otimes y, \quad (B_4) x \ll y \text{ and } y \ll x \text{ imply } x = y,$$

$$(B_2) (x \otimes y) \otimes z = (x \otimes z) \otimes y,$$

$$(B_3) x \ll x, \quad (B_5) 0 \otimes (0 \otimes x) \ll x, x \neq 0,$$

where for every $A, B \subseteq H$, $A \ll B$ if and only if for each $a \in A$, there exists $b \in B$ such that $0 \in a \otimes b$. In particular, for every $x, y \in H$, $x \ll y$ if and only if $0 \in x \otimes y$. In such case, we call “ \ll ” the *hyper-order* in H . A hyper BCI-algebra $(H, \otimes, 0)$ is said to be *ordered* if for $x, y, z \in H$, $x \ll y$ and $y \ll z$ implies $x \ll z$.

All throughout, we denote a hyper BCI-algebra $(H, \otimes, 0)$ by H , unless otherwise specified.

Let H be a hyper BCI-algebra and $A \subseteq H$. In [11], the set $L_H(A)$ is given by $L_H(A) = \{x \in H \mid x \ll a, \forall a \in A\} = \{x \in H \mid 0 \in x \circledast a, \forall a \in A\}$. If $A = \{a\}$, we write $L_H(\{a\}) = L_H(a)$. An element a of H is called a *hyperatom* if for each $x \in H$, $x \ll a$ implies $x = 0$ or $x = a$. Denote by $A(H)$ the set of all hyperatoms of H , and by $A^*(H)$ the set of all nonzero hyperatoms of H ; that is, $A^*(H) = A(H) \setminus \{0\}$. H is said to be *hyperatomic* if each element of H is a hyperatom, that is, $A(H) = H$. It is shown in [11] that H is hyperatomic if and only if $L_H(x) = \{x\}$ or $L_H(x) = \{0, x\}$ for each $x \in H$.

3. Results

The following result gives some properties of the operator L_H .

Proposition 1. [11] *Let A and B be subsets of H . Then the following hold:*

- (i) $L_H(\emptyset) = H$
- (ii) $L_H(\{0\}) = \{0\}$
- (iii) If $A \subseteq B$, then $L_H(B) \subseteq L_H(A)$.
- (iv) $L_H(A) = \bigcap_{a \in A} L_H(\{a\})$
- (v) If $x \in H$, then $x \in L_H(\{x\})$. Furthermore, $L_H(\{x\}) = \{0\}$ if and only if $x = 0$.

Theorem 1. [2] *Let (X, τ) be a topological space and (Y, τ_Y) be a subspace. If $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a basis (subbasis) for τ , $\{Y \cap U_\alpha \mid \alpha \in \mathcal{A}\}$ is a basis (subbasis) for τ_Y .*

Lemma 1. *Let $\{A_\alpha : \alpha \in I\}$ be a collection of subsets of a hyper BCI-algebra H . Then*

$$\bigcap_{\alpha \in I} L_H(A_\alpha) = L_H\left(\bigcup_{\alpha \in I} A_\alpha\right).$$

Proof. If $\bigcap_{\alpha \in I} L_H(A_\alpha) = \emptyset$, then by Proposition 1(iii), $L_H\left(\bigcup_{\alpha \in I} A_\alpha\right) \subseteq \bigcap_{\alpha \in I} L_H(A_\alpha) = \emptyset$. Thus, $L_H\left(\bigcup_{\alpha \in I} A_\alpha\right) = \emptyset$. If $\bigcap_{\alpha \in I} L_H(A_\alpha) \neq \emptyset$, then

$$\begin{aligned} x \in \bigcap_{\alpha \in I} L_H(A_\alpha) &\Leftrightarrow x \in L_H(A_\alpha) \text{ for all } \alpha \in I \\ &\Leftrightarrow x \ll a \text{ for all } a \in A_\alpha \text{ and for all } \alpha \in I \\ &\Leftrightarrow x \ll a \text{ for all } a \in \bigcup_{\alpha \in I} A_\alpha \\ &\Leftrightarrow x \in L_H\left(\bigcup_{\alpha \in I} A_\alpha\right). \end{aligned}$$

This proves the assertion. □

Theorem 2. *Let H be a hyper BCI-algebra. Then the family $\mathcal{B}_L(H) = \{L_H(A) : \emptyset \neq A \subseteq H\}$ is a basis for some topology on H .*

Proof. Clearly, $H = \bigcup_{a \in H} L_H(a)$. Let A and B be nonempty subsets of H . Then by Lemma 1, $L_H(A) \cap L_H(B) = L_H(A \cup B) \in \mathcal{B}_L(H)$. Therefore, $\mathcal{B}_L(H)$ is a basis for some topology on H . \square

Denote by $\tau_L(H)$ the topology generated by $\mathcal{B}_L(H)$.

Example 1. *Consider $H := [0, \infty)$ with the hyperoperation “ \otimes ”, defined in [13]:*

$$x \otimes y := \begin{cases} [0, x], & \text{if } x \leq y, \\ (0, y], & \text{if } x > y \neq 0, \\ \{x\}, & \text{if } y = 0 \end{cases}$$

for all $x, y \in H$. Then $(H, \otimes, 0)$ is a hyper BCI-algebra. Now, let $k \in H$. Then $L_H(k) = [0, k]$. Let $\emptyset \neq A \subseteq H$ and let $p = \inf A$. Since $L_H(A) = \bigcap_{a \in A} L_H(a) = L_H(p)$, it follows

that $L_H(A) = [0, p] = L_H(p)$. Let $\emptyset \neq G \in \tau_L(H)$. Then $G = \bigcup_{p \in \mathcal{K}} L_H(p)$, where $\mathcal{K} \subseteq H$.

Suppose first that $|G| < \infty$ and let $q = \sup G$. Then $G = L_H(q)$. Suppose $q > 0$. Then $G = L_H(q) = [0, q]$, a contradiction. Thus, $q = 0$, that is, $G = L_H(0) = \{0\}$. Next, suppose that G is an infinite set. If \mathcal{K} is infinite, then $G = \bigcup_{p \in \mathcal{K}} [0, p] = H$. Suppose

\mathcal{K} is finite. Since G is infinite, $0 < m = \max \mathcal{K}$. Hence, $G = [0, m]$. Consequently, $\tau_L(H) = \{\emptyset, H\} \cup \{[0, p] : p \in H\}$.

Example 2. *Consider $H = \{0, a, b\}$ with the hyperoperation “ \otimes ” defined as follows:*

\otimes	0	a	b
0	$\{0, a\}$	$\{0, a\}$	$\{b\}$
a	$\{a\}$	$\{0, a\}$	$\{b\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

Then H is a hyper BCI-algebra. By Theorem 2, $\mathcal{B}_L(H) = \{L_H(A) : \emptyset \neq A \subseteq H\} = \{\{0\}, \{0, a\}, \{b\}, \emptyset\}$. Thus, $\tau_L(H) = \{\{0\}, \{0, a\}, \{b\}, \{0, b\}, \emptyset, H\}$.

Observe that in Example 1, $(H, \tau_L(H))$ is connected, however, in Example 2, $H = \{0, a\} \cup \{b\}$. Hence, $(H, \tau_L(H))$ is disconnected.

Lemma 2. *Let H be an ordered hyper BCI-algebra and let $x \in H$. If $z \in L_H(x)$, then $L_H(z) \subseteq L_H(x)$.*

Proof. Suppose that $z \in L_H(x)$ and let $w \in L_H(z)$. Then $w \ll z$. Since $z \ll x$ and H is ordered, $w \ll x$; that is, $w \in L_H(x)$. Therefore, $L_H(z) \subseteq L_H(x)$. \square

An ordered hyper BCI-algebra H is said to be L_H -0 hereditary if $0 \in L_H(z)$ for all $z \in L_H(x)$ whenever $x \in H$ with $0 \in L_H(x)$.

Example 3. The hyper BCI-algebra H in Example 2 is L_H -0 hereditary.

Theorem 3. Let H be an L_H -0 hereditary hyper BCI-algebra. Then $(H, \tau_L(H))$ is connected if and only if $0 \in L_H(x)$ for all $x \in H$.

Proof. Suppose that $(H, \tau_L(H))$ is connected and suppose that there exists $x \in H \setminus \{0\}$ such that $0 \notin L_H(x)$. Set $D_1 = \{z \in H : 0 \notin L_H(z)\}$ and $D_2 = H \setminus D_1$. Since $0 \notin L_H(x)$ and $0 \in L_H(0)$, $D_1 \neq \emptyset$ and $D_2 \neq \emptyset$. Let $z \in D_1$ and let $w \in L_H(z)$. Then $L_H(w) \subseteq L_H(z)$ by Lemma 2. Since $0 \notin L_H(z)$, $0 \notin L_H(w)$; that is, $w \in D_1$. Thus, $z \in L_H(z) \subseteq D_1$ and so, D_1 is $\tau_L(H)$ -open. Next, let $y \in D_2$ and let $v \in L_H(y)$. Since $0 \in L_H(y)$ and H is L_H -0 hereditary, it follows that $0 \in L_H(x)$; that is, $v \in D_2$. Hence, $y \in L_H(y) \subseteq D_2$ and so, D_2 is $\tau_L(H)$ -open. Since $D_1 \cap D_2 = \emptyset$ and $D_1 \cup D_2 = H$, the space is disconnected, contrary to our assumption.

For the converse, let G be a non-empty open subset of H . Then there exists $A \subseteq H$ such that $L_H(A) \subseteq G$. Since $0 \in L_H(x)$ for all $x \in H$, $0 \in L_H(A)$. Thus, $0 \in G$. It follows that $(H, \tau_L(H))$ is connected. □

The next result follows from Theorem 2 and the definition of discrete topology.

Proposition 2. Let H be a hyper BCI-algebra. Then $\tau_L(H)$ is the discrete topology \mathcal{D} on H if and only if for each $x \in H$, there exists $A_x \subseteq H$ such that $L_H(A_x) = \{x\}$.

Corollary 1. Let H be a hyper BCI-algebra. If $L_H(x) = \{x\}$ for each $x \in H$, then $\tau_L(H)$ is the discrete topology \mathcal{D} on H . In particular, $\mathcal{B}_L(H) = \{\{a\} : a \in H\}$.

Proof. Suppose that for each $x \in H$, $L_H(x) = \{x\}$. Then by Proposition 2, $\tau_L(H)$ is the discrete topology \mathcal{D} on H . Furthermore, for any $A \subseteq H$ with $|A| \geq 2$, $L_H(A) = \emptyset$. Therefore, $\mathcal{B}_L(H) = \{\{a\} : a \in H\}$. □

Example 4. Consider $H = \{0, a, b\}$ with the hyperoperation “ \otimes ” defined as follows:

\otimes	0	a	b
0	{0}	{b}	{a}
a	{a}	{0}	{b}
b	{b}	{a}	{0}

Then H is a hyper BCI-algebra. By Theorem 2,

$$\mathcal{B}_L(H) = \{L_H(A) : \emptyset \neq A \subseteq H\} = \{\{0\}, \{a\}, \{b\}, \emptyset\}.$$

Thus, $\tau_L(H) = \{\{0\}, \{a\}, \{b\}, \{0, a\}, \{0, b\}, \{a, b\}, \emptyset, H\} = \mathcal{D}$.

Theorem 4. If H is a finite hyper BCI-algebra, then the family $\mathcal{S}_L(H) = \{L_H(a) : a \in H\}$ is a subbase of $\tau_L(H)$.

Proof. That $\mathcal{S}_L(H) \subseteq \tau_L(H)$ is evident. Since $L_H(A) = \bigcap_{a \in A} L_H(\{a\})$ for each nonempty $A \subseteq H$, it follows that every element of $\mathcal{B}_L(H)$ is a finite intersection of members of $\mathcal{S}_L(H)$. Hence, $\mathcal{S}_L(H)$ is a subbase of $\tau_L(H)$. □

Proposition 3. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then*

$$\mathcal{B}_L(H) = \{\{a\} : a \in A^*(H), 0 \notin L_H(a)\} \cup \{\{0, a\} : a \in A^*(H), 0 \in L_H(a)\} \\ \cup \{L_H(A) : A \cap A^*(H) = \emptyset\}.$$

Proof. For each $a \in A^*(H)$, either $L_H(a) = \{a\}$ or $L_H(a) = \{0, a\}$. Let A be a nonempty subset of H such that $A \cap A^*(H) \neq \emptyset$, say $q \in A \cap A^*(H)$. Since $L_H(A) \subseteq L_H(q)$ (by Proposition 1(iii)) and $L_H(q) \in \{\{q\}, \{0, q\}\}$, it follows that $L_H(A) \in \{\{0\}, \{q\}, \{0, q\}\}$. □

Corollary 2. *Let H be a hyper BCI-algebra such that $0 \in L_H(x)$ for each $x \in H \setminus \{0\}$ with $|H| \geq 2$. Then $\mathcal{B}_L(H) = \{\{0, a\} : a \in A^*(H)\} \cup \{L_H(A) : A \cap A^*(H) = \emptyset\}$.*

Corollary 3. *Let H be a hyper BCI-algebra such that $0 \in L_H(x)$ for each $x \in H \setminus \{0\}$ with $|H| \geq 2$. If $A^*(H) = \{a\}$, then $\mathcal{B}_L(H) = \{\{0, a\}\} \cup \{L_H(A) : a \notin A\}$.*

Theorem 5. *Let H be a hyper BCI-algebra with $|H| \geq 2$. Then $\mathcal{B}_L(H) = \{\{0\}\} \cup \{\{a\} : a \in H \setminus \{0\}, 0 \notin L_H(a)\} \cup \{\{0, a\} : a \in H \setminus \{0\}, 0 \in L_H(a)\}$ if and only if H is hyperatomic.*

Proof. Suppose H is hyperatomic. Then for any nonempty subset A of H such that $A \neq \{0\}$, $A \cap A^*(H) \neq \emptyset$. Thus, $\{L_H(A) : A \neq \emptyset \text{ and } A \cap A^*(H) = \emptyset\} = \{\{0\}\}$. The result then follows from Proposition 3.

For the converse, suppose that $\mathcal{B}_L(H)$ is the given family of subsets of H . Let $a \in H \setminus \{0\}$. Then either $L_H(a) = \{a\}$ or $L_H(a) = \{0, a\}$. Hence, if $x \in H$ and $x \ll a$, then either $x = a$ or $x = 0$. Thus, $a \in A(H)$. Therefore, H is hyperatomic. □

Example 5. Refer to Example 2. It is easy to verify that H is hyperatomic.

Corollary 4. *Let H be a hyper BCI-algebra such that $0 \in L_H(x)$ for each $x \in H$ with $|H| \geq 2$. Then $\mathcal{B}_L(H) = \{\{0\}\} \cup \{\{0, a\} : a \in H \setminus \{0\}\}$ if and only if H is hyperatomic.*

Theorem 6. *Let H be a hyperatomic hyper BCI-algebra. Then $A \in \tau_L(H)$ if and only if $A = \emptyset$ or $0 \in A$ or $0 \notin L_H(a)$ for all $a \in A$.*

Proof. Let $A \in \tau_L(H) \setminus \{\emptyset\}$ and let $a \in A$. Since $\mathcal{B}_L(H)$ is a basis for $\tau_L(H)$, there exists $B_a \subseteq H$ such that $a \in L_H(B_a) \subseteq A$. Since H is hyperatomic, $L_H(b) = \{b\}$ or $\{0, b\}$ for each $b \in B_a$. If $a = 0$, then $0 \in A$. Suppose that $a \neq 0$ and let $b \in B_a$. Then $b \neq 0$ and $a \in L_H(b)$. Hence, $a = b$; that is, $B_a = \{a\}$. Thus, $L_H(B_a) = L_H(a) = \{a\}$. Therefore, either $0 \in A$ or $0 \notin A$ and $L_H(a) = \{a\}$ for each $a \in A$.

For the converse, suppose first that $0 \notin L_H(a)$ for each $a \in A$. Then $A = \bigcup_{a \in A} L_H(a) \in \tau_L(H)$. Next, suppose that $0 \in A$. Since $L_H(x) = \{x\}$ or $\{0, x\}$ for all $x \in H$, it follows that $L_H(a) \subseteq A$ for all $a \in A$. Thus,

$$A = \left(\bigcup_{\substack{a \in A \\ 0 \in L_H(a)}} L_H(a) \right) \cup \left(\bigcup_{\substack{a \in A \\ 0 \notin L_H(a)}} L_H(a) \right) \in \tau_L(H).$$

This proves the assertion. □

Recall that for a nonempty set X and a fixed $p \in X$, the topology τ_p given by $\tau_p = \{\emptyset\} \cup \{A \subseteq X : p \in A\}$ is called the *particular point p topology* on X (see [12]). The next result gives a characterization of $\tau_L(H)$ involving a particular point topology.

Theorem 7. *Let H be a hyper BCI-algebra such that $L_H(\{x, y\}) = \{0\}$ for every pair of distinct points x and y of H . Then $\tau_L(H)$ is the particular point 0 topology τ_0 on H if and only if H is hyperatomic.*

Proof. Suppose that H is hyperatomic. Then by Corollary 4, $\mathcal{B}_L(H) = \{\{0\}\} \cup \{\{0, a\} : a \in H \setminus \{0\}\}$. Since $\mathcal{B}_L(H)$ is a basis for $\tau_L(H)$,

$$\begin{aligned} A \in \tau_L(H) &\Leftrightarrow A = \emptyset \text{ or } A = \{0\} \text{ or } A = \bigcup_{a \in A} \{0, a\} \\ &\Leftrightarrow A = \emptyset \text{ or } 0 \in A \\ &\Leftrightarrow A \in \tau_0. \end{aligned}$$

Thus, $\tau_L(H) = \tau_0$.

For the converse, suppose that $\tau_L(H) = \tau_0$ and let $x \in H \setminus \{0\}$. Then $\{0, x\} \in \tau_L(H)$. Since $\mathcal{B}_L(H)$ is a basis for $\tau_L(H)$, there exists a subset A of H such that $x \in L_H(A) \subseteq \{0, x\}$. Hence, $L_H(A) = \{x\}$ or $L_H(A) = \{0, x\}$. Now, since $0 \in L_H(a)$ for each $a \in A$, $L_H(A) = \{0, x\}$. If $A = \emptyset$, then by Proposition 1(i), $L_H(A) = H = \{0, x\}$. Hence, H is hyperatomic. If $A \neq \emptyset$, then $|A| = 1$ (otherwise, $L_H(A) = \{0\}$ which is a contradiction). Therefore, since $y \in L_H(y)$ for each $y \in H$, $A = \{x\}$, that is, $L_H(A) = L_H(x) = \{0, x\}$. This shows that H is hyperatomic. □

Remark 1. *The condition $L_H(\{x, y\}) = \{0\}$ for each pair $(x, y) \in H \times H$, where $x \neq y$, cannot be omitted. The hyper BCI-algebra in Example 2 is hyperatomic but does not satisfy this condition. Hence, $\tau_L(H) \neq \tau_0$.*

Theorem 8. *Let H be a hyperatomic hyper BCI-algebra and let $A, F \subseteq H$. Then with respect to $\tau_L(H)$,*

(i)

$$int(A) = \begin{cases} A & \text{if } A = \emptyset \text{ or } 0 \in A \\ & \text{or } 0 \notin L_H(a) \forall a \in A, \\ A \setminus \{a \in A : 0 \in L_H(a)\} & \text{otherwise; and} \end{cases}$$

(ii)

$$\overline{F} = \begin{cases} F & \text{if } 0 \in F \text{ and } 0 \notin L_H(x) \forall x \in H \setminus F \\ & \text{or } 0 \notin F, \\ F \cup \{x \in H \setminus F : 0 \in L_H(x)\} & \text{otherwise.} \end{cases}$$

Proof.

- (i) If $A = \emptyset$ or $0 \in A$ or $0 \notin L_H(a)$ for all $a \in A$, then $A \in \tau_L(H)$ by Theorem 6. Thus, $\text{int}A = A$. Now, suppose $A \notin \tau_L(H)$. Then $A \neq \emptyset$, $0 \notin A$, and there exists $a \in A$ such that $0 \in L_H(a)$ by Theorem 6. Let $B_A = A \setminus \{x \in A : 0 \in L_H(x)\}$. Clearly, $B_A \subsetneq A$. Let $z \in B_A$. Then $0 \notin L_H(z)$. By Theorem 6, $B_A \in \tau_L(H)$.

Next, let $G \in \tau_L(H)$ such that $G \subseteq A$ and let $v \in G$. Since $0 \notin A$, $0 \notin G$. Hence, by Theorem 6, $0 \notin L_H(v)$, that is, $v \in B_A$. Therefore, $\text{int}A = B_A$.

- (ii) Suppose first that $0 \notin F$. Then $0 \in H \setminus F = F^c$; hence $F^c \in \tau_L(H)$ by Theorem 6. If $0 \in F$ and $0 \notin L_H(x)$ for all $x \in F^c$, then by Theorem 6, $F^c \in \tau_L(H)$. Thus, in both cases, F is a $\tau_L(H)$ -closed set. Therefore, $\overline{F} = F$.

Next, suppose that $0 \in F$ and there exists $x \in H \setminus F$ such that $0 \in L_H(x)$. Let $Q = F \cup \{z \in H \setminus F : 0 \in L_H(z)\}$ and let $q \in Q^c$. Then $0 \notin Q^c$ and $0 \notin L_H(q)$. By Theorem 6, $Q^c \in \tau_L(H)$, that is, Q is $\tau_L(H)$ -closed. Now, let $w \in H \setminus F$ such that $0 \notin L_H(w)$. Then $L_H(w) = \{w\}$ is a neighborhood of w with $L_H(w) \cap F = \emptyset$. Thus, $w \notin \overline{F}$. Therefore, the smallest closed set containing F is Q , that is, $\overline{F} = Q$. \square

Theorem 9. *Let H be a hyper BCI-algebra and let $D \subsetneq H$.*

- (i) *If $0 \in L_H(x)$ for all $x \in H$, then D is dense in H if and only if $0 \in D$.*
- (ii) *If H is hyperatomic, then D is dense if and only if $0 \in D$ and $0 \in L_H(x)$ for all $x \in H \setminus D$.*

Proof.

- (i) If D is dense in H , then $L_H(0) \cap D \neq \emptyset$. Hence, $0 \in D$. Next, suppose that $0 \in D$ and $A \subseteq H$ with $L_H(A) \neq \emptyset$. Since $0 \in L_H(a)$ for all $a \in A$, $0 \in L_H(A)$. Thus, $L_H(A) \cap D \neq \emptyset$. Therefore, D is dense in H .
- (ii) Suppose D is dense in H . Then $0 \in D$. Since $D \neq H$, D is not $\tau_L(H)$ -closed (otherwise, $\overline{D} = D \neq H$, a contradiction.) Thus, by Theorem 8 and the assumption that D is dense, $\overline{D} = D \cup \{x \in H \setminus D : 0 \in L_H(x)\} = H$. Therefore, $0 \in L_H(x)$ for all $x \in H \setminus D$. For the converse, suppose that the given conditions hold. By Theorem 8, $\overline{D} = H$. Thus, D is dense in H . \square

Lemma 3. *Let K be a hyper subalgebra of a hyper BCI-algebra H . Then*

- (i) $A^*(H) \cap K \subseteq A^*(K)$; and
- (ii) $L_K(D) = L_H(D) \cap K$ for every $D \subseteq K$.
- (iii) $L_H(A) \cap K \subseteq L_H(A \cap K)$ for any $A \subseteq H$.

Proof.

- (i) Let $a \in A^*(H) \cap K$. Then $a \in K$ and for all $x \in H$, $x \ll a$ implies that $x = a$ or $x = 0$. In particular, for all $y \in K$, $y \ll a$ implies $y = 0$ or $y = a$. Thus, $a \in A^*(K)$.
- (ii) Let $D \subseteq K$. Then $z \in L_K(D)$ if and only if $z \in K$ and $z \ll d$ for all $d \in D$. Thus, $z \in L_K(D)$ if and only if $z \in K \cap L_H(d)$ for each $d \in D \subseteq K \subseteq H$. Consequently, $L_K(D) = K \cap L_H(D)$.
- (iii) Let $A \subseteq H$. Since $A \cap K \subseteq A$, by Proposition 1(iii), $L_H(A) \subseteq L_H(A \cap K)$. Thus, $L_H(A) \cap K \subseteq L_H(A \cap K) \cap K = L_K(A \cap K)$, by (ii). Hence, $L_H(A) \cap K \subseteq L_K(A \cap K)$. \square

Lemma 4. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H . Then for any $\emptyset \neq A \subseteq H$, $L_H(A) \cap K = \bigcup_{x \in L_H(A) \cap K} L_K(x)$.*

Proof. Let $\emptyset \neq A \subseteq H$ and $x \in L_H(A) \cap K$. Then $x \ll a$ for all $a \in A$ and $x \in K$. Let $y \in L_H(x) \cap K$. Then $y \ll x$ and $y \in K$. Since H is ordered, $y \ll a$ for all $a \in A$. Hence, $y \in L_H(A) \cap K$ showing that $L_H(x) \cap K \subseteq L_H(A) \cap K$. Consequently, $\bigcup_{x \in L_H(A) \cap K} (L_H(x) \cap K) \subseteq L_H(A) \cap K$.

Next, let $z \in L_H(A) \cap K$. By Proposition 1(v), $z \in L_H(z)$. It follows that $z \in L_H(z) \cap K$ showing that $L_H(A) \cap K \subseteq L_H(z) \cap K$. Thus, $L_H(A) \cap K \subseteq \bigcup_{x \in L_H(A) \cap K} (L_H(x) \cap K)$.

Therefore, by Lemma 3(ii),

$$L_H(A) \cap K = \bigcup_{x \in L_H(A) \cap K} (L_H(x) \cap K) = \bigcup_{x \in L_H(A) \cap K} L_K(x).$$

This proves the assertion. \square

Theorem 10. *Let K be a hyper subalgebra of an ordered hyper BCI-algebra H with $|K| \geq 2$. Then $\tau_L(K)$ coincides with the relative topology τ_K on K .*

Proof. By Theorem 1 and Theorem 2, bases for τ_K and $\tau_L(K)$ are given by the families $\mathcal{B}_K = \{L_H(A) \cap K : \emptyset \neq A \subseteq H\}$ and $\mathcal{B}_L(K) = \{L_K(A) : \emptyset \neq A \subseteq K\}$, respectively.

Let $U = L_H(A) \cap K \in \mathcal{B}_K$ and let $x \in U$. Since $x \in L_H(x)$ and $x \in K$, $x \in L_H(x) \cap K = L_K(x)$ by Lemma 3. By Lemma 4, $L_K(x) \subseteq \bigcup_{y \in L_H(A) \cap K} L_K(y) = L_H(A) \cap K$.

Take $U' = L_H(x)$. It follows that $\tau_K \subseteq \tau_L(K)$.

To show the other inclusion, let $U \in \mathcal{B}_L(K)$. Then there exists $B \subseteq K$ such that $U = L_K(B)$. By Lemma 3, $U = L_K(B) = L_H(B) \cap K \in \mathcal{B}_K$. Hence, $\mathcal{B}_L(K) \subseteq \mathcal{B}_K$, that is, $\tau_L(K) \subseteq \tau_K$. Therefore, $\tau_L(K) = \tau_K$. \square

Conclusion: An operator on the power set of a hyper BCI-algebra into the family of its nonempty subsets had been defined via left application of the hyper-order associated with the hyper BCI-algebra. The collection of images of subsets under this operator turned out to be a basis for some topology on the given hyperstructure. The topological space generated in this way enabled us to look into the topological structure of hyper BCI-algebra in many ways. In particular, under some conditions on the hyper BCI-algebra, elementary concepts associated with the space such as open, closed, density, closure, interior, and relative space had been described or characterized.

The topological space generated in this study may be studied further for other topological aspects such as connectedness and compactness. Also, if it were possible to define hyper-orders on the sum (or join) and product of two hyper BCI-algebras so as to obtain two hyper BCI-algebras, it would be interesting to know what the respective bases would be for the sum and product. Further, it may be worthwhile to investigate whether or not the right application of the hyper-order or the combination of the left and right applications will also give rise to a topological space. If any of these does, then one needs to know if the resulting space is the same (or homeomorphic) to the one generated in this study.

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